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ON CERTAIN SUBCLASS OF DIRICHLET SERIES ABSOLUTELY CONVERGENT IN HALF-PLANE

M. M. Sheremeta. *On certain subclass of Dirichlet series absolutely convergent in half-plane*, Mat. Stud. **57** (2022), 32-44.

Denote by \mathfrak{D}_0 a class of absolutely convergent in half-plane $\Pi_0 = \{s: \text{Re } s < 0\}$ Dirichlet series $F(s) = e^{sh} - \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k + h)\}$, $s = \sigma + it$, where $h > 0$, $h < \lambda_k \uparrow +\infty$ and $f_k > 0$. For $0 \leq \alpha < h$ and $l \geq 0$ we say that F belongs to the class $\mathfrak{D}\mathfrak{F}_h(l, \alpha)$ if and only if $\text{Re}\{e^{-hs}((1-l)F(s) + \frac{l}{h}F'(s))\} > \frac{\alpha}{h}$, and belongs to the class $\mathfrak{D}\mathfrak{G}_h(l, \alpha)$ if and only if $\text{Re}\{e^{-hs}((1-l)F'(s) + \frac{l}{h}F''(s))\} > \alpha$ for all $s \in \Pi_0$. It is proved that $F \in \mathfrak{D}\mathfrak{F}_h(l, \alpha)$ if and only if $\sum_{k=1}^{\infty} (h+l\lambda_k)f_k \leq h-\alpha$, and $F \in \mathfrak{D}\mathfrak{G}_h(l, \alpha)$ if and only if $\sum_{k=1}^{\infty} (h+l\lambda_k)(\lambda_k+h)f_k \leq h(h-\alpha)$.

If $F_j \in \mathfrak{D}\mathfrak{F}_h(l_j, \alpha_j)$, $j = 1, 2$, where $l_j \geq 0$ and $0 \leq \alpha_j < h$, then Hadamard composition $(F_1 * F_2) \in \mathfrak{D}\mathfrak{F}_h(l, \alpha)$, where $l = \min\{l_1, l_2\}$ and $\alpha = h - \frac{(h-\alpha_1)(h-\alpha_2)}{h+l\lambda_1}$. Similar statement is correct for the class $F_j \in \mathfrak{D}\mathfrak{G}_h(l, \alpha)$.

For $j > 0$ and $\delta > 0$ the neighborhood of the function $F \in \mathfrak{D}_0$ is defined as follows $O_{j,\delta}(F) = \{G(s) = e^s - \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\} \in \mathfrak{D}_0: \sum_{k=1}^{\infty} \lambda_k^j |g_k - f_k| \leq \delta\}$. It is described the neighborhoods of functions from classes $\mathfrak{D}\mathfrak{F}_h(l, \alpha)$ and $\mathfrak{D}\mathfrak{G}_h(l, \alpha)$.

Conditions on real parameters $\gamma_0, \gamma_1, \gamma_2, a_1$ and a_2 of the differential equation $w'' + (\gamma_0 e^{2hs} + \gamma_1 e^{hs} + \gamma_2)w = a_1 e^{hs} + a_2 e^{2hs}$ are found, under which this equation has a solution either in $\mathfrak{D}\mathfrak{F}_h(l, \alpha)$ or in $\mathfrak{D}\mathfrak{G}_h(l, \alpha)$.

1. Introduction. Let $p \in \mathbb{N}$, $0 \leq \alpha < p$ and $l \geq 0$. We say that an analytic in the disk $\mathbb{D} = \{z: |z| < 1\}$ function

$$f(z) = z^p - \sum_{n=1}^{\infty} a_n z^{n+p}, \quad a_n \geq 0, \tag{1}$$

belongs to the class $\mathfrak{F}_p(l, \alpha)$ if and only if

$$\text{Re} \left\{ (1-l) \frac{f(z)}{z^p} + l \frac{f'(z)}{pz^{p-1}} \right\} > \frac{\alpha}{p} \quad (z \in \mathbb{D}), \tag{2}$$

and belongs to the class $\mathfrak{G}_p(l, \alpha)$ if and only if

$$\text{Re} \left\{ (p+l(1-p)) \frac{f'(z)}{pz^{p-1}} + l \frac{f''(z)}{pz^{p-2}} \right\} > \alpha \quad (z \in \mathbb{D}). \tag{3}$$

The class $\mathfrak{F}_p(l, \alpha)$ was introduced and studied earlier by S. K. Lee, S. Owa and H. M. Srivastava [1] and was further investigated by M. K. Aouf and H. E. Darwish [2]. The class $\mathfrak{G}_p(l, \alpha)$

2010 *Mathematics Subject Classification*: 30B45; 30D50.

Keywords: Dirichlet series; Hadamard composition; neighborhood of the function; differential equation.

doi:10.30970/ms.57.1.32-44

was studied recently by M. K. Aouf [3]. In particular, the class $\mathfrak{G}_1(l, \alpha)$ was considered earlier by O. Altintas [4-5]. The research of classes $\mathfrak{F}_p(l, \alpha)$ and $\mathfrak{G}_p(l, \alpha)$ was continued in the works [6-7].

Absolutely convergent in half-plane $\Pi_0 = \{s: \text{Re } s < 0\}$ Dirichlet series with positive increasing to $+\infty$ exponents are direct generalizations of analytic functions in the disk \mathbb{D} . Therefore, it is quite natural to introduce analogues of the classes $\mathfrak{F}_p(l, \alpha)$ and $\mathfrak{G}_p(l, \alpha)$ for such series of Dirichlet and explore their properties.

So, let $h > 0$, (λ_k) be an increasing to $+\infty$ sequence of positive numbers ($\lambda_1 > h$) and \mathfrak{D}_0 be a class of Dirichlet series

$$F(s) = e^{sh} - \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k + h)\}, \quad f_k > 0, \quad s = \sigma + it, \quad (4)$$

absolutely convergent in half-plane Π_0 . For $0 \leq \alpha < h$ and $l \geq 0$ we say that F belongs to the class $\mathfrak{D}\mathfrak{F}_h(l, \alpha)$ if and only if

$$\text{Re} \left\{ e^{-hs} \left((1-l)F(s) + \frac{l}{h}F'(s) \right) \right\} > \frac{\alpha}{h} \quad (s \in \Pi_0), \quad (5)$$

and belongs to the class $\mathfrak{D}\mathfrak{G}_h(l, \alpha)$ if and only if

$$\text{Re} \left\{ e^{-hs} \left((1-l)F'(s) + \frac{l}{h}F''(s) \right) \right\} > \alpha \quad (s \in \Pi_0). \quad (6)$$

It follows from (5) and (6) that

$$F \in \mathfrak{D}\mathfrak{G}_h(l, \alpha) \iff \frac{F'}{h} \in \mathfrak{D}\mathfrak{F}_h(l, \alpha). \quad (7)$$

2. Properties of coefficients. In order to derive various properties associated with the classes $\mathfrak{D}\mathfrak{F}_h(l, \alpha)$ and $\mathfrak{D}\mathfrak{G}_h(l, \alpha)$ we shall need the following theorems.

Theorem 1. *Let the function F be define by (4). Then $F \in \mathfrak{D}\mathfrak{F}_h(l, \alpha)$ if and only if*

$$\sum_{k=1}^{\infty} (h + l\lambda_k)f_k \leq h - \alpha. \quad (8)$$

Proof. Since

$$\begin{aligned} & e^{-hs} \left((1-l)F(s) + \frac{l}{h}F'(s) \right) = \\ &= e^{-hs} \left\{ (1-l) \left(e^{sh} - \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k + h)\} \right) + \frac{l}{h} \left(h e^{sh} - \sum_{k=1}^{\infty} (\lambda_k + h) f_k \exp\{s(\lambda_k + h)\} \right) \right\} = \\ &= (1-l) \left(1 - \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\} \right) + \frac{l}{h} \left(h - \sum_{k=1}^{\infty} (\lambda_k + h) f_k \exp\{s\lambda_k\} \right) = \\ &= 1 - \sum_{k=1}^{\infty} (1-l)f_k \exp\{s\lambda_k\} - \sum_{k=1}^{\infty} \frac{l}{h} (\lambda_k + h) f_k \exp\{s\lambda_k\} = 1 - \sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h} f_k \exp\{s\lambda_k\}, \end{aligned}$$

condition (5) holds if and only if

$$\operatorname{Re} \left\{ 1 - \sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h} f_k \exp\{s\lambda_k\} \right\} > \frac{\alpha}{h} \quad (s \in \Pi_0),$$

i. e.

$$\operatorname{Re} \left\{ \sum_{k=1}^{\infty} (h + l\lambda_k) f_k \exp\{s\lambda_k\} \right\} < h - \alpha \quad (s \in \Pi_0). \quad (9)$$

Since $f_k > 0$ and (9) holds for all $s \in \Pi_0$, for all $\sigma < 0$ we have

$$\sum_{k=1}^{\infty} (h + l\lambda_k) f_k \exp\{\sigma\lambda_k\} < h - \alpha. \quad (10)$$

As $\sigma \rightarrow 0$ from hence we get a condition (8).

On the contrary, the condition (8) implies (10) for all $\sigma < 0$. Therefore, for all $s \in \Pi_0$ we have

$$\begin{aligned} \operatorname{Re} \left\{ \sum_{k=1}^{\infty} (h + l\lambda_k) f_k \exp\{s\lambda_k\} \right\} &\leq \left| \sum_{k=1}^{\infty} (h + l\lambda_k) f_k \exp\{s\lambda_k\} \right| \leq \\ &\leq \sum_{k=1}^{\infty} (h + l\lambda_k) f_k \exp\{\sigma\lambda_k\} < h - \alpha, \end{aligned}$$

i. e. (9) holds. □

Since for function (4) we have

$$\frac{F'(s)}{h} = e^{sh} - \sum_{k=1}^{\infty} \frac{\lambda_k + h}{h} f_k \exp\{s(\lambda_k + h)\},$$

by Theorem 1 we get $F'/h \in \mathfrak{D}\mathfrak{F}_h(l, \alpha)$ if and only if

$$\sum_{k=1}^{\infty} (h + l\lambda_k) \frac{\lambda_k + h}{h} f_k \leq h - \alpha.$$

Therefore, in view of (7) we obtain the following statement.

Corollary 1. *Let the function F be define by (4). Then $F \in \mathfrak{D}\mathfrak{G}_h(l, \alpha)$ if and only if*

$$\sum_{k=1}^{\infty} (h + l\lambda_k) (\lambda_k + h) f_k \leq h(h - \alpha). \quad (11)$$

3. Hadamard compositions. For power series $f_j(z) = \sum_{k=0}^{\infty} f_{k,j} z^k$ ($j = 1, 2$) the series $(f_1 * f_2)(z) = \sum_{k=0}^{\infty} f_{k,1} f_{k,2} z^k$ is called the Hadamard composition (product)[8]. Obtained by J.Hadamard properties of this composition find the applications [9–10] in the theory of the analytic continuation of the functions represented by power series. We remark also that singular points of the Hadamard composition are investigated in the article [11]. L. Zalzman [12] studied Hadamard compositions of univalent functions. For the functions $f_j(z) = 1/z +$

$\sum_{k=1}^{\infty} f_{k,j} z^k$ ($j = 1, 2$) M. L. Mogra [13] defined Hadamard composition as $(f_1 * f_2)(z) = 1/z + \sum_{k=1}^{\infty} f_{k,1} f_{k,2} z^k$ and proved, for example, that if the functions f_j are meromorphically starlike of the order $\alpha_j \in [0, 1)$ and $f_{k,j} \geq 0$ for all $k \geq 1$ then $f_1 * f_2$ is meromorphically starlike of the order $\alpha = \max\{\alpha_1, \alpha_2\}$. Hadamard compositions of analytic and meromorphic functions in \mathbb{D} studied also by J. H. Choi, Y. C. Kim and S. Owa [14], M. K. Aouf and H. Silverman [15], J. Liu and R. Srivastava [16], S. Ruscheweyh [17] and many other mathematicians. Hadamard products of functions from classes $\mathfrak{F}_p(l, \alpha)$ and $\mathfrak{G}_p(l, \alpha)$ was studied in [7]. For absolutely convergent in half-plane Π_0 Dirichlet series Hadamard compositions were used in [18].

So, let the functions $F_j, j = 1, 2$, be defined by

$$F_j(s) = e^{sh} - \sum_{k=1}^{\infty} f_{k,j} \exp\{s(\lambda_k + h)\}, \quad f_{k,j} > 0. \tag{12}$$

Then the Hadamard composition of F_1 and F_2 is defined by

$$(F_1 * F_2)(s) = e^{sh} - \sum_{k=1}^{\infty} f_{k,1} f_{k,2} \exp\{s(\lambda_k + h)\}. \tag{13}$$

Theorem 2. Let $F_j \in \mathfrak{D}\mathfrak{F}_h(l_j, \alpha_j), j = 1, 2$, where $l_j \geq 0$ and $0 \leq \alpha_j < h$. Then $(F_1 * F_2) \in \mathfrak{D}F_h(l, \alpha)$, where $l = \min\{l_1, l_2\}$ and

$$\alpha = h - \frac{(h - \alpha_1)(h - \alpha_2)}{h + l\lambda_1}. \tag{14}$$

Proof. Since $F_j \in \mathfrak{D}\mathfrak{F}_h(l_j, \alpha_j)$ and $l = \min\{l_1, l_2\}$, by Theorem 1

$$\sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h - \alpha_j} f_{k,j} \leq \sum_{k=1}^{\infty} \frac{h + l_j\lambda_k}{h - \alpha_j} f_{k,j} \leq 1.$$

and by Cauchy-Schwarz inequality we have

$$\sum_{k=1}^{\infty} \sqrt{\frac{h + l\lambda_k}{h - \alpha_1} f_{k,1} \frac{h + l\lambda_k}{h - \alpha_2} f_{k,2}} \leq \sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h - \alpha_1} f_{k,1} \sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h - \alpha_2} f_{k,2} \leq 1,$$

i. e.

$$\sum_{k=1}^{\infty} \frac{h + l\lambda_k}{\sqrt{(h - \alpha_1)(h - \alpha_2)}} \sqrt{f_{k,1} f_{k,2}} \leq 1, \tag{15}$$

whence

$$\frac{h + l\lambda_k}{\sqrt{(h - \alpha_1)(h - \alpha_2)}} \sqrt{f_{k,1} f_{k,2}} \leq 1,$$

i. e.

$$\sqrt{f_{k,1} f_{k,2}} \leq \frac{\sqrt{(h - \alpha_1)(h - \alpha_2)}}{h + l\lambda_k}.$$

Therefore, in view of (14) and (15)

$$\sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h - \alpha} f_{k,1} f_{k,2} = \sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h - \alpha} \sqrt{f_{k,1} f_{k,2}} \sqrt{f_{k,1} f_{k,2}} \leq$$

$$\begin{aligned} &\leq \sum_{k=1}^{\infty} \frac{\sqrt{(h-\alpha_1)(h-\alpha_2)}}{h-\alpha} \sqrt{f_{k,1}f_{k,2}} = \sum_{k=1}^{\infty} \frac{h+l\lambda_1}{\sqrt{(h-\alpha_1)(h-\alpha_2)}} \sqrt{f_{k,1}f_{k,2}} \leq \\ &\leq \sum_{k=1}^{\infty} \frac{h+l\lambda_k}{\sqrt{(h-\alpha_1)(h-\alpha_2)}} \sqrt{f_{k,1}f_{k,2}} \leq 1, \end{aligned}$$

i. e.

$$\sum_{k=1}^{\infty} (h+l\lambda_k) f_{k,1} f_{k,2} \leq h-\alpha.$$

Thus, by Theorem 1 $(F_1 * F_2) \in \mathfrak{D}\mathfrak{F}_h(l, \alpha)$. \square

Corollary 2. Let $F_j \in \mathfrak{D}\mathfrak{F}_h(l, \alpha)$, $j = 1, 2$, where $l \geq 0$ and $0 \leq \alpha < h$. Then $(F_1 * F_2) \in \mathfrak{D}\mathfrak{F}_h(l, \beta)$, where $\beta = h - \frac{(h-\alpha)^2}{h+l\lambda_1}$.

For the class $\mathfrak{D}\mathfrak{G}_h(l, \alpha)$, the next analogue of Theorem 2 is true.

Theorem 3. Let $F_j \in \mathfrak{D}\mathfrak{G}_h(l_j, \alpha_j)$, $j = 1, 2$, where $l_j \geq 0$ and $0 \leq \alpha_j < h$. Then $(F_1 * F_2) \in \mathfrak{D}\mathfrak{G}_h(l, \alpha)$, where $l = \min\{l_1, l_2\}$ and

$$\alpha = h - \frac{h(h-\alpha_1)(h-\alpha_2)}{(h+l\lambda_1)(h+l\lambda_2)}. \quad (16)$$

Proof. Since $F_j \in \mathfrak{D}\mathfrak{G}_h(l_j, \alpha_j)$ and $l = \min\{l_1, l_2\}$, by Corollary 1 as above we have

$$\sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha_j)} f_{k,j} \leq 1, \quad j = 1, 2.$$

and by Cauchy-Schwarz inequality

$$\begin{aligned} &\sum_{k=1}^{\infty} \sqrt{\frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha_1)} f_{k,1} \frac{(h+l\lambda_k)\lambda_k+h}{h(h-\alpha_2)} f_{k,2}} \leq \\ &\leq \sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha_1)} f_{k,1} \sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha_2)} f_{k,2} \leq 1, \end{aligned}$$

i. e.

$$\sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h\sqrt{(h-\alpha_1)(h-\alpha_2)}} \sqrt{f_{k,1}f_{k,2}} \leq 1, \quad (17)$$

whence as above

$$\sqrt{f_{k,1}f_{k,2}} \leq \frac{h\sqrt{(h-\alpha_1)(h-\alpha_2)}}{(h+l\lambda_k)(\lambda_k+h)}.$$

Therefore, in view of (16) and (17)

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha)} f_{k,1} f_{k,2} = \sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha)} \sqrt{f_{k,1}f_{k,2}} \sqrt{f_{k,1}f_{k,2}} \leq \\ &\leq \sum_{k=1}^{\infty} \frac{\sqrt{(h-\alpha_1)(h-\alpha_2)}}{h-\alpha} \sqrt{f_{k,1}f_{k,2}} = \sum_{k=1}^{\infty} \frac{(h+l\lambda_1)(\lambda_1+h)}{h\sqrt{(h-\alpha_1)(h-\alpha_2)}} \sqrt{f_{k,1}f_{k,2}} \leq \end{aligned}$$

$$\leq \sum_{k=1}^{\infty} \frac{(h + l\lambda_k)(\lambda_k + h)}{h\sqrt{(h - \alpha_1)(h - \alpha_2)}} \sqrt{f_{k,1}f_{k,2}} \leq 1,$$

i. e. by Corollary 1 ($F_1 * F_2$) $\in \mathfrak{D}\mathfrak{G}_h(l, \alpha)$. \square

Corollary 3. Let $F_j \in \mathfrak{D}\mathfrak{G}_h(l, \alpha)$, $j = 1, 2$, where $l \geq 0$ and $0 \leq \alpha < h$. Then $(F_1 * F_2) \in \mathfrak{D}\mathfrak{G}_h(l, \beta)$, where $\beta = h - \frac{h(h-\alpha)^2}{(h+l\lambda_1)(h+\lambda_1)}$.

4. Neighborhoods of Diriclet series. For the function $f(z) = \sum_{k=0}^{\infty} f_k z^k$, following A. W. Goodman [19] and S. Ruscheweyh [17], its neighborhood is called a set

$$N_{\delta}(f) = \left\{ g(z) = z + \sum_{k=2}^{\infty} g_k z^k : \sum_{k=2}^{\infty} k |g_k - f_k| \leq \delta \right\}.$$

The neighborhoods of various classes of analytical in \mathbb{D} functions were studied by many authors (we indicate here only on articles [20–26]). Here for $j > 0$ and $\delta > 0$ we define the neighborhood of $F \in \mathfrak{D}_0$ as follows

$$O_{j,\delta}(F) = \left\{ G(s) = e^s - \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\} \in \mathfrak{D}_0 : \sum_{k=1}^{\infty} \lambda_k^j |g_k - f_k| \leq \delta \right\}. \quad (18)$$

The following two theorems describe the neighborhoods of functions from classes $\mathfrak{D}\mathfrak{F}_h(l, \alpha)$ and $\mathfrak{D}\mathfrak{F}_h(l, \alpha)$.

Theorem 4. Let $l > 0$, $0 \leq \beta < \alpha < h$ and $F \in \mathfrak{D}\mathfrak{G}_h(l, \alpha)$. If $G \in O_{2,\delta}(F)$ with $\delta = \delta_1 := \frac{h(\alpha-\beta)\lambda_1}{(h+l\lambda_1)(h+\lambda_1)}$ then $G \in \mathfrak{D}\mathfrak{G}_h(l, \beta)$. On the contrary, if $G \in \mathfrak{D}\mathfrak{G}_h(l, \beta)$ then $G \in O_{2,\delta}(F)$ with $\delta = \delta_2 := \frac{2h-\alpha-\beta}{l}$.

Proof. Since $F \in \mathfrak{D}\mathfrak{G}_h(l, \alpha)$, by Corollary 1 condition (8) holds. Therefore, for $\beta < \alpha$ in view of (18) we have

$$\begin{aligned} \sum_{k=1}^{\infty} (h + l\lambda_k)g_k &\leq \sum_{k=1}^{\infty} (h + l\lambda_k)f_k + \sum_{k=1}^{\infty} (h + l\lambda_k)|g_k - f_k| \leq \\ &\leq \sum_{k=1}^{\infty} \frac{h + l\lambda_k}{\lambda_k} \lambda_k |g_k - f_k| \leq h - \alpha + \frac{h + l\lambda_1}{\lambda_1} \sum_{k=1}^{\infty} \lambda_k |g_k - f_k| \leq \\ &\leq h - \alpha + \frac{h + l\lambda_1}{\lambda_1} \delta = h - \alpha + \frac{h + l\lambda_1}{\lambda_1} \frac{(\alpha - \beta)\lambda_1}{h + l\lambda_1} = h - \beta \end{aligned}$$

and, thus, $G \in \mathfrak{D}\mathfrak{F}_h(l, \beta)$.

On the contrary, if $G \in \mathfrak{D}\mathfrak{F}_h(l, \beta)$ then

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k |g_k - f_k| &= \sum_{k=1}^{\infty} \frac{\lambda_k}{h + l\lambda_k} (h + l\lambda_k) |g_k - f_k| \leq \frac{1}{l} \sum_{k=1}^{\infty} (h + l\lambda_k) |g_k - f_k| \leq \\ &\leq \frac{1}{l} \left(\sum_{k=1}^{\infty} (h + l\lambda_k)g_k + \sum_{k=1}^{\infty} (h + l\lambda_k)f_k \right) \leq \frac{h - \alpha + h - \beta}{l} = \delta_2, \end{aligned}$$

that is $G \in O_{1,\delta}(F)$ with $\delta = \delta_2$. \square

Theorem 5. Let $l > 0$, $0 \leq \beta < \alpha < h$ and $F \in \mathfrak{D}G_h(l, \alpha)$. If $G \in O_{2,\delta}(F)$ with $\delta = \delta_3 := \frac{h(\alpha-\beta)\lambda_1^2}{(h+l\lambda_1)(h+\lambda_1)}$ then $G \in \mathfrak{D}\mathfrak{G}_h(l, \beta)$. On the contrary, if $G \in \mathfrak{D}\mathfrak{G}_h(l, \beta)$ then $G \in O_{2,\delta}(F)$ with $\delta = \delta_4 := \frac{h(2h-\alpha-\beta)}{l}$.

Proof. Since $F \in \mathfrak{D}\mathfrak{G}_h(l, \alpha)$, by Corollary 1 condition (11) holds. Therefore, for $\beta < \alpha$ in view of (18) we have

$$\begin{aligned} \sum_{k=1}^{\infty} (h+l\lambda_k)(\lambda_k+h)g_k &\leq \sum_{k=1}^{\infty} (h+l\lambda_k)(\lambda_k+h)f_k + \sum_{k=1}^{\infty} (h+l\lambda_k)(\lambda_k+h)|g_k-f_k| \leq \\ &\leq h(h-\alpha) + \sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{\lambda_k^2} \lambda_k^2 |g_k-f_k| \leq h(h-\alpha) + \\ &+ \frac{(h+l\lambda_1)(\lambda_1+h)}{\lambda_1^2} \sum_{k=1}^{\infty} \lambda_k^2 |g_k-f_k| \leq h(h-\alpha) + \frac{(h+l\lambda_1)(\lambda_1+h)}{\lambda_1^2} \delta_3 = h(h-\beta) \end{aligned}$$

and, thus, by Corollary 1 $G \in \mathfrak{D}\mathfrak{G}_h(l, \beta)$.

On the contrary, if $G \in \mathfrak{D}\mathfrak{G}_h(l, \beta)$ then as above

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k^2 |g_k-f_k| &= \sum_{k=1}^{\infty} \frac{\lambda_k^2}{(h+l\lambda_k)(h+\lambda_k)} (h+l\lambda_k)(h+\lambda_k) |g_k-f_k| \leq \\ &\leq \frac{1}{l} \sum_{k=1}^{\infty} (h+l\lambda_k)(h+\lambda_k) |g_k-f_k| \leq \frac{h(h-\alpha) + h(h-\beta)}{l} = \delta_4 \end{aligned}$$

that is $G \in O_{2,\delta}(F)$ with $\delta = \delta_4$. □

5. Differential equations of second order. S. M. Shah [27] indicated conditions on real parameters $\gamma_0, \gamma_1, \gamma_2$ of the differential equation $z^2 w'' + zw' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0$, under which there exists an entire transcendental solution f that together with its derivatives are close-to-convex in \mathbb{D} . The investigations are continued by Z. M. Sheremeta (see, for example, [28–39]). Substituting $z = e^s$ we obtain the differential equation

$$\frac{d^2 w}{ds^2} + (\gamma_0 e^{2hs} + \gamma_1 e^{hs} + \gamma_2)w = 0 \quad (19)$$

with $h = 1$. In [32] and [33, p. 150] conditions found, under which this equation has solutions pseudoconvex or close-to-pseudoconvex in Π_0 . Here we consider a differential equation

$$\frac{d^2 w}{ds^2} + (\gamma_0 e^{2hs} + \gamma_1 e^{hs} + \gamma_2)w = a_1 e^{hs} + a_2 e^{2hs}, \quad (20)$$

where $\gamma_0 < 0$, $\gamma_1 < 0$, $a_1 > 0$ and $a_2 < 0$. Suppose that function (4) satisfies (20), i. e.

$$\begin{aligned} h^2 e^{sh} + \gamma_2 e^{sh} + \gamma_1 e^{2hs} + \gamma_0 e^{3hs} - \sum_{k=1}^{\infty} ((\lambda_k+h)^2 + \gamma_2) f_k \exp\{s(\lambda_k+h)\} - \\ - \gamma_1 \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k+2h)\} - \gamma_0 \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k+3h)\} = a_1 e^{hs} + a_2 e^{2hs}. \end{aligned} \quad (21)$$

As $\sigma \rightarrow -\infty$ from hence we obtain $(h^2 + \gamma_2)e^{sh}(1 + o(1)) = a_1e^{sh}(1 + o(1))$, that is $h^2 + \gamma_2 = a_1 > 0$. Therefore, from (21) we get

$$\begin{aligned} & \gamma_1 e^{2hs} + \gamma_0 e^{3hs} - ((\lambda_1 + h)^2 + \gamma_2) f_1 \exp\{s(\lambda_1 + h)\} - \sum_{k=2}^{\infty} ((\lambda_k + h)^2 + \gamma_2) f_k \exp\{s(\lambda_k + h)\} - \\ & - \gamma_1 \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k + 2h)\} - \gamma_0 \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k + 3h)\} = a_2 e^{2hs}. \end{aligned} \quad (22)$$

Since $(\lambda_1 + h)^2 + \gamma_2 > 0$, from hence it follows that

$$((\lambda_1 + h)^2 + \gamma_2) f_1 \exp\{s(\lambda_1 + h)\} (1 + o(1)) = (\gamma_1 - a_2) e^{2hs} (1 + o(1)), \quad \sigma \rightarrow -\infty$$

and, thus, $\lambda_1 = h$ and

$$f_1 = \frac{\gamma_1 - a_2}{4h^2 + \gamma_2} = \frac{|a_2| - |\gamma_1|}{4h^2 + \gamma_2} > 0 \quad (23)$$

provided $|a_2| - |\gamma_1| > 0$. Therefore, (22) implies

$$\begin{aligned} & \gamma_0 e^{3hs} - ((\lambda_2 + h)^2 + \gamma_2) f_2 \exp\{s(\lambda_2 + h)\} - \sum_{k=3}^{\infty} ((\lambda_k + h)^2 + \gamma_2) f_k \exp\{s(\lambda_k + h)\} - \\ & - \gamma_1 f_1 \exp\{s(\lambda_1 + 2h)\} - \gamma_1 \sum_{k=2}^{\infty} f_k \exp\{s(\lambda_k + 2h)\} - \gamma_0 \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k + 3h)\} = 0, \end{aligned} \quad (24)$$

whence $(\gamma_0 - \gamma_1 f_1) e^{3hs} = (1 + o(1)) ((\lambda_2 + h)^2 + \gamma_2) f_2 \exp\{s(\lambda_2 + h)\}$ as $\sigma \rightarrow -\infty$ and, thus, $\lambda_2 = 2h$ and

$$f_2 = \frac{\gamma_0 - \gamma_1 f_1}{9h^2 + \gamma_2} = \frac{|\gamma_1| (|a_2| - |\gamma_1|) - |\gamma_0| (4h^2 + \gamma_2)}{(9h^2 + \gamma_2)(4h^2 + \gamma_2)} > 0 \quad (25)$$

provided $|\gamma_0| (4h^2 + \gamma_2) < |\gamma_1| (|a_2| - |\gamma_1|)$. Therefore, from (24) it follows that

$$\begin{aligned} & ((\lambda_3 + h)^2 + \gamma_2) f_3 + \sum_{k=4}^{\infty} ((\lambda_k + h)^2 + \gamma_2) f_k \exp\{s(\lambda_k + h)\} + \gamma_1 f_2 \exp\{s(\lambda_2 + 2h)\} + \\ & + \gamma_1 \sum_{k=3}^{\infty} f_k \exp\{s(\lambda_k + 2h)\} + \gamma_0 f_1 \exp\{s(\lambda_1 + 3h)\} + \gamma_0 \sum_{k=2}^{\infty} f_k \exp\{s(\lambda_k + 3h)\} = 0, \end{aligned} \quad (26)$$

whence $((\lambda_3 + h)^2 + \gamma_2) f_3 \exp\{s(\lambda_3 + h)\} (1 + o(1)) = -(\gamma_1 f_2 + \gamma_0 f_1) e^{4hs}$ as $\sigma \rightarrow -\infty$ and, thus, $\lambda_3 = 3h$ and

$$f_3 = -\frac{\gamma_1 f_2 + \gamma_0 f_1}{16h^2 + \gamma_2} = \frac{|\gamma_1| f_2 + |\gamma_0| f_1}{16h^2 + \gamma_2} > 0$$

Continuing this process, we will come to the formulas

$$\lambda_k = kh, \quad f_k = -\frac{\gamma_1 f_{k-1} + \gamma_0 f_{k-2}}{(k+1)^2 h^2 + \gamma_2} = \frac{|\gamma_1| f_{k-1} + |\gamma_0| f_{k-2}}{(k+1)^2 h^2 + \gamma_2} > 0 \quad (k \geq 3). \quad (27)$$

We remark that the condition $|\gamma_0| (4h^2 + \gamma_2) < |\gamma_1| (|a_2| - |\gamma_1|)$ holds if and if $|a_2| > |\gamma_0| (4h^2 + \gamma_2) / (|\gamma_1| + |\gamma_1|)$.

Thus, the following statement is true.

Lemma 1. *Let $\gamma_0 < 0$, $\gamma_1 < 0$, $\gamma_2 < 0$, $a_1 = h^2 + \gamma_2 > 0$, $a_2 < 0$ and $|a_2| > > |\gamma_0|(4h^2 + \gamma_2)/|\gamma_1| + |\gamma_1|$. Then differential equation (20) has a solution*

$$F(s) = e^{sh} - \sum_{k=1}^{\infty} f_k \exp\{s(k+1)h\}, \quad (28)$$

where the coefficients f_k are determined by formulas (23), (25) and (27).

Using Lemma 1, we prove the following theorem.

Theorem 6. *Let the parameters a_1 , a_2 , γ_0 , γ_1 and γ_2 satisfy the conditions of Lemma 1. Then differential equation (20) has a solution (28) such that:*

1) if

$$\begin{aligned} & \frac{(l+1)(|a_2| - |\gamma_1|)}{4h^2 + \gamma_2} - \frac{(2l+1)|\gamma_0|}{9h^2 + \gamma_2} \leq \\ & \leq (h - \alpha) \left(1 - \frac{(2l+1)|\gamma_1|}{(l+1)(9h^2 + \gamma_2)} - \frac{(3l+1)|\gamma_0|}{(l+1)(16h^2 + \gamma_2)} \right) \end{aligned} \quad (29)$$

then $F \in \mathfrak{D}\mathfrak{F}_h(l, \alpha)$;

2) if

$$\begin{aligned} & h \left(\frac{2(l+1)(|a_2| - |\gamma_1|)}{4h^2 + \gamma_2} - \frac{3(2l+1)|\gamma_0|}{9h^2 + \gamma_2} \right) \leq \\ & \leq (h - \alpha) \left(1 - \frac{3(2l+1)|\gamma_1|}{2(l+1)(9h^2 + \gamma_2)} - \frac{4(3l+1)|\gamma_0|}{2(l+1)(16h^2 + \gamma_2)} \right) \end{aligned} \quad (30)$$

then $F \in \mathfrak{D}\mathfrak{G}_h(l, \alpha)$;

3) Dirichlet series (28) is entire and $\ln \ln M(\sigma, F) = (1 + o(1))h\sigma$ as $0 \leq \sigma \rightarrow +\infty$, where $M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$.

Proof. For function (27) in view of (27), (25) and (23) we have

$$\begin{aligned} & \sum_{k=1}^{\infty} (h + l\lambda_k) f_k = \sum_{k=1}^{\infty} (lk+1) h f_k = (l+1) h f_1 + (2l+1) h f_2 + \sum_{k=3}^{\infty} (lk+1) h f_k = \\ & = (l+1) h f_1 + (2l+1) h f_2 + \sum_{k=3}^{\infty} (lk+1) h \frac{|\gamma_1| f_{k-1} + |\gamma_0| f_{k-2}}{(k+1)^2 h^2 + \gamma_2} f_k = \\ & = (l+1) h f_1 + (2l+1) h f_2 + \sum_{k=3}^{\infty} (lk+1) h \frac{|\gamma_1| f_{k-1}}{(k+1)^2 h^2 + \gamma_2} + \sum_{k=3}^{\infty} (lk+1) h \frac{|\gamma_0| f_{k-2}}{(k+1)^2 h^2 + \gamma_2} = \\ & = (l+1) h f_1 + (2l+1) h f_2 + \sum_{k=2}^{\infty} (lk+l+1) h \frac{|\gamma_1| f_k}{(k+2)^2 h^2 + \gamma_2} + \\ & + \sum_{k=1}^{\infty} (lk+2l+1) h \frac{|\gamma_0| f_k}{(k+3)^2 h^2 + \gamma_2} = (l+1) h f_1 + (2l+1) h f_2 - (2l+1) h \frac{|\gamma_1| f_1}{9h^2 + \gamma_2} + \\ & + h \sum_{k=1}^{\infty} \left(\frac{(lk+l+1)|\gamma_1| f_k}{(k+2)^2 h^2 + \gamma_2} + \frac{(lk+2l+1)|\gamma_0| f_k}{(k+3)^2 h^2 + \gamma_2} \right) \leq \end{aligned}$$

$$\begin{aligned} &\leq h \left(\frac{(l+1)(|a_2| - |\gamma_1|)}{4h^2 + \gamma_2} - \frac{(2l+1)|\gamma_0|}{9h^2 + \gamma_2} \right) + \\ &+ \sum_{k=1}^{\infty} \left(\frac{(2l+1)|\gamma_1|}{(l+1)(9h^2 + \gamma_2)} + \frac{(3l+1)|\gamma_0|}{(l+1)(16h^2 + \gamma_2)} \right) (lk+1)hf_k. \end{aligned} \quad (31)$$

From (29) it follows that

$$\frac{(2l+1)|\gamma_1|}{(l+1)(9h^2 + \gamma_2)} + \frac{(3l+1)|\gamma_0|}{(l+1)(16h^2 + \gamma_2)} < 1$$

and, thus, (31) implies

$$\begin{aligned} &\left(1 - \frac{(2l+1)|\gamma_1|}{(l+1)(9h^2 + \gamma_2)} - \frac{(3l+1)|\gamma_0|}{(l+1)(16h^2 + \gamma_2)} \right) \sum_{k=1}^{\infty} (lk+1)hf_k \leq \\ &\leq h \left(\frac{(l+1)(|a_2| - |\gamma_1|)}{4h^2 + \gamma_2} - \frac{(2l+1)|\gamma_0|}{9h^2 + \gamma_2} \right). \end{aligned}$$

In view of condition (29) from hence we get

$$\sum_{k=1}^{\infty} (lk+1)hf_k \leq \frac{h \left(\frac{(l+1)(|a_2| - |\gamma_1|)}{4h^2 + \gamma_2} - \frac{(2l+1)|\gamma_0|}{9h^2 + \gamma_2} \right)}{h \left(1 - \frac{(2l+1)|\gamma_1|}{(l+1)(9h^2 + \gamma_2)} - \frac{(3l+1)|\gamma_0|}{(l+1)(16h^2 + \gamma_2)} \right)} \leq h - \alpha$$

and by Theorem 1 $F \in \mathfrak{D}\mathfrak{F}_h(l, \alpha)$. The first part of Theorem 6 is proved.

Similarly,

$$\begin{aligned} &\sum_{k=1}^{\infty} (h + l\lambda_k)(h + \lambda_k)f_k = \sum_{k=1}^{\infty} (lk+1)(k+1)h^2f_k = \\ &= 2(l+1)h^2f_1 + 3(2l+1)h^2f_2 + \sum_{k=3}^{\infty} (lk+1)(k+1)h^2f_k = 2(l+1)h^2f_1 + \\ &+ 3(2l+1)h^2f_2 + \sum_{k=3}^{\infty} (lk+1)(k+1)h^2 \left(\frac{|\gamma_1|f_{k-1}}{(k+1)^2h^2 + \gamma_2} + \frac{|\gamma_0|f_{k-2}}{(k+1)^2h^2 + \gamma_2} \right) = \\ &= (2(l+1)f_1 + 3(2l+1)f_2)h^2 + h^2 \sum_{k=3}^{\infty} \frac{(lk+1)(k+1)|\gamma_1|}{(k+1)^2h^2 + \gamma_2} f_{k-1} + \\ &+ h^2 \sum_{k=3}^{\infty} \frac{(lk+1)(k+1)|\gamma_0|}{(k+1)^2h^2 + \gamma_2} f_{k-2} = (2(l+1)f_1 + 3(2l+1)f_2)h^2 - h^2 \frac{3(2l+1)|\gamma_1|}{9h^2 + \gamma_2} f_1 + \\ &+ h^2 \sum_{k=1}^{\infty} \frac{((k+1)l+1)(k+2)|\gamma_1|}{(k+2)^2h^2 + \gamma_2} f_k + h^2 \sum_{k=1}^{\infty} \frac{((k+2)l+1)(k+3)|\gamma_0|}{(k+3)^2h^2 + \gamma_2} f_k = \\ &= h^2 \left(2(l+1)f_1 + 3(2l+1) \frac{\gamma_0 - \gamma_1 f_1}{9h^2 + \gamma_2} - \frac{3(2l+1)|\gamma_1|}{9h^2 + \gamma_2} f_1 \right) + \\ &+ \sum_{k=1}^{\infty} \left(\frac{((k+1)l+1)(k+2)|\gamma_1|}{(lk+1)(k+1)((k+2)^2h^2 + \gamma_2)} + \frac{((k+2)l+1)(k+3)|\gamma_0|}{(lk+1)(k+1)((k+3)^2h^2 + \gamma_2)} \right) \times \end{aligned}$$

$$\begin{aligned} & \times (lk + 1)(k + 1)h^2 f_k \leq h^2 \left(\frac{2(l+1)(|a_2| - |\gamma_1|)}{4h^2 + \gamma_2} - \frac{3(2l+1)|\gamma_0|}{9h^2 + \gamma_2} \right) + \\ & + \sum_{k=1}^{\infty} \left(\frac{3(2l+1)|\gamma_1|}{2(l+1)(9h^2 + \gamma_2)} + \frac{4(3l+1)|\gamma_0|}{2(l+1)(16h^2 + \gamma_2)} \right) (lk + 1)(k + 1)h^2 f_k. \end{aligned}$$

Since (30) implies

$$\frac{3(2l+1)|\gamma_1|}{2(l+1)(9h^2 + \gamma_2)} + \frac{4(3l+1)|\gamma_0|}{2(l+1)(16h^2 + \gamma_2)} < 1,$$

as above we have

$$\begin{aligned} & \left(1 - \frac{3(2l+1)|\gamma_1|}{2(l+1)(9h^2 + \gamma_2)} - \frac{4(3l+1)|\gamma_0|}{2(l+1)(16h^2 + \gamma_2)} \right) \sum_{k=1}^{\infty} (lk + 1)(k + 1)h^2 f_k \leq \\ & \leq h^2 \left(\frac{2(l+1)(|a_2| - |\gamma_1|)}{4h^2 + \gamma_2} - \frac{3(2l+1)|\gamma_0|}{9h^2 + \gamma_2} \right) \end{aligned}$$

and, thus, in view of (30)

$$\sum_{k=1}^{\infty} (lk + 1)(k + 1)h^2 f_k \leq \frac{h^2 \left(\frac{2(l+1)(|a_2| - |\gamma_1|)}{4h^2 + \gamma_2} - \frac{3(2l+1)|\gamma_0|}{9h^2 + \gamma_2} \right)}{\left(1 - \frac{3(2l+1)|\gamma_1|}{2(l+1)(9h^2 + \gamma_2)} - \frac{4(3l+1)|\gamma_0|}{2(l+1)(16h^2 + \gamma_2)} \right)} \leq h(h - \alpha)$$

and by Corollary 1 $F \in \mathfrak{DG}_h(l, \alpha)$. The second part of Theorem 6 is proved.

Finally, since for every $\sigma \in \mathbb{R}$ there exists $k_0 = k_0(\sigma) \geq 3$ such that

$$\frac{|\gamma_1|e^{\sigma h}}{(k+1)^2 h^2 + \gamma_2} + \frac{|\gamma_0|e^{\sigma h}}{(k+3)^2 h^2 + \gamma_2} \leq \frac{1}{2} \quad (k \geq k_0),$$

we have as above

$$\begin{aligned} & \sum_{k=k_0}^{\infty} f_k \exp\{\sigma(k+1)h\} = \sum_{k=k_0}^{\infty} \left(\frac{|\gamma_1|f_{k-1} + |\gamma_0|f_{k-2}}{(k+1)^2 h^2 + \gamma_2} \right) \exp\{\sigma(k+1)h\} = \\ & = \sum_{k=k_0-1}^{\infty} \frac{|\gamma_1|f_k e^{\sigma(k+2)h}}{(k+2)^2 h^2 + \gamma_2} + \sum_{k=k_0-2}^{\infty} \frac{|\gamma_0|f_k e^{\sigma(k+3)h}}{(k+3)^2 h^2 + \gamma_2} = \frac{|\gamma_1|f_{k_0} e^{\sigma(k_0+2)h}}{(k_0+2)^2 h^2 + \gamma_2} + \frac{|\gamma_0|f_{k_0} e^{\sigma(k_0+3)h}}{(k_0+3)^2 h^2 + \gamma_2} + \\ & + \frac{|\gamma_0|f_{k_0-1} e^{\sigma(k_0+2)h}}{(k_0+2)^2 h^2 + \gamma_2} + \sum_{k=k_0}^{\infty} \left(\frac{|\gamma_1|e^{\sigma h}}{(k+1)^2 h^2 + \gamma_2} + \frac{|\gamma_0|e^{\sigma h}}{(k+3)^2 h^2 + \gamma_2} \right) f_k \exp\{\sigma(k+1)h\} \leq \\ & \leq \frac{|\gamma_1|f_{k_0} e^{\sigma(k_0+2)h}}{(k_0+2)^2 h^2 + \gamma_2} + \frac{|\gamma_0|f_{k_0} e^{\sigma(k_0+3)h}}{(k_0+3)^2 h^2 + \gamma_2} + \frac{|\gamma_0|f_{k_0-1} e^{\sigma(k_0+2)h}}{(k_0+2)^2 h^2 + \gamma_2} + \frac{1}{2} \sum_{k=k_0}^{\infty} f_k \exp\{\sigma(k+1)h\}, \end{aligned}$$

whence it follows that $\sum_{k=1}^{\infty} f_k \exp\{\sigma(k+1)h\} < +\infty$, i. e. Dirichlet series (28) is absolutely convergent in \mathbb{C} and, thus, entire. \square

For the proof of the equality $\ln \ln M(\sigma, F) = (1 + o(1))h\sigma$ as $0 \leq \sigma \rightarrow +\infty$ we use the following lemma [34, p. 123] (see also [35]).

Lemma 2. *If $\sum_{k=1}^{\infty} \frac{1}{k\lambda_k} < +\infty$ then for an entire Dirichlet series $F(s) = \sum_{k=0}^{\infty} a_k \exp s\lambda_k$*

$$F''(s) = \lambda_{\nu}^2(F(s) + o(M(\sigma, F))), \quad \nu = \nu(\sigma, F), \quad (32)$$

outside some set $C = \bigcup_{n \geq 1} [\sigma'_n, \sigma''_n]$ of finite measure, where $\nu(\sigma, F) = \max\{k : |a_k| \exp\{\sigma\lambda_k\} = \mu(\sigma, F)\}$ is the central index and $\mu(\sigma, F) = \max\{|a_k| \exp\{\sigma\lambda_k\} : k \geq 0\}$ is the maximal term.

Substituting (32) in (20) we get

$$\lambda_{\nu}^2(F(s) + o(M(\sigma, F))) + (\gamma_0 e^{2hs} + \gamma_1 e^{hs} + \gamma_2)F(s) = a_1 e^{hs} + a_2 e^{2hs},$$

and if $s = \sigma + it_0$ is such that $|F(s)| = (1 + o(1))M(\sigma, F)$ then $\lambda_{\nu}^2 F(s) = (1 + o(1))|\gamma_0|e^{2hs}F(s)$, i. e. $\lambda_{\nu(\sigma)} = (1 + o(1))\sqrt{|\gamma_0|}e^{h\sigma}$ as $0 \leq \sigma \rightarrow +\infty$ outside C . If $\sigma \in C$, that is $\sigma'_n \leq \sigma < \sigma''_n$, then

$$\lambda_{\nu(\sigma)} \leq \lambda_{\nu(\sigma''_n)} = (1 + o(1))\sqrt{|\gamma_0|}e^{h\sigma''_n} = (1 + o(1))\sqrt{|\gamma_0|}e^{h\sigma'_n}e^{h(\sigma''_n - \sigma'_n)} \leq (1 + o(1))\sqrt{|\gamma_0|}e^{h\sigma}e^{hK},$$

where K is the measure of C . Similarly,

$$\lambda_{\nu(\sigma)} \geq \lambda_{\nu(\sigma'_n)} = (1 + o(1))\sqrt{|\gamma_0|}e^{h\sigma'_n} = (1 + o(1))\sqrt{|\gamma_0|}e^{h\sigma''_n}e^{h(\sigma'_n - \sigma''_n)} \geq (1 + o(1))\sqrt{|\gamma_0|}e^{h\sigma}e^{-hK},$$

Therefore, $\lambda_{\nu(\sigma)} \asymp e^{h\sigma}$ and, since [34, p.17] $\ln \mu(\sigma) = \ln \mu(\sigma) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(x)} dx$ for $-\infty < \sigma_0 \leq \sigma < +\infty$, we get $\ln \mu(\sigma) \asymp e^{h\sigma}$. Finally, since $\ln k = o((k+1)h)$ as $k \rightarrow \infty$, we have [34, p.22] $\mu(\sigma) \leq M(\sigma) \leq \mu(\sigma + \varepsilon)$ for every $\varepsilon > 0$ and all $\sigma \geq \sigma_0(\varepsilon)$ and, thus, $\ln M(\sigma) \asymp e^{h\sigma}$. Hence it follows that $\ln \ln M(\sigma, F) = (1 + o(1))h\sigma$ as $0 \leq \sigma \rightarrow +\infty$. The proof of Theorem 6 is complete.

Remark. Now suppose that $a_1 = a_2 = 0$, i. e. function (4) satisfies (19). Then, as above, $\gamma_2 = -h^2$, $\lambda_1 = h$ and $f_1 = \frac{\gamma_1}{3h^2}$, $\lambda_2 = 2h$ and $f_2 = \frac{\gamma_0 - \gamma_1 f_1}{8h^2}$, $\lambda_3 = 3h$ and $f_3 = -\frac{\gamma_1 f_2 + \gamma_0 f_1}{15h^2}$.

Let's show that these equality contradict to the condition $f_k > 0$ for all $k \geq 1$. Indeed, $f_1 > 0$ if and only if $\gamma_1 > 0$, and $f_2 > 0$ if and only if $\gamma_0 - \gamma_1 f_1 > 0$, whence $\gamma_0 > \gamma_1 f_1 > 0$. Then $\gamma_1 f_2 + \gamma_0 f_1 > 0$ and $f_3 < 0$.

Thus, differential equation (19) has no solution in the class \mathfrak{D}_0 .

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Received 25.09.2021

Revised 25.03.2022