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THE NONLOCAL MULTIPOINT PROBLEM WITH DIRICHLET-TYPE CONDITIONS FOR AN ORDINARY DIFFERENTIAL EQUATION OF EVEN ORDER WITH INVOLUTION

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The spectral properties of the nonself-adjoint problem with multipoint perturbations of the Dirichlet conditions for differential operator of order $2n$ with involution are investigated. The system of eigenfunctions of a multipoint problem is constructed. Sufficient conditions have been established, under which this system is complete and, under some additional assumptions, forms the Riesz basis. The research is structured as follows. In section 2 we investigate the properties of the Sturm-type conditions and nonlocal problem with self-adjoint boundary conditions for the equation

$$(-1)^n y^{(2n)}(x) + a_0 y^{(2n-1)}(x) + a_1 y^{(2n-1)}(1-x) = f(x),\ x \in (0,1).$$

In section 3 we study the spectral properties for nonlocal problem with nonself-adjoint boundary conditions for this equation. In sections 4 we construct a commutative group of transformation operators. Using spectral properties of multipoint problem and conditions for completeness the basis properties of the systems of eigenfunctions are established in section 5. In section 6 some analogous results are obtained for multipoint problems generated by differential equations with an involution and are proved the main theorems.

1. Introduction and main results. During recent years the number of publications with the use of an involution operator in the theory of PT-symmetric operators, PT-symmetric quantum theory (see [11], [12]), various sections of the theory of ordinary differential equations (see [1], [4], [15], [17], [19], [21]–[24], [27], [31], [32]), partial differential equations (see [6], [10], [14], [20], [25]) and differential equations with operator coefficients (see [2], [3], [7], [8]) increased significantly.

The spectral properties of boundary-value problems with strongly regular conditions by Birkhoff were studied in [9], [19], [23]. The properties of problems with regular but not strongly regular conditions by Birkhoff were studied in the papers [5], [24], [29]. Problems with irregular conditions by Birkhoff were studied in [16], [29].

This paper is a continuation of the research [2]–[10].

Let

$$W_2^{2n}(0,1) := \left\{ y \in L_2(0,1) \colon\ y^{(m)} \in AC[0,1],\, y^{(2n)} \in L_2(0,1),\, m = 0,1,...,2n-1 \right\},$$

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W^* be the space of linear and continuous functionals over $W_2^{2n}(0,1)$,

$$L_{2,j}(0,1) := \{y \in L_2(0,1) \colon \ y(x) = (-1)^j y(1-x)\}, \ j = 0, 1,$$

$[L_2(0,1)]$ the algebra of linear and bounded operators $A\colon\ L_2(0,1) \to L_2(0,1)$.

We are considering the multipoint problem

$$(-1)^n y^{(2n)}(x) + a_0 y^{(2n-1)}(x) + a_1 y^{(2n-1)}(1-x) = f(x), \, x \in (0,1), \tag{1}$$

$$\ell_j y := y^{(j-1)}(0) - (-1)^j y^{(j-1)}(1) = 0, \ j = 1, 2, ..., n, \tag{2}$$

$$\ell_{n+j} y := y^{(j-1)}(0) + (-1)^j y^{(j-1)}(1) + \ell_j^1 y = 0, \ j = 1, 2, ..., n, \tag{3}$$

where

$$\ell_j^1 y := \sum_{s=0}^{k_0} \sum_{m=0}^{k_j} b_{s,m,j} y^{(m)}(x_s), \tag{4}$$

$a_0, \ a_1, \ b_{s,m,j} \in \mathbb{R}, \, s = 0, 1, ..., k_0, \, 0 = x_0 < x_1 < ... < x_{k_0} = 1, \ m = 0, 1, ..., k_j, \ j = 1, 2, ..., n, \ p = 0, 1.$

Definition 1. The function $y(x)$ from the space $W_2^{2n}(0,1)$ that satisfies nonlocal conditions (2)–(4) and equation (1) in the sense of equality in space $L_2(0,1)$ is called a *solution of problem* (1)–(4).

Let L be the operator of problem (1)–(4):

$$Ly := (-1)^n y^{(2n)}(x) + a_0 y^{(2n-1)}(x) + a_1 y^{(2n-1)}(1-x), \, y \in D(L),$$
$$D(L) := \left\{ y \in W_2^{2n}(0,1) \colon \ell_s y = 0, \ s = 1, 2, ..., 2n \right\},$$
$$V(L) := \{v_{r,q}(x, L) \in L_2(0,1), \ r = 0, 1, \ q = 1, 2, ...\}$$

the system of eigenfunctions of the operator L.

Let us consider the following assumptions, that are necessary for further investigation.

Assumption P_1 :

$$b_{s,m,j} = (-1)^m b_{k_0 - s, m, j} \in \mathbb{R}, \ x_s = 1 - x_{k_0 - s},$$

where $s = 0, 1, ..., k_0, \ m = 0, 1, ..., k_j, \ j = 1, 2, ..., n.$

Assumption P_2 : $a_1 = -a_0$.

Assumption P_3 : $k_j \leq j - 1, \ j = 1, 2, ..., n.$

The following theorems are the main result of the paper.

Theorem 1. *1. Let Assumptions $P_1, \ P_2$ hold. Then the operator L has discrete spectrum and complete and minimal system of eigenfunctions $V(L)$ in the space $L_2(0,1)$.*

2. If Assumptions P_1-P_3 hold, then the system $V(L)$ is a Riesz basis of the space $L_2(0,1)$.

Theorem 2. *If Assumptions P_1-P_3 hold, then for arbitrary function*

$$f = \sum_{r=0}^{1} \sum_{q=1}^{\infty} f_{r,q} v_{r,q}(x, L) \in L_2(0,1)$$

the unique solution $u \in W_2^{2n}(0,1)$ of problem (5)–(7) exists.

Our research is structured as follows. In section 2 we investigate the properties of the Sturm-type conditions and nonlocal problem with self-adjoint boundary conditions for the equation (1). In section 3 we study the spectral properties for nonlocal problem with nonself-adjoint boundary conditions for the equation (1). In section 4 we construct a commutative group of transformation operators. Using spectral properties of multipoint problem and conditions for completeness the basis properties of the systems of eigenfunctions are established in section 5. In section 6 some analogous results are obtained for multipoint problems generated by differential equations with an involution and proved the main theorems.

2. Self-adjoint boundary value problems. Let us consider for the equation

$$(-1)^n y^{(2n)}(x) = f(x),\ x \in (0,1), \tag{5}$$

the problem with boundary conditions

$$\ell_{0,j} y := y^{(j-1)}(0) - (-1)^j y^{(j-1)}(1) = 0,\ j = 1,2,...,n, \tag{6}$$

$$\ell_{0,n+j} y := y^{(j-1)}(0) + (-1)^j y^{(j-1)}(1) = 0,\ j = 1,2,...,n. \tag{7}$$

Let L_0 be the operator of problem (5)–(7):

$$L_0 y := (-1)^n y^{(2n)}(x),\ y \in D(L_0),\quad D(L_0) := \left\{ y \in W_2^{2n}(0,1) \colon\ \ell_{0,s} y = 0,\ s = 1,2,...,2n \right\}.$$

Remark 1. The boundary conditions (6)–(7) are chosen so that the relations

$$\ell_{0,j} \in W_0^*,\ \ell_{0,n+j} \in W_1^*,\ j = 1,2,...,n, \tag{8}$$

hold, where $W_p^*(0,1) := \{\ell \in W^* \colon L_{2,1-p}(0,1) \subset ker(\ell)\},\ p = 0,1.$

Remark 2. The boundary conditions (6)–(7) are equivalent to self-adjoint conditions of Sturm-type (see [26]) and strongly regular by Birkhoff [19], [23], [26]:

$$y^{(j-1)}(0) = y^{(j-1)}(1) = 0,\ j = 1,2,...,n.$$

Therefore, the conditions (6)–(7) are self-adjoint.

Thus, the operator L_0 is generated by expression $(-1)^n y^{(2n)}(x)$ and boundary conditions (6)–(7) and is self-adjoint.

We define the eigenfunctions of operator L_0. The roots $\rho_j,\ j = 1,2,...,n$, of characteristic equation $(-1)^n \rho^{2n} = \lambda$, $|\arg \rho| \le \frac{1}{2n}\pi$, which corresponds to the differential equation

$$(-1)^n y^{(2n)} - \lambda y = 0,\ \lambda \in \mathbb{R}, \tag{9}$$

are determined by the relation $\rho_j = \omega_j \rho$, where

$$(-1)^n \omega_j^{2n} = 1,\ \omega_1 = \imath,\ \omega_j = \omega_1 \exp\left(\frac{\pi \imath}{n}(j-1)\right),\ j = 2,3,...,n.$$

Let us consider the fundamental system of differential equation (9):

$$y_j(x,\rho) := e^{\omega_j \rho x} + e^{\omega_j \rho (1-x)} \in L_{2,0}(0,1),\ j = 1,2,...,n, \tag{10}$$

$$y_{n+j}(x,\rho) := e^{\omega_j \rho x} - e^{\omega_j \rho (1-x)} \in L_{2,1}(0,1),\ j = 1,2,...,n. \tag{11}$$

Substituting the general solution

$$y(x,\rho)=\sum_{s=1}^{2n} C_s y_s(x,\rho),\ C_s\in\mathbb{C}, \tag{12}$$

of differential equation (9) into boundary conditions (6)–(7), we obtain the equation for determining the eigenvalues of the operator L_0 :

$$\Delta(\rho,L_0):=\det[\ell_{0,r}y_s]_{r,s=\overline{1,2n}}=0.$$

From the relations (8), (10), (11) we obtain the equalities

$$\ell_{0,j}y_{n+r}(x,\rho)=0,\ \ell_{0,n+j}y_r(x,\rho)=0,\ j,r=1,2,...,n.$$

Therefore, $\Delta(\rho,L_0)=\Delta_0(\rho,L_0)\Delta_1(\rho,L_0)$, where

$$\Delta_0(\rho,L_0):=\det[\ell_{0,j}y_r]_{r,j=\overline{1,n}},\quad \Delta_1(\rho,L_0):=\det[\ell_{0,n+j}y_{n+r}]_{r,j=\overline{1,n}}.$$

Let $\rho_{s,q}$ be solutions of the equation $\Delta_s(\rho,L_0)=0$ and $\lambda_{s,q}:=(-1)^n\rho_{s,q}^{2n}$ are corresponding eigenvalues of operator L_0, that numbered in ascending order, $s=0,1,\ q=1,2,\ldots$.

We construct the system of eigenfunctions of the operator L_0.

By the elements of systems (10), (11) define the functions

$$v_{0,q}(x,L_0):=\gamma_{0,q}\left\|\begin{matrix} y_1(x,\rho_{0,q}) & ... & y_n(x,\rho_{0,q})\\ \ell_{0,2}y_1 & ... & \ell_{0,2}y_n\\ ... & ... & ...\\ \ell_{0,n}y_1 & ... & \ell_{0,n}y_n\end{matrix}\right\|,\ \ q=1,2,...,$$

which after some transformations take the form

$$v_{0,q}(x,L_0)=\gamma_{1,q}\times$$

$$\times\left\|\begin{matrix} y_1(x,\rho_{0,q}) & ... & y_r(x,\rho_{0,q}) & ... & y_n(x,\rho_{0,q})\\ \omega_1(1-e^{\omega_1\rho_{0,q}}) & ... & \omega_r(1-e^{\omega_r\rho_{0,q}}) & ... & \omega_n(1-e^{\omega_n\rho_{0,q}})\\ ... & ... & ... & ... & ...\\ \omega_1^{n-1}(1-(-1)^n e^{\omega_1\rho_{0,q}}) & ... & \omega_r^{n-1}(1-(-1)^n e^{\omega_r\rho_{0,q}}) & ... & \omega_n^{n-1}(1-(-1)^n e^{\omega_n\rho_{0,q}})\end{matrix}\right\|, \tag{13}$$

where $q=1,2,\ldots$.

Note, that we select the parameters $\gamma_{1,q}$ so that $\|v_{0,q}(x,L_0)\|_{L_2(0,1)}=1,\ q=1,2,\ldots$.

Similarly, define eigenfunctions $v_{1,q}(x,L_0)\in L_{2,1}(0,1)$:

$$v_{1,q}(x,L_0):=\gamma_{2,q}\left\|\begin{matrix} y_{n+1}(x,\rho_{1,q}) & ... & y_{2n}(x,\rho_{1,q})\\ \ell_{0,n+2}y_{n+1} & ... & \ell_{0,n+2}y_{2n}\\ ... & ... & ...\\ \ell_{0,2n}y_{n+1} & ... & \ell_{0,2n}y_{2n}\end{matrix}\right\|,\ q=1,2,..., \tag{14}$$

where parameters $\gamma_{2,q}$ are selected so that equalities $\|v_{1,q}(x,L_0)\|_{L_2(0,1)}=1,\ q=1,2,...$, are valid.

Thus, the system of eigenfunctions

$$V(L_0):=\{v_{s,q}(x,L_0)\in L_2(0,1):\ \ s=0,1,\ q=1,2,...\}$$

of the self-adjoint operator L_0 is an orthonormal basis of space $L_2(0,1)$.

The self-adjoint operator L_0 has only its eigenfunctions, each of which is defined by formulas (13) or (14).

Therefore, the integration of systems

$$V_s(L_0) := \{v_{s,q}(x, L_0) \in L_2(0,1),\ q = 1, 2, ...\},\ s = 0, 1,$$

is an orthonormal basis of space $L_2(0,1)$.

Considering the ratio $L_2(0,1) = L_{2,0}(0,1) \oplus L_{2,1}(0,1)$ we obtain the following conclusions:

1. The systems of functions

$$V_s(L_0) := \{v_{s,q}(x, L_0) \in L_2(0,1),\ q = 1, 2, ...\},\ s = 0, 1,$$

are orthonormal bases in $L_{2,0}(0,1)$ and $L_{2,1}(0,1)$, respectively.

2. $L_0 \colon L_{2,s}(0,1) \to L_{2,s}(0,1),\ s = 0, 1$.

Let us determine eigenfunctions (13), (14) by the relations

$$v_{0,q}(x, L_0) = \gamma_{1,q} \sum_{r=1}^{n} \Delta^0_{0,r}(\rho_{0,q}) y_r(x, \rho_{0,q}),$$

$$\Delta^0_{0,r}(\rho_{0,q}) :=$$

$$= \left\| \begin{matrix} \omega_1(1 - e^{\omega_1\rho_{0,q}}) & ... & \omega_{r-1}(1 - e^{\omega_{r-1}\rho_{0,q}}) & \omega_{r+1}(1 - e^{\omega_{r+1}\rho_{0,q}}) & ... & \omega_n^1(1 - e^{\omega_n\rho_{0,q}}) \\ \omega_1^2(1 + e^{\omega_1\rho_{0,q}}) & ... & \omega_{r-1}^2(1 + e^{\omega_{r-1}\rho_{0,q}}) & \omega_{r+1}^2(1 + e^{\omega_{r+1}\rho_{0,q}}) & ... & \omega_n^2(1 + e^{\omega_n\rho_{0,q}}) \\ ... & ... & ... & ... & ... & ... \\ \omega_1^{n-1}(1-(-1)^n e^{\omega_1\rho_{0,q}}) & ... & \omega_{r-1}^{n-1}(1-(-1)^n e^{\omega_{r-1}\rho_{0,q}}) & \omega_{r+1}^{n-1}(1-(-1)^n e^{\omega_{r+1}\rho_{0,q}}) & ... & \omega_n^{n-1}(1-(-1)^n e^{\omega_n\rho_{0,q}}) \end{matrix} \right\|,$$

$$v_{1,q}(x, L_0) = \gamma_{2,q} \sum_{r=1}^{n} \Delta^0_{1,r}(\rho_{1,q}) y_{n+r}(x, \rho_{1,q}),$$

$$\Delta^0_{1,r}(\rho_{1,q}) :=$$

$$= \left\| \begin{matrix} \omega_1(1 + e^{\omega_1\rho_{1,q}}) & ... & \omega_{r-1}(1 + e^{\omega_{r-1}\rho_{1,q}}) & \omega_{r+1}(1 + e^{\omega_{r+1}\rho_{1,q}}) & ... & \omega_n^1(1 + e^{\omega_n\rho_{1,q}}) \\ \omega_1^2(1 - e^{\omega_1\rho_{1,q}}) & ... & \omega_{r-1}^2(1 - e^{\omega_{r-1}\rho_{1,q}}) & \omega_{r+1}^2(1 - e^{\omega_{r+1}\rho_{1,q}}) & ... & \omega_n^2(1 - e^{\omega_n\rho_{1,q}}) \\ ... & ... & ... & ... & ... & ... \\ \omega_1^{n-1}(1+(-1)^n e^{\omega_1\rho_{1,q}}) & ... & \omega_{r-1}^{n-1}(1+(-1)^n e^{\omega_{r-1}\rho_{1,q}}) & \omega_{r+1}^{n-1}(1+(-1)^n e^{\omega_{r+1}\rho_{1,q}}) & ... & \omega_n^{n-1}(1+(-1)^n e^{\omega_n\rho_{1,q}}) \end{matrix} \right\|,$$

where $r = 1, 2, ..., n,\ \ q = 1, 2, ...$.

For equation (5) consider the self-adjoint problem with boundary conditions:

$$\ell_{1,p,j}y := y^{(j-1)}(0) - (-1)^j y^{(j-1)}(1) = 0,\ j = 1, 2, ..., n, \tag{15}$$

$$\ell_{1,p,n+p}y := y^{(2n-p)}(0) - (-1)^p y^{(2n-p)}(1) = 0, \tag{16}$$

$$\ell_{1,p,n+j}y := y^{(j-1)}(0) + (-1)^j y^{(j-1)}(1) = 0,\ j \neq p,\ j = 1, 2, ..., n. \tag{17}$$

Let $L_{1,p}$ be the operator of problem (5), (16)–(17):

$$L_{1,p}y := (-1)^n y^{(2n)}(x),\ y \in D(L_{1,p}),$$
$$D(L_{1,p}) := \{y \in W_2^{2n}(0,1) :\ \ell_{1,p,s}y = 0,\ s = 1, 2, ..., 2n\}.$$

Substituting the general solution (12) of the equation (5) into boundary conditions (15)–(17) we obtain the equation

$$\Delta(\rho, L_{1,p}) = \Delta_0(\rho, L_{1,p})\Delta_1(\rho, L_{1,p}) = 0$$

to determine the eigenvalues of the self-adjoint operator $L_{1,p}$, where

$$\Delta(\rho, L_{1,p}) = \det[\ell_{1,p,s}y_r(x,\rho)]_{s,r=1}^{2n}, \quad \Delta_0(\rho, L_{1,p}) = \det[\ell_{1,p,s}y_r(x,\rho)]_{r,s=1}^{n},$$
$$\Delta_1(\rho, L_{1,p}) = \det[\ell_{1,p,n+s}y_{n+r}(x,\rho)]_{s,r=1}^{n}.$$

The boundary conditions (15)–(17) are chosen so that the relations

$$\ell_{1,j} \in W_0^*, \ \ell_{1,n+p} \in W_1^*, \ j = 1, 2, ..., n,$$

hold. Therefore $\Delta(\rho, L_{1,p}) = \Delta_0(\rho, L_{1,p})\Delta_1(\rho, L_{1,p})$.

Since boundary conditions (6) and (15) coincide, then

$$\Delta_0(\rho, L_0) \equiv \Delta_0(\rho, L_{1,1}) \equiv ... \equiv \Delta_0(\rho, L_{1,n}).$$

Substituting functions $v_{0,q}(x, L_0)$ into boundary conditions (16)–(17), we conclude that $v_{0,q}(x, L_0) \in D(L_{1,p}), \ q = 1, 2, ...$.

Therefore, the self-adjoint operators $L_0, \ L_{1,p}, \ p \in \{1, 2, ..., n\}$, have the same part of eigenvalues $\sigma_0(L_0) = \{\lambda_{0,q}(L_0), \ q = 1, 2, ...\}$ and eigenfunctions $V_0(L_0)$, and systems $V_r(L_{1,p}) \subset L_{2,r}(0,1)$ are orthonormal bases in $L_{2,r}(0,1), \ r = 0, 1$.

3. Nonlocal boundary problems. For equation (5) we consider the problem with boundary conditions:

$$\ell_{2,p,j}y := y^{(j-1)}(0) - (-1)^j y^{(j-1)}(1) = 0, \ j = 1, 2, ..., n, \tag{18}$$
$$\ell_{2,p,n+p}y := y^{(2n-p)}(0) - (-1)^p y^{(2n-p)}(1) + b_p\big(y^{(2n-p)}(0) + (-1)^p y^{(2n-p)}(1)\big) = 0, \tag{19}$$
$$\ell_{2,p,n+j}y := y^{(j-1)}(0) + (-1)^{j-1} y^{(j-1)}(1) = 0, \ j \neq p, \ j = 1, 2, ..., n, \tag{20}$$

for $b_p \in \mathbb{R}, \ p \in \{1, 2, ..., n\}$.

Let $L_{2,p}$ be the operator of problem (5), (18)–(20):

$$L_{2,p}y := (-1)^n y^{(2n)}(x), \ y \in D(L_{2,p}),$$
$$D(L_{2,p}) := \{y \in W_2^{2n}(0,1): \ \ell_{2,p,s}y = 0, \ s = 1, 2, ..., 2n\}.$$

Lemma 1. *The eigenvalues of operators $L_{1,p}$ and $L_{2,p}$ coincide. The system of eigenfunctions $V(L_{2,p})$ of the operator $L_{2,p}$ is a Riesz basis of the space $L_2(0,1)$.*

Proof. Let us consider the spectral problem (8), (18)–(20).

Substituting the general solution (9) of differential equation (10) into boundary conditions (18)–(20), we obtain the equation to determine the eigenvalues of the operator $L_{2,p}$:

$$\Delta(\rho, L_{2,p}) := \det[\ell_{2,p,r}y_k]_{r,k=\overline{1,2n}} = 0.$$

From (8), (10), (11), (18)–(20) we have equalities

$$\ell_{2,p,n+p}y_{n+r}(x,\rho) = \ell_{1,p,n+p}y_{n+r}(x,\rho), \quad \ell_{2,p,n+j}y_r(x,\rho) = \ell_{1,p,n+j}y_r(x,\rho) = 0, \ p \neq j,$$
$$\ell_{2,p,j}y_{n+r}(x,\rho) = 0, \ p, j, r = 1, 2, ..., n.$$

Therefore, $\Delta(\rho, L_{2,p}) = \Delta(\rho, L_{1,p})$, where

$$\Delta(\rho, L_{1,p}) = \Delta_0(\rho, L_{1,p})\Delta_1(\rho, L_{1,p}), \quad \Delta_0(\rho, L_{1,p}) = \det[\ell_{1,p,r}y_j]_{r,j=\overline{1,n}},$$

$$\Delta_1(\rho, L_{1,p}) = \det[\ell_{1,p,n+r} y_{n+j}]_{r,j=\overline{1,n}}.$$

Consequently, the operators $L_{1,p}$ and $L_{2,p}$ have the same eigenvalues $\lambda_{s,q}(L_{1,p}) = \lambda_{s,q}(L_{2,p})$, $s = 0, 1$, $q = 1, 2, \ldots$, and the same system of eigenfunctions $V_1(L_0) \subset L_{2,1}(0, 1)$.

We construct the eigenfunctions $v_{0,q}(x, L_{2,p})$ of the operator $L_{2,p}$, that correspond to eigenvalues $\lambda_{0,q}(L_{1,p}) = \lambda_{0,q}(L_{2,p})$, $q = 1, 2, \ldots$, respectively.

Consider the functions

$$y_{1,s}(x, \rho_{0,q}) := e^{\omega_s \rho_{0,q} x} - e^{\omega_s \rho_{0,q}(1-x)} \in L_{2,1}(0, 1),\ s = 1, 2, \ldots, n,$$

$$y_{2,p}(x, \rho_{0,q}) := \sum_{s=1}^{n} \Delta_{p,s}(\rho_{0,q}) y_{1,s}(x, \rho_{0,q}), \tag{21}$$

where

$$\Delta_{p,s}(\rho_{0,q}) := \det[\ell_{2,p,n+j} y_{1,r}(x, \rho_{0,q})]_{j,r=\overline{1,n}},\ j \neq p,\ r \neq s.$$

If we substitute functions (21) into boundary conditions (18)–(20), then we obtain

$$\ell_{2,p,j} y_{2,p}(x, \rho_{0,q}) = 0,\ j \neq n+p,\ j = 1, 2, \ldots, 2n,\ p = 1, 2, \ldots, n. \tag{22}$$

The eigenfunctions $v_{0,q}(x, L_{2,p})$ of the operator $L_{2,p}$ are defined by expression

$$v_{0,q}(x, L_{2,p}) := v_{0,q}(x, L_0) + \eta_{p,q} y_{2,p}(x, \rho_{0,q}),\ q = 1, 2, \ldots. \tag{23}$$

Substituting expression (23) into boundary conditions (19), we have

$$\eta_{p,q} = -(\ell_{2,p,n+p} y_{2,p}(x, \rho_{0,q}))^{-1} \ell_{2,p,n+p} v_{0,q}(x, L_0).$$

Let $y_{3,p}(x, \rho_{0,q}) := \eta_{p,q} y_{2,p}(x, \rho_{0,q})$ for $b_p = 1$. Then from the formula (23) we obtain

$$v_{0,q}(x, L_{2,p}) = v_{0,q}(x, L_0) + b_p y_{3,p}(x, \rho_{0,q}),\ q = 1, 2, \ldots.$$

Therefore, the operator $L_{2,p}$ has the system of eigenfunctions

$$V(L_{2,p}) := \{v_{s,q}(x, L_{2,p}) \in L_2(0, 1),\ s = 0, 1,\ q = 1, 2, \ldots\},$$

where

$$v_{1,q}(x, L_{2,p}) = v_{1,q}(x, L_0),\ q = 1, 2, \ldots. \tag{24}$$

The conditions (18)–(20) are strongly regular by Birkhoff [26]. Thus, the system $V(L_{2,p})$ is a Riesz basis in the space $L_2(0, 1)$. □

Remark 3. The Riesz basis is almost normalized system [16]. Therefore, taking into account the expression

$$\|v_{0,q}(x, L_{2,p})\|_{L_2(0,1)} = 1 + |b_p|\, \|y_{3,p}(x, \rho_{0,q})\|_{L_2(0,1)} < +\infty,$$

we obtain the boundedness of number sequence

$$C_{0,p} < \|y_{3,p}(x, \rho_{0,q})\|_{L_2(0,1)} \leq C_{1,p} < +\infty,\ p = 1, 2, \ldots, n,\ q = 1, 2, \ldots.$$

4. Transformation operators. We choose arbitrary number sequence $\Theta := \{\theta_q\}_{q=1}^{\infty} \subset \mathbb{R}$ and consider the operator $B_{p,\Theta}$, defined in space $L_2(0,1)$, whose eigenvalues coincide with the eigenvalues of the operator L_0 and eigenfunctions are determined by the relations

$$v_{1,q}(x, B_{p,\Theta}) := v_{1,q}(x, L_0),\ q = 1, 2, \dots, \tag{25}$$

$$v_{0,q}(x, B_{p,\Theta}) := v_{0,q}(x, L_0) + \theta_q y_{3,p}(x, \rho_{0,q}),\ q = 1, 2, \dots . \tag{26}$$

Let
$R(B_{p,\Theta}):\ L_2(0,1) \to L_2(0,1)$ and $R(B_{p,\Theta})v_{s,q}(x, L_0) := v_{s,q}(x, B_{p,\Theta}), s = 0, 1, q = 1, 2, \dots .$
From the definition of the operator $R(B_{p,\Theta}) := E + S(B_{p,\Theta})$ we obtain

$$S(B_{p,\Theta}):\ L_{2,0}(0,1) \to L_{2,1}(0,1),\ S(B_{p,\Theta}):\ L_{2,1}(0,1) \to 0,\ S^2(B_{p,\Theta}) = 0.$$

Then the operator $R^{-1}(B_{p,\Theta}) = E - S(B_{p,\Theta})$ exists.

Lemma 2. *For any sequence $\{\theta_q\}_{q=1}^{\infty} \subset \mathbb{R}$ the system of functions $V(B_{p,\Theta})$ is complete and minimal in $L_2(0,1)$, $p = 1, 2, \dots, n$.*

Proof. We prove on the contrary that the system of functions $V(B_{p,\Theta})$ is total (complete) in the space $L_2(0,1)$.

Let a function $h = h_0 + h_1$, $h_s \in L_{2,s}(0,1)$, exist, that is orthogonal to all elements of the system $V(B_{p,\Theta})$. Taking into account, that the system (25) is an orthonormal basis of space $L_{2,1}(0,1)$, we obtain $h_1 \equiv 0$.

Therefore $h \in L_{2,0}(0,1)$.

Assuming the orthogonality of the function h to the elements of the system $V(B_{p,\Theta})$, we have equality

$$\big(h, v_{0,q}(x, B_{p,\Theta})\big)_{L_2(0,1)} = \big(h, v_{0,q}(x, L_0)\big)_{L_2(0,1)} = 0,\ q = 1, 2, \dots .$$

Taking into account, that the system $V_0(L_0)$ is an orthonormal basis of space $L_{2,0}(0,1)$, we obtain $h \equiv 0$. □

We consider the operators $L_{0,\Theta}$, $L_{p,\Theta}$ that are defined in space $L_2(0,1)$ and whose eigenfunctions and eigenvalues are determined by the relations

$$\lambda_{r,q}(L_{0,\Theta}) := \lambda_{r,q}(L_{p,\Theta}) := |\theta_q|\lambda_{r,q}(L_0),\ r = 0, 1,\ q = 1, 2, \dots,$$
$$v_{r,q}(x, L_{0,\Theta}) := v_{r,q}(x, L_0),\ r = 0, 1,\ q = 1, 2, \dots,$$
$$v_{r,q}(x, L_{p,\Theta}) := v_{r,q}(x, L_{2,p}),\ r = 0, 1,\ q = 1, 2, \dots .$$

Let

$$H(L_{p,\Theta}) := \big\{v \in L_2(0,1):\ L_{p,\Theta}v \in L_2(0,1)\big\},$$
$$\big(u, v\big)_{H(L_{p,\Theta})} := \big(u, v\big)_{L_2(0,1)} + \big(L_{p,\Theta}u, L_{p,\Theta}v\big)_{L_2(0,1)},\ p = 0, 1, \dots, n.$$

Taking into account Lemma 1, we obtain the inequality

$$\|L_{p,\Theta}v\|_{H(L_{0,\Theta})} \le \|[L_{2,p}]\|\|v\|_{H(L_{0,\Theta})},\ v \in H(L_{0,\Theta}).$$

From the definition of functions $v_{0,q}(x, B_{p,\Theta})$ we obtain

$$v_{0,q}(x, B_{p,\Theta}) = b_p^{-1}(\theta_q v_{0,q}(x, L_{2,p}) + (1 - \theta_q)v_{0,q}(x, L_0)),\ q = 1, 2, \dots .$$

Therefore, for any $h = \sum\limits_{r=0}^{1}\sum\limits_{q=1}^{\infty} h_{r,q} v_{r,q}(x, L_0) \in H(L_{0,\Theta})$ we obtain the inequality

$$\|R(B_{p,\Theta})h\|^2_{L_2(0,1)} \le 2b_p^{-2}((1 + \|[R(L_{2,p})]\|^2)\|h\|^2_{H(L_{0,\Theta})} + 2\|h\|^2_{L_2(0,1)}) < \infty.$$

Consider the relation that defines the conjugate operator (see [14]):

$$\big(R(B_{p,\Theta})h, g\big)_{L_2(0,1)} = \big(h, R^*(B_{p,\Theta})g\big)_{L_2(0,1)},\ h \in D(R(B_{p,\Theta})),\ g \in D(R^*(B_{p,\Theta})).$$

Therefore, the operator $(R^*(B_{p,\Theta}))^{-1} = E - S^*(B_{p,\Theta})$ exists and the system of functions

$$W(B_{p,\Theta}) := \{w_{r,q}(x, L_{p,\Theta}) \in L_2(0,1) :$$
$$w_{r,q}(x, L_{p,\Theta}) := (E - S^*(B_{p,\Theta}))v_{r,q}(x, L_0),\ r = 0, 1,\ q = 1, 2,\ ...\}$$

is biorthogonal to the system $V(B_{p,\Theta})$. □

Lemma 3. *The system of functions $V(B_{p,\Theta})$ is a Riesz basis in $L_2(0,1)$ if and only if the sequence $\{\theta_q\}_{q=1}^{\infty}$ is bounded.*

Proof. Necessity. If the system of functions $V(B_{p,\Theta})$ is a Riesz basis, then it is almost normalized.

From the contrary, if $|\theta_q| \to \infty$ for $q \to \infty$, then, taking into account (24)–(26) and Remark 3, we obtain

$$\|v_{0,q}(x, B_{p,\Theta})\|_{L_2(0,1)} = 1 + b_p^{-1}|\theta_q|\|y_{3,p}(x, \rho_{0,q})\|_{L_2(0,1)} \to \infty,\ q \to \infty.$$

Sufficiency. A according to account (25), (26), for arbitrary function $h \in L_2(0,1)$ we obtain the following relations

$$\sum_{q=0}^{\infty}\sum_{s=0}^{1} |(R^*(B_{p,\Theta})h, v_{s,q}(x, L_0))|^2_{L_2(0,1)} = \sum_{q=0}^{\infty}\sum_{s=0}^{1} |(h, v_{s,q}(x, B_{p,\Theta}))|^2_{L_2(0,1)}.$$

$$\sum_{q=0}^{\infty}\sum_{s=0}^{1} |(h, v_{s,q}(x, B_{p,\Theta}))|^2_{L_2(0,1)} \le \|[B_{p,\Theta}]\|^2\|h\|^2_{L_2(0,1)}.$$

Therefore, the operators $R^*(B_{p,\Theta})$ and $(R^*(B_{p,\Theta}))^{-1}\colon\ L_2(0,1) \to L_2(0,1)$ are bounded. Then the system of functions $W(B_{p,\Theta})$ and $V(B_{p,\Theta})$ is Riesz basis by definition (see [15]).□

The set of operators $B_{p,\Theta}$, eigenfunctions of which are determined by formulas (25), (26) is denoted by $\Gamma_p(L_0)$. The corresponding set of transformation operators $R(B_{p,\Theta}) = E + S(B_{p,\Theta})$, we denote by $\Phi_p(L_0)$.

We introduce the operation of multiplication of transformation operators on the set $\Phi_p(L_0)$:

$$R(B_{p,\Theta,1})R(B_{p,\Theta,2}) := E + S(B_{p,\Theta,1}) + S(B_{p,\Theta,2}) = R(B_{p,\Theta,2})R(B_{p,\Theta,1}),\ R(B_{p,\Theta,s}) \in \Phi_p(L_0),$$

and the inverse operator $R^{-1}(B_{p,\Theta}) = E - S(B_{p,\Theta})$. Therefore, $\Phi_p(L_0)$ is an Abelian group with respect to multiplication.

Let us choose n sequences of real numbers $\{\theta_{p,q}\}_{q=1}^{\infty}$, $p = 1, 2, ..., n$, that are denoted by Θ, and consider the operator B_Θ, whose eigenvalues coincide with the eigenvalues of the operator L_0 and whose eigenfunctions are determined by the relations:

$$v_{1,q}(x, B_\Theta) = v_{1,q}(x, L_0),\ q = 1, 2, \dots, \tag{27}$$

$$v_{0,q}(x, B_\Theta) = v_{0,q}(x, L_0) + \sum_{p=1}^{n} \theta_{p,q} y_{3,p}(x, \rho_{0,q}),\ q = 1, 2, \dots . \tag{28}$$

We consider the transformation operator $R(B_\Theta) := E + S(B_\Theta)\colon\ L_2(0, 1) \to L_2(0, 1)$,

$$R(B_\Theta)v_{s,q}(x, L_0) := v_{s,q}(x, B_\Theta),\ s = 0, 1,\ q = 1, 2, \dots .$$

From the definition of the operator B_Θ we obtain that $B_\Theta = \prod_{p=1}^{n} B_{p,\Theta}$ and

$$S(B_\Theta)\colon\ L_{2,0}(0, 1) \to L_{2,1}(0, 1), \quad L_{2,1}(0, 1) \to 0, \quad S^2(B_\Theta) = 0.$$

Therefore, the inverse operator $R^{-1}(B_\Theta) = E - S(B_\Theta)$ exists.

Lemma 4. *For arbitrary sequences $\{\theta_{p,q}\}_{q=1}^{\infty}$, $p = 1, 2, ..., n$, the system of eigenfunctions $V(B_\Theta)$ of the operator B_Θ is complete and minimal in the space $L_2(0, 1)$.*

The system of eigenfunctions $V(B_\Theta)$ is a Riesz basis in the space $L_2(0, 1)$ if and only if the all sequences $\{\theta_{p,q}\}_{q=1}^{\infty}$, $p = 1, 2, ..., n$, are bounded.

Proof of Lemma 4 is similar to that of Lemma 3. □

The set of operators B_Θ whose eigenfunctions are defined by formulas (26), (27) are denoted by $\Gamma(L_0)$. We also denote the corresponding set of transformation operators by $\Phi(L_0)$.

Remark 4. On the set $\Phi(L_0)$ we can define a multiplication operation and prove that $\Phi(L_0)$ is an Abelian group.

5. Multipoint problems. For $p \in \{1, 2, ..., n\}$ we consider the nonlocal problem with multipoint conditions for equation (5):

$$\ell_{3,p,j} y := y^{(j-1)}(0) - (-1)^j y^{(j-1)}(1) = 0,\ j = 1, 2, ..., n, \tag{29}$$

$$\ell_{3,p,n+p} y := y^{(p-1)}(0) + (-1)^p y^{(p-1)}(1) + \sum_{s=0}^{k_0} \sum_{m=0}^{k_p} b_{p,m,s} y^{(m)}(x_s) = 0, \tag{30}$$

$$\ell_{3,p,n+j} y := y^{(j-1)}(0) + (-1)^j y^{(j-1)}(1) = 0,\ j \neq p,\ j = 1, 2, ..., n. \tag{31}$$

Let $L_{3,p}$ be the operator of the problem (5), (29)–(31):

$$L_{3,p} y := (-1)^n y^{(2n)}(x),\ y \in D(L_{3,p}),$$
$$D(L_{3,p}) := \left\{ y \in W_2^{2n}(0, 1) :\ \ell_{3,p,s} y = 0,\ s = 1, 2, ..., 2n \right\}.$$

Lemma 5. *Let Assumption P_1 hold for condition* (30) *for $p \in \{1, 2, ..., n\}$. Then the eigenvalues of the operators L_0 and $L_{3,p}$ coincide and the system of eigenfunctions $V(L_{3,p})$ of the operator $L_{3,p}$ is complete and minimal in space $L_2(0, 1)$.*

If Assumptions P_1-P_2 hold, then the system $V(L_{3,p})$ is a Riesz basis of the space $L_2(0, 1)$.

Proof. The isospectrality of the operators L_0 and $L_{3,p}$ is determined by substituting the general solution (12) of differential equation (9) into multipoint conditions (29), (31).

Resulting system of order $2n$ has the coefficient matrix that contains a minor of order n, all elements $\ell_{3,p,n+j}y_s(x,\rho_{0,q})$ of which are equal to zero for $j,\ s=1,2,\ldots,n$.

Then

$$\det\left[\ell_{3,p,j}y_r\right]_{j,r=\overline{1,2n}} = \det\left[\ell_{0,m}y_s\right]_{m,s=\overline{1,n}} \det\left[\ell_{0,n+m}y_{n+s}\right]_{m,s=\overline{1,n}}.$$

By direct substitution we can see that

$$v_{1,q}(x,L_0)\in D(L_{3,p}),\ q=1,2,\ldots.$$

Therefore, $v_{1,q}(x,L_{3,p})=v_{1,q}(x,L_0)$, $q=1,2,\ldots$.

Taking into account the conditions (22), the eigenfunctions $v_{0,q}(x,L_{3,p})$ of the operator $L_{3,p}$ we define by formulas

$$v_{0,q}(x,L_{3,p}) := v_{0,q}(x,L_0)+\eta_{1,p,q}y_{3,p}(x,\rho_{0,q}),\quad q=1,2,\ldots.$$

Substituting this expression into a multipoint condition (29), we obtain

$$\eta_{1,p,q} = -(\ell_{3,p,n+p}y_{3,p}(x,\rho_{0,q}))^{-1}\ell_{3,p,n+p}v_{0,q}(x,L_0),\ q=1,2,\ldots. \tag{32}$$

Therefore, the transformation operator $R(L_{3,p})\colon\ L_2(0,1)\to L_2(0,1)$,

$$R(L_{3,p})\colon\ v_{r,q}(x,L_0)=v_{r,q}(x,L_{3,p}),\ r=0,1,\ q=1,2,\ldots,$$

is an element of the set $\Phi_p(L_0)$.

Then, in view of Lemma 2, we obtain the completeness and the minimality of the system of functions $V(L_{3,p})$ in the space $L_2(0,1)$.

In the case when Assumptions P_1 and P_3 hold, by direct calculations we establish boundedness of the sequences $\{\eta_{1,p,q}\}_{q=1}^{\infty}$.

Therefore, using Lemma 3, we obtain the second statement of Lemma 5. □

We consider for equation (5) the nonlocal problem with multipoint conditions (2), (4).

Let $L_4\colon L_2(0,1)\in L_2(0,1)$ be the operator of the problem (5), (2)–(4):

$$L_4y := (-1)^n y^{(2n)}(x),\quad y\in D(L_4),\quad D(L_4):=\{y\in W_2^{2n}(0,1)\colon\ \ell_s y=0,\ s=1,2,\ldots,2n\}.$$

Lemma 6. *Let Assumption P_1 hold. Then operator L_4 has complete and minimal system of eigenfunctions $V(L_4)$ in the space $L_2(0,1)$.*

If Assumptions $P_1,\ P_2$ hold, then the system $V(L_4)$ is a Riesz basis of the space $L_2(0,1)$.

Proof. The isospectrality of operators L_0 and L_4 is established by the considerations of Lemma 5. By direct substitution we can see that $v_{1,q}(x,L_0)\in D(L_4)$, $q=1,2,\ldots$.

Therefore, $v_{1,q}(x,L_4)=v_{1,q}(x,L_0)$, $q=1,2,\ldots$.

The eigenfunctions $v_{0,q}(x,L_4)$ of operator L_4 we define by formulas

$$v_{0,q}(x,L_4) := v_{0,q}(x,L_0)+\sum_{p=1}^{n}\eta_{1,p,q}y_{3,p}(x,\rho_{0,q}),\ q=1,2,\ldots.$$

Given conditions (28), we obtain equality (32) for the parameters $\eta_{1,p,q}$.

Therefore, the transformation operator $R(L_4)\colon\ L_2(0,1) \to L_2(0,1)$:

$$R(L_4)\colon\ v_{r,q}(x,L_0) = v_{r,q}(x,L_4),\ r=0,1,\ q=1,2,\ \dots,$$

is an element of the set $\Phi(L_0)$.

So, according to Lemma 4, we obtain that the system of eigenfunctions $V(L_4)$ is complete and minimal in space $L_2(0,1)$.

If Assumptions P_1, P_2 hold, then transformation operators $R(L_{3,p})$ are bounded. Therefore, taking into account the equality $R(L_4) = \prod_{p=1}^{n} R(L_{3,p})$, we obtain that $R(L_4) \in [L_2(0,1)]$.

Then, taking into account the second statement of Lemma 4, we find that system of eigenfunctions $V(L_4)$ is a Riesz basis in the space $L_2(0,1)$. □

6. Proof of the main results. *Proof of Theorem 1.* The isospectrality of the operators L_0 and L is established by the considerations of Lemma 6. By direct substitution we can see that $v_{1,q}(x,L_0) \in D(L)$.

Therefore,

$$v_{1,q}(x,L) = v_{1,q}(x,L_0),\ q=1,2,\ \dots .$$

Let us consider the functions

$$y_4(x,\rho_{0,q}) := (2x-1)v_{0,q}(x,L_0) \in L_{2,1}(0,1),\ q=1,2,\ \dots . \tag{33}$$

The eigenfunctions $v_{0,q}(x,L)$ of the operator L are defined by formulas

$$v_{0,q}(x,L) := v_{0,q}(x,L_0) + c_q y_4(x,\rho_{0,q}) + \sum_{p=1}^{n} \eta_{2,p,q} y_{3,p}(x,\rho_{0,q}),\ q=1,2,\ \dots . \tag{34}$$

From Assumption P_2 and relations (21), (33) we have

$$L y_4(x,\rho_{0,q}) = \lambda_{0,q} y_4(x,\rho_{0,q}) + (-1)^n 4n v_{0,q}^{(2n-1)}(x,L_0),$$
$$L \sum_{p=1}^{n} \eta_{2,p,q} y_{3,p}(x,\rho_{0,q}) = \lambda_{0,q} \sum_{p=1}^{n} \eta_{2,p,q} y_{3,p}(x,\rho_{0,q}).$$

Therefore, substituting expression (34) into following equation

$$L v_{0,q}(x,L) = \lambda_{0,q} v_{0,q}(x,L),$$

we obtain

$$2a_0 v_{0,q}^{(2n-1)}(x,L_0) = (-1)^n 4n c_q v_{0,q}^{(2n-1)}(x,L_0),\ c_q = (-1)^{n-1}(2na_0)^{-1},\ q=1,2,\ \dots .$$

Substituting expression (34) into boundary conditions (2), (4), we define the following parameters

$$\eta_{2,p,q} = \eta_{1,p,q} - c_q(\ell_{n+p} y_{3,p,q}(x,\rho_{0,q}))^{-1} \ell_{n+p} y_4(x,\rho_{0,q}),\ q=1,2,\ \dots .$$

We choose $n+1$ sequences of real numbers $\{\tau_{p,q}\}_{q=1}^{\infty}$, $p=0,1,\ \dots, n$, that are denoted by T.

Consider the operator B_T, whose eigenvalues coincide with the eigenvalues of the operator L_0, and whose eigenfunctions are determined by the relations

$$v_{1,q}(x,B_T) = v_{1,q}(x,L_0),\ q=1,2,\ \dots, \tag{35}$$

$$v_{0,q}(x,B_T) = v_{0,q}(x,L_0) + \tau_{0,q} y_4(x,\rho_{0,q}) + \sum_{p=1}^{n} \tau_{p,q} y_{3,p}(x,\rho_{0,q}),\ q=1,2,\dots.$$

Let us consider the transformation operator $R(B_T) := E + S(B_T)\colon\ L_2(0,1) \to L_2(0,1)$:

$$R(B_T) v_{s,q}(x,L_0) := v_{s,q}(x,B_T),\ s=0,1,\ q=1,2,\dots.$$

From the definition of the operator B_T we obtain

$$S(B_T)\colon\ L_{2,0}(0,1) \to L_{2,1}(0,1),\ L_{2,1}(0,1) \to 0.$$

Therefore, $S^2(B_T) = 0$ and the operator $R^{-1}(B_T) = E - S(B_T)$ exists.

Let $\Phi(B_T)\colon\ L_2(0,1) \to L_2(0,1)$ belong to the set of transformation operators $R(B_T)$. On the set $\Phi(B_T)$ we can define the multiplication operation and prove that this set is an Abelian group.

Lemma 7. *For any sequences $\{\tau_{r,q}\}_{q=1}^{\infty}$, $r=0,1,\dots,n$, the system of eigenfunctions $V(B_T)$ of the operator B_T is complete and minimal in space $L_2(0,1)$.*

If the sequences $\{\tau_{r,q}\}_{q=1}^{\infty}$, $r=0,1,\dots,n$, are bounded, then the system of eigenfunctions $V(B_T)$ is a Riesz basis in the space $L_2(0,1)$.

Proof. As in Lemma 2, we prove that the system of eigenfunctions of the operator B_T is complete and minimal in space $L_2(0,1)$.

Consider the operator B as a partial case of the operator B_T, when $\tau_{p,q} = 0$, $p = 1,2,\dots,n$:

$$v_{1,q}(x,B) = v_{1,q}(x,L_0),\ q=1,2,\dots, \tag{36}$$

$$v_{0,q}(x,B) = v_{0,q}(x,L_0) + \tau_{0,q} y_4(x,\rho_{0,q}),\ q=1,2,\dots. \tag{37}$$

Taking into account the definition (33) of the function $y_4(x,\rho_{0,q})$ and orthonormality of the system $V_0(L_0)$ in the space $L_2(0,1)$, we obtain the boundedness of transformation operator $R(B)\colon\ L_2(0,1) \to L_2(0,1)$:

$$R(B)\colon\ v_{r,q}(x,L_0) := v_{r,q}(x,B),\ r=0,1,\ q=1,2,\dots.$$

Consider the relations $R(B_T) = R(B)R(B_\Theta)$. If $|\tau_{s,q}| < C < \infty$, $s=0,1,\dots,n$, $q = 1,2,\dots$, then using Lemma 7 we obtain $R(B)$, where $R(B_\Theta) \in [L_2(0,1)]$. □

Taking into account (34) we obtain that the transformation operator $R(L)\colon\ L_2(0,1) \to L_2(0,1)$ is the element of group $\Phi(B_T)$. Then the statement of Theorem 1 follows from Lemma 7. □

Proof of Theorem 2. Substituting the expansions

$$f = \sum_{r=0}^{1}\sum_{q=1}^{\infty} f_{r,q} v_{r,q}(x,L), \quad u = \sum_{r=0}^{1}\sum_{q=1}^{\infty} u_{r,q} v_{r,q}(x,L),$$

into equation (1) we obtain that $u_{r,k} = \lambda_{r,k}^{-1} f_{r,k}$, $r=0,1$, $q=1,2,\dots$.

Now let us show that $u \in W_2^{2n}(0,1)$. Differentiating the function (36), (37), we obtain

$$(-1)^n \lambda_{1,k}^{-1} f_{1,k} v_{1,q}^{(2n)}(x,L) = f_{1,k} v_{1,q}^{(2n)}(x,L),\ q=1,2,\dots,$$

$$(-1)^n \lambda_{0,k}^{-1} f_{0,k} v_{0,q}^{(2n)}(x,L) = f_{0,k}\big(v_{0,q}^{(2n)}(x,L) + (-1)^{n-1} a_0 \lambda_{0,k}^{-1} v_{0,q}^{(2n-1)}(x,L_0)\big),\ q=1,2,\dots.$$

Taking into account (13) for some $C>0$ we obtain the inequality

$$\lambda_{0,k}^{-1} \|v_{0,q}^{(2n-1)}(x,L_0)\|_{L_2(0,1)} \le C q^{-1},\ q=1,2,\dots.$$

Therefore, for some $C_1 > 0$ we obtain the inequality $\|u\|_{W_2^{2n}(0,1)} \le C_1 \|f\|_{L_2(0,1)}$. □

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