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ON CERTAIN SUBCLASS OF DIRICHLET SERIES ABSOLUTELY CONVERGENT IN HALF-PLANE

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Denote by $\mathfrak{D}_0$ the class of absolutely convergent in the half-plane $\Pi_0=\{s\colon \operatorname{Re} s<0\}$ Dirichlet series $F(s)=e^{sh}-\sum_{k=1}^{\infty} f_k\exp\{s(\lambda_k+h)\}$, $s=\sigma+it$, where $h>0$, $h<\lambda_k\uparrow+\infty$ and $f_k>0$. For $0\le\alpha<h$ and $l\ge0$ we say that F belongs to the class $\mathfrak{D}\mathfrak{F}_h(l,\alpha)$ if and only if $\operatorname{Re}\{e^{-hs}((1-l)F(s)+\frac{l}{h}F'(s))\}>\frac{\alpha}{h}$, and belongs to the class $\mathfrak{D}\mathfrak{G}_h(l,\alpha)$ if and only if $\operatorname{Re}\{e^{-hs}((1-l)F'(s)+\frac{l}{h}F''(s))\}>\alpha$ for all $s\in\Pi_0$. It is proved that $F\in\mathfrak{D}\mathfrak{F}_h(l,\alpha)$ if and only if $\sum_{k=1}^{\infty}(h+l\lambda_k)f_k\le h-\alpha$, and $F\in\mathfrak{D}\mathfrak{G}_h(l,\alpha)$ if and only if $\sum_{k=1}^{\infty}(h+l\lambda_k)(\lambda_k+h)f_k\le h(h-\alpha)$.

If $F_j\in\mathfrak{D}\mathfrak{F}_h(l_j,\alpha_j)$, $j=1,2$, where $l_j\ge0$ and $0\le\alpha_j<h$, then Hadamard composition $(F_1*F_2)\in\mathfrak{D}F_h(l,\alpha)$, where $l=\min\{l_1,l_2\}$ and $\alpha=h-\frac{(h-\alpha_1)(h-\alpha_2)}{h+l\lambda_1}$. Similar statement is correct for the class $F_j\in\mathfrak{D}\mathfrak{G}_h(l,\alpha)$.

For $j>0$ and $\delta>0$ the neighborhood of the function $F\in\mathfrak{D}_0$ is defined as follows $O_{j,\delta}(F)=\{G(s)=e^s-\sum_{k=1}^{\infty} g_k\exp\{s\lambda_k\}\in\mathfrak{D}_0\colon \sum_{k=1}^{\infty}\lambda_k^j|g_k-f_k|\le\delta\}$. It is described the neighborhoods of functions from classes $\mathfrak{D}\mathfrak{F}_h(l,\alpha)$ and $\mathfrak{D}\mathfrak{G}_h(l,\alpha)$.

Conditions on real parameters γ_0, γ_1, γ_2, a_1 and a_2 of the differential equation $w''+(\gamma_0e^{2hs}+\gamma_1e^{hs}+\gamma_2)w=a_1e^{hs}+a_2e^{2hs}$ are found, under which this equation has a solution either in $\mathfrak{D}\mathfrak{F}_h(l,\alpha)$ or in $\mathfrak{D}\mathfrak{G}_h(l,\alpha)$.

1. Introduction. Let $p\in\mathbb{N}$, $0\le\alpha<p$ and $l\ge0$. We say that an analytic in the disk $\mathbb{D}=\{z\colon |z|<1\}$ function

$$f(z)=z^p-\sum_{n=1}^{\infty}a_nz^{n+p},\quad a_n\ge0, \tag{1}$$

belongs to the class $\mathfrak{F}_p(l,\alpha)$ if and only if

$$\operatorname{Re}\left\{(1-l)\frac{f(z)}{z^p}+l\frac{f'(z)}{pz^{p-1}}\right\}>\frac{\alpha}{p}\quad (z\in\mathbb{D}), \tag{2}$$

and belongs to the class $\mathfrak{G}_p(l,\alpha)$ if and only if

$$\operatorname{Re}\left\{(p+l(1-p))\frac{f'(z)}{pz^{p-1}}+l\frac{f''(z)}{pz^{p-2}}\right\}>\alpha\quad (z\in\mathbb{D}). \tag{3}$$

The class $\mathfrak{F}_p(l,\alpha)$ was introduced and studied earlier by S. K. Lee, S. Owa and H. M. Srivastava [1] and was further investigated by M. K. Aouf and H. E. Darwish [2]. The class $\mathfrak{G}_p(l,\alpha)$

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was studied recently by M. K. Aouf [3]. In particular, the class $\mathfrak{G}_1(l,\alpha)$ was considered earlier by O. Altintas [4-5]. The research of classes $\mathfrak{F}_p(l,\alpha)$ and $\mathfrak{G}_p(l,\alpha)$ was continued in the works [6-7].

Absolutely convergent in the half-plane $\Pi_0=\{s\colon \operatorname{Re} s<0\}$ Dirichlet series with positive increasing to $+\infty$ exponents are direct generalizations of analytic functions in the disk $\mathbb{D}$. Therefore, it is quite natural to introduce analogues of the classes $\mathfrak{F}_p(l,\alpha)$ and $\mathfrak{G}_p(l,\alpha)$ for such series of Dirichlet and explore their properties.

So, let $h>0$, (λ_k) be an increasing to $+\infty$ sequence of positive numbers ($\lambda_1>h$) and $\mathfrak{D}_0$ be the class of Dirichlet series

$$F(s)=e^{sh}-\sum_{k=1}^{\infty} f_k\exp\{s(\lambda_k+h)\},\quad f_k>0,\quad s=\sigma+it, \tag{4}$$

absolutely convergent in half-plane Π_0. For $0\le\alpha<h$ and $l\ge 0$ we say that F belongs to the class $\mathfrak{D}\mathfrak{F}_h(l,\alpha)$ if and only if

$$\operatorname{Re}\left\{e^{-hs}\left((1-l)F(s)+\frac{l}{h}F'(s)\right)\right\}>\frac{\alpha}{h}\quad (s\in\Pi_0), \tag{5}$$

and belongs to the class $\mathfrak{D}\mathfrak{G}_h(l,\alpha)$ if and only if

$$\operatorname{Re}\left\{e^{-hs}\left((1-l)F'(s)+\frac{l}{h}F''(s)\right)\right\}>\alpha\quad (s\in\Pi_0). \tag{6}$$

It follows from (5) and (6) that

$$F\in\mathfrak{D}\mathfrak{G}_h(l,\alpha)\Longleftrightarrow\frac{F'}{h}\in\mathfrak{D}\mathfrak{F}_h(l,\alpha). \tag{7}$$

2. Properties of coefficients. In order to derive various properties associated with the classes $\mathfrak{D}\mathfrak{F}_h(l,\alpha)$ and $\mathfrak{D}\mathfrak{G}_h(l,\alpha)$ we shall need the following theorems.

Theorem 1. *Let a function F be defined by (4). Then $F\in\mathfrak{D}\mathfrak{F}_h(l,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}(h+l\lambda_k)f_k\le h-\alpha. \tag{8}$$

Proof. Since

$$e^{-hs}\left((1-l)F(s)+\frac{l}{h}F'(s)\right)=$$

$$=e^{-hs}\left\{(1-l)\left(e^{sh}-\sum_{k=1}^{\infty}f_k\exp\{s(\lambda_k+h)\}\right)+\frac{l}{h}\left(he^{sh}-\sum_{k=1}^{\infty}(\lambda_k+h)f_k\exp\{s(\lambda_k+h)\}\right)\right\}=$$

$$=(1-l)\left(1-\sum_{k=1}^{\infty}f_k\exp\{s\lambda_k\}\right)+\frac{l}{h}\left(h-\sum_{k=1}^{\infty}(\lambda_k+h)f_k\exp\{s\lambda_k\}\right)=$$

$$=1-\sum_{k=1}^{\infty}(1-l)f_k\exp\{s\lambda_k\}-\sum_{k=1}^{\infty}\frac{l}{h}(\lambda_k+h)f_k\exp\{s\lambda_k\}=1-\sum_{k=1}^{\infty}\frac{h+l\lambda_k}{h}f_k\exp\{s\lambda_k\},$$

condition (5) holds if and only if

$$\mathrm{Re}\left\{1-\sum_{k=1}^{\infty}\frac{h+l\lambda_k}{h}f_k\exp\{s\lambda_k\}\right\}>\frac{\alpha}{h}\quad(s\in\Pi_0),$$

i. e.

$$\mathrm{Re}\left\{\sum_{k=1}^{\infty}(h+l\lambda_k)f_k\exp\{s\lambda_k\}\right\}<h-\alpha\quad(s\in\Pi_0). \tag{9}$$

Since $f_k>0$ and (9) holds for all $s\in\Pi_0$, for all $\sigma<0$ we have

$$\sum_{k=1}^{\infty}(h+l\lambda_k)f_k\exp\{\sigma\lambda_k\}<h-\alpha. \tag{10}$$

As $\sigma\to 0$ from hence we get a condition (8).

On the contrary, the condition (8) implies (10) for all $\sigma<0$. Therefore, for all $s\in\Pi_0$ we have

$$\mathrm{Re}\left\{\sum_{k=1}^{\infty}(h+l\lambda_k)f_k\exp\{s\lambda_k\}\right\}\le\left|\sum_{k=1}^{\infty}(h+l\lambda_k)f_k\exp\{s\lambda_k\}\right|\le$$
$$\le\sum_{k=1}^{\infty}(h+l\lambda_k)f_k\exp\{\sigma\lambda_k\}<h-\alpha,$$

i. e. (9) holds. □

Since for function (4) we have

$$\frac{F'(s)}{h}=e^{sh}-\sum_{k=1}^{\infty}\frac{\lambda_k+h}{h}f_k\exp\{s(\lambda_k+h)\},$$

by Theorem 1 we get $F'/h\in\mathfrak{D}\mathfrak{F}_h(l,\alpha)$ if and only if

$$\sum_{k=1}^{\infty}(h+l\lambda_k)\frac{\lambda_k+h}{h}f_k\le h-\alpha.$$

Therefore, in view of (7) we obtain the following statement.

Corollary 1. *Let the function F be define by* (4). *Then $F\in\mathfrak{D}\mathfrak{G}_h(l,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}(h+l\lambda_k)(\lambda_k+h)f_k\le h(h-\alpha). \tag{11}$$

3. Hadamard compositions. For power series $f_j(z)=\sum_{k=0}^{\infty}f_{k,j}z^k$ $(j=1,2)$ the series $(f_1*f_2)(z)=\sum_{k=0}^{\infty}f_{k,1}f_{k,2}z^k$ is called the Hadamard composition (product)[8]. Properties of this composition obtained by J. Hadamard find applications [9–10] in the theory of the analytic continuation of the functions represented by power series. We remark also that singular points of the Hadamard composition are investigated in the article [11]. L. Zalzman [12] studied Hadamard compositions of univalent functions. For the functions $f_j(z)=1/z+$

$\sum_{k=1}^{\infty} f_{k,j} z^k$ $(j = 1, 2)$ M. L. Mogra [13] defined Hadamard composition as $(f_1 * f_2)(z) = 1/z + \sum_{k=1}^{\infty} f_{k,1} f_{k,2} z^k$ and proved, for example, that if the functions f_j are meromorphically starlike of order $\alpha_j \in [0,\ 1)$ and $f_{k,j} \geq 0$ for all $k \geq 1$ then $f_1 * f_2$ is meromorphically starlike of order $\alpha = \max\{\alpha_1,\ \alpha_2\}$. Hadamard compositions of analytic and meromorphic functions in $\mathbb{D}$ studied also by J. H. Choi, Y. C. Kim and S. Owa [14], M. K. Aouf and H. Silverman [15], J. Liu and R. Srivastava [16], S. Ruscheweyh [17] and many other mathematicians. Hadamard products of functions from classes $\mathfrak{F}_p(l, \alpha)$ and $\mathfrak{G}_p(l, \alpha)$ were studied in [7]. For Dirichlet series absolutely convergent in half-plane Π_0 Hadamard compositions were used in [18].

So, let functions F_j, $j = 1, 2$, be defined by

$$F_j(s) = e^{sh} - \sum_{k=1}^{\infty} f_{k,j} \exp\{s(\lambda_k + h)\}, \quad f_{k,j} > 0. \tag{12}$$

Then the Hadamard composition of F_1 and F_2 is defined by

$$(F_1 * F_2)(s) = e^{sh} - \sum_{k=1}^{\infty} f_{k,1} f_{k,2} \exp\{s(\lambda_k + h)\}. \tag{13}$$

Theorem 2. *Let $F_j \in \mathfrak{D}\mathfrak{F}_h(l_j, \alpha_j)$, $j = 1, 2$, where $l_j \geq 0$ and $0 \leq \alpha_j < h$. Then $(F_1 * F_2) \in \mathfrak{D}F_h(l, \alpha)$, where $l = \min\{l_1, l_2\}$ and*

$$\alpha = h - \frac{(h - \alpha_1)(h - \alpha_2)}{h + l\lambda_1}. \tag{14}$$

Proof. Since $F_j \in \mathfrak{D}\mathfrak{F}_h(l_j, \alpha_j)$ and $l = \min\{l_1, l_2\}$, by Theorem 1

$$\sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h - \alpha_j} f_{k,j} \leq \sum_{k=1}^{\infty} \frac{h + l_j\lambda_k}{h - \alpha_j} f_{k,j} \leq 1.$$

and by the Cauchy-Schwarz inequality we have

$$\sum_{k=1}^{\infty} \sqrt{\frac{h + l\lambda_k}{h - \alpha_1} f_{k,1} \frac{h + l\lambda_k}{h - \alpha_2} f_{k,2}} \leq \sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h - \alpha_1} f_{k,1} \sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h - \alpha_2} f_{k,2} \leq 1,$$

i. e.

$$\sum_{k=1}^{\infty} \frac{h + l\lambda_k}{\sqrt{(h - \alpha_1)(h - \alpha_2)}} \sqrt{f_{k,1} f_{k,2}} \leq 1, \tag{15}$$

whence

$$\frac{h + l\lambda_k}{\sqrt{(h - \alpha_1)(h - \alpha_2)}} \sqrt{f_{k,1} f_{k,2}} \leq 1,$$

i. e.

$$\sqrt{f_{k,1} f_{k,2}} \leq \frac{\sqrt{(h - \alpha_1)(h - \alpha_2)}}{h + l\lambda_k}.$$

Therefore, in view of (14) and (15)

$$\sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h - \alpha} f_{k,1} f_{k,2} = \sum_{k=1}^{\infty} \frac{h + l\lambda_k}{h - \alpha} \sqrt{f_{k,1} f_{k,2}} \sqrt{f_{k,1} f_{k,2}} \leq$$

$$\leq \sum_{k=1}^{\infty} \frac{\sqrt{(h-\alpha_1)(h-\alpha_2)}}{h-\alpha} \sqrt{f_{k,1}f_{k,2}} = \sum_{k=1}^{\infty} \frac{h+l\lambda_1}{\sqrt{(h-\alpha_1)(h-\alpha_2)}} \sqrt{f_{k,1}f_{k,2}} \leq$$
$$\leq \sum_{k=1}^{\infty} \frac{h+l\lambda_k}{\sqrt{(h-\alpha_1)(h-\alpha_2)}} \sqrt{f_{k,1}f_{k,2}} \leq 1,$$

i. e.

$$\sum_{k=1}^{\infty} (h+l\lambda_k) f_{k,1}f_{k,2} \leq h-\alpha.$$

Thus, by Theorem 1 $(F_1 * F_2) \in \mathfrak{DF}_h(l,\alpha)$. □

Corollary 2. *Let $F_j \in \mathfrak{DF}_h(l,\alpha)$, $j=1,2$, where $l \geq 0$ and $0 \leq \alpha < h$. Then $(F_1 * F_2) \in \mathfrak{DF}_h(l,\beta)$, where $\beta = h - \frac{(h-\alpha)^2}{h+l\lambda_1}$.*

For the class $\mathfrak{DG}_h(l,\alpha)$, the next analogue of Theorem 2 is true.

Theorem 3. *Let $F_j \in \mathfrak{DG}_h(l_j,\alpha_j)$, $j=1,2$, where $l_j \geq 0$ and $0 \leq \alpha_j < h$. Then $(F_1 * F_2) \in \mathfrak{DG}_h(l,\alpha)$, where $l = \min\{l_1,l_2\}$ and*

$$\alpha = h - \frac{h(h-\alpha_1)(h-\alpha_2)}{(h+l\lambda_1)(h+\lambda_1)}. \tag{16}$$

Proof. Since $F_j \in \mathfrak{DG}_h(l_j,\alpha_j)$ and $l = \min\{l_1,l_2\}$, by Corollary 1 as above we have

$$\sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha_j)} f_{k,j} \leq 1, \quad j=1,2.$$

and by the Cauchy-Schwarz inequality

$$\sum_{k=1}^{\infty} \sqrt{\frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha_1)} f_{k,1} \frac{(h+l\lambda_k)\lambda_k+h)}{h(h-\alpha_2)} f_{k,2}} \leq$$
$$\leq \sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha_1)} f_{k,1} \sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha_2)} f_{k,2} \leq 1,$$

i. e.

$$\sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h\sqrt{(h-\alpha_1)(h-\alpha_2)}} \sqrt{f_{k,1}f_{k,2}} \leq 1, \tag{17}$$

whence as above

$$\sqrt{f_{k,1}f_{k,2}} \leq \frac{h\sqrt{(h-\alpha_1)(h-\alpha_2)}}{(h+l\lambda_k)(\lambda_k+h)}.$$

Therefore, in view of (16) and (17)

$$\sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha)} f_{k,1}f_{k,2} = \sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h(h-\alpha)} \sqrt{f_{k,1}f_{k,2}}\sqrt{f_{k,1}f_{k,2}} \leq$$
$$\leq \sum_{k=1}^{\infty} \frac{\sqrt{(h-\alpha_1)(h-\alpha_2)}}{h-\alpha} \sqrt{f_{k,1}f_{k,2}} = \sum_{k=1}^{\infty} \frac{(h+l\lambda_1)(\lambda_1+h)}{h\sqrt{(h-\alpha_1)(h-\alpha_2)}} \sqrt{f_{k,1}f_{k,2}} \leq$$

$$\le \sum_{k=1}^{\infty} \frac{(h+l\lambda_k)(\lambda_k+h)}{h\sqrt{(h-\alpha_1)(h-\alpha_2)}}\sqrt{f_{k,1}f_{k,2}} \le 1,$$

i. e. by Corollary 1 $(F_1 * F_2) \in \mathfrak{DG}_h(l,\alpha)$. □

Corollary 3. *Let* $F_j \in \mathfrak{DG}_h(l,\alpha)$, $j = 1, 2$, *where* $l \ge 0$ *and* $0 \le \alpha < h$. *Then* $(F_1 * F_2) \in \mathfrak{DG}_h(l,\beta)$, *where* $\beta = h - \frac{h(h-\alpha)^2}{(h+l\lambda_1)(h+\lambda_1)}$.

4. Neighborhoods of Diriclet series. For the function $f(z) = \sum_{k=0}^{\infty} f_k z^k$, following A. W. Goodman [19] and S. Ruscheweyh [17], the set

$$N_\delta(f) = \left\{ g(z) = z + \sum_{k=2}^{\infty} g_k z^k \colon \sum_{k=2}^{\infty} k|g_k - f_k| \le \delta \right\}$$

is called δ-neighborhood of f. The neighborhoods of various classes of analytical in $\mathbb{D}$ functions were studied by many authors (we indicate here only on articles [20–26]). Here for $j > 0$ and $\delta > 0$ we define the neighborhood of $F \in \mathfrak{D}_0$ as follows

$$O_{j,\delta}(F) = \left\{ G(s) = e^s - \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\} \in \mathfrak{D}_0 \colon \sum_{k=1}^{\infty} \lambda_k^j |g_k - f_k| \le \delta \right\}. \tag{18}$$

The following two theorems describe the neighborhoods of functions from classes $\mathfrak{DF}_h(l,\alpha)$ and $\mathfrak{DF}_h(l,\alpha)$.

Theorem 4. *Let* $l > 0$, $0 \le \beta < \alpha < h$ *and* $F \in \mathfrak{DG}_h(l,\alpha)$. *If* $G \in O_{2,\delta}(F)$ *with* $\delta = \delta_1 := \frac{h(\alpha-\beta)\lambda_1}{(h+l\lambda_1)(h+\lambda_1)}$ *then* $G \in \mathfrak{DG}_h(l,\beta)$. *On the contrary, if* $G \in \mathfrak{DG}_h(l,\beta)$ *then* $G \in O_{2,\delta}(F)$ *with* $\delta = \delta_2 := \frac{2h-\alpha-\beta}{l}$.

Proof. Since $F \in \mathfrak{DG}_h(l,\alpha)$, by Corollary 1 condition (8) holds. Therefore, for $\beta < \alpha$ in view of (18) we have

$$\sum_{k=1}^{\infty}(h+l\lambda_k)g_k \le \sum_{k=1}^{\infty}(h+l\lambda_k)f_k + \sum_{k=1}^{\infty}(h+l\lambda_k)|g_k - f_k| \le$$
$$\le \sum_{k=1}^{\infty} \frac{h+l\lambda_k}{\lambda_k}\lambda_k|g_k - f_k| \le h - \alpha + \frac{h+l\lambda_1}{\lambda_1}\sum_{k=1}^{\infty}\lambda_k|g_k - f_k| \le$$
$$\le h - \alpha + \frac{h+l\lambda_1}{\lambda_1}\delta = h - \alpha + \frac{h+l\lambda_1}{\lambda_1}\frac{(\alpha-\beta)\lambda_1}{h+l\lambda_1} = h - \beta$$

and, thus, $G \in \mathfrak{DF}_h(l,\beta)$.

On the contrary, if $G \in \mathfrak{DF}_h(l,\beta)$ then

$$\sum_{k=1}^{\infty}\lambda_k|g_k - f_k| = \sum_{k=1}^{\infty}\frac{\lambda_k}{h+l\lambda_k}(h+l\lambda_k)|g_k - f_k| \le \frac{1}{l}\sum_{k=1}^{\infty}(h+l\lambda_k)|g_k - f_k| \le$$
$$\le \frac{1}{l}\left(\sum_{k=1}^{\infty}(h+l\lambda_k)g_k + \sum_{k=1}^{\infty}(h+l\lambda_k)f_k\right) \le \frac{h-\alpha+h-\beta}{l} = \delta_2,$$

that is $G \in O_{1,\delta}(F)$ with $\delta = \delta_2$. □

Theorem 5. *Let $l>0$, $0\le\beta<\alpha<h$ and $F\in\mathfrak{D}G_h(l,\alpha)$. If $G\in O_{2,\delta}(F)$ with $\delta=\delta_3:=$ $=\frac{h(\alpha-\beta)\lambda_1^2}{(h+l\lambda_1)(h+\lambda_1)}$ then $G\in\mathfrak{D}\mathfrak{G}_h(l,\beta)$. On the contrary, if $G\in\mathfrak{D}\mathfrak{G}_h(l,\beta)$ then $G\in O_{2,\delta}(F)$ with $\delta=\delta_4:=\frac{h(2h-\alpha-\beta)}{l}$.*

Proof. Since $F\in\mathfrak{D}\mathfrak{G}_h(l,\alpha)$, by Corollary 1 condition (11) holds. Therefore, for $\beta<\alpha$ in view of (18) we have

$$\sum_{k=1}^{\infty}(h+l\lambda_k)(\lambda_k+h)g_k\le\sum_{k=1}^{\infty}(h+l\lambda_k)(\lambda_k+h)f_k+\sum_{k=1}^{\infty}(h+l\lambda_k)(\lambda_k+h)|g_k-f_k|\le$$
$$\le h(h-\alpha)+\sum_{k=1}^{\infty}\frac{(h+l\lambda_k)(\lambda_k+h)}{\lambda_k^2}\lambda_k^2|g_k-f_k|\le h(h-\alpha)+$$
$$+\frac{(h+l\lambda_1)(\lambda_1+h)}{\lambda_1^2}\sum_{k=1}^{\infty}\lambda_k^2|g_k-f_k|\le h(h-\alpha)+\frac{(h+l\lambda_1)(\lambda_1+h)}{\lambda_1^2}\delta_3=h(h-\beta)$$

and, thus, by Corollary 1 $G\in\mathfrak{D}\mathfrak{G}_h(l,\beta)$.

On the contrary, if $G\in\mathfrak{D}\mathfrak{G}_h(l,\beta)$ then as above

$$\sum_{k=1}^{\infty}\lambda_k^2|g_k-f_k|=\sum_{k=1}^{\infty}\frac{\lambda_k^2}{(h+l\lambda_k)(h+\lambda_k)}(h+l\lambda_k)(h+\lambda_k)|g_k-f_k|\le$$
$$\le\frac{1}{l}\sum_{k=1}^{\infty}(h+l\lambda_k)(h+\lambda_k)|g_k-f_k|\le\frac{h(h-\alpha)+h(h-\beta)}{l}=\delta_4$$

that is $G\in O_{2,\delta}(F)$ with $\delta=\delta_4$. □

5. Differential equations of second order. S. M. Shah [27] indicated conditions on real parameters γ_0, γ_1, γ_2 of the differential equation $z^2w''+zw'+(\gamma_0z^2+\gamma_1z+\gamma_2)w=0$, under which there exists an entire transcendental solution f that together with its derivatives are close-to-convex in $\mathbb{D}$. The investigations are continued by Z. M. Sheremeta (see, for example, [28–39]). Substituting $z=e^s$ we obtain the differential equation

$$\frac{d^2w}{ds^2}+(\gamma_0e^{2hs}+\gamma_1e^{hs}+\gamma_2)w=0 \tag{19}$$

with $h=1$. In [32] and [33, p. 150] conditions found, under which this equation has solutions pseudoconvex or close-to-pseudoconvex in Π_0. Here we consider a differential equation

$$\frac{d^2w}{ds^2}+(\gamma_0e^{2hs}+\gamma_1e^{hs}+\gamma_2)w=a_1e^{hs}+a_2e^{2hs}, \tag{20}$$

where $\gamma_0<0$, $\gamma_0<0$, $a_1>0$ and $a_2<0$. Suppose that function (4) satisfies (20), i. e.

$$h^2e^{sh}+\gamma_2e^{sh}+\gamma_1e^{2hs}+\gamma_0e^{3hs}-\sum_{k=1}^{\infty}((\lambda_k+h)^2+\gamma_2)f_k\exp\{s(\lambda_k+h)\}-$$
$$-\gamma_1\sum_{k=1}^{\infty}f_k\exp\{s(\lambda_k+2h)\}-\gamma_0\sum_{k=1}^{\infty}f_k\exp\{s(\lambda_k+3h)\}=a_1e^{hs}+a_2e^{2hs}. \tag{21}$$

As $\sigma \to -\infty$ from hence we obtain $(h^2+\gamma_2)e^{sh}(1+o(1)) = a_1 e^{sh}(1+o(1)$, that is $h^2+\gamma_2 = a_1 > 0$. Therefore, from (21) we get

$$\gamma_1 e^{2hs} + \gamma_0 e^{3hs} - ((\lambda_1+h)^2+\gamma_2) f_1 \exp\{s(\lambda_1+h)\} - \sum_{k=2}^{\infty}((\lambda_k+h)^2+\gamma_2) f_k \exp\{s(\lambda_k+h)\} - $$
$$-\gamma_1 \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k+2h)\} - \gamma_0 \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k+3h)\} = a_2 e^{2hs}. \quad (22)$$

Since $(\lambda_1+h)^2+\gamma_2 > 0$, it follows that

$$((\lambda_1+h)^2+\gamma_2) f_1 \exp\{s(\lambda_1+h)\}(1+o(1)) = (\gamma_1 - a_2)e^{2hs}(1+o(1)), \quad \sigma \to -\infty$$

and, thus, $\lambda_1 = h$ and

$$f_1 = \frac{\gamma_1 - a_2}{4h^2+\gamma_2} = \frac{|a_2| - |\gamma_1|}{4h^2+\gamma_2} > 0 \quad (23)$$

provided $|a_2| - |\gamma_1| > 0$. Therefore, (22) implies

$$\gamma_0 e^{3hs} - ((\lambda_2+h)^2+\gamma_2) f_2 \exp\{s(\lambda_2+h)\} - \sum_{k=3}^{\infty}((\lambda_k+h)^2+\gamma_2) f_k \exp\{s(\lambda_k+h)\} - $$
$$-\gamma_1 f_1 \exp\{s(\lambda_1+2h)\} - \gamma_1 \sum_{k=2}^{\infty} f_k \exp\{s(\lambda_k+2h)\} - \gamma_0 \sum_{k=1}^{\infty} f_k \exp\{s(\lambda_k+3h)\} = 0, \quad (24)$$

whence $(\gamma_0 - \gamma_1 f_1)e^{3hs} = (1+o(1))((\lambda_2+h)^2+\gamma_2) f_2 \exp\{s(\lambda_2+h)\}$ as $\sigma \to -\infty$ and, thus, $\lambda_2 = 2h$ and

$$f_2 = \frac{\gamma_0 - \gamma_1 f_1}{9h^2+\gamma_2} = \frac{|\gamma_1|(|a_2| - |\gamma_1|) - |\gamma_0|(4h^2+\gamma_2)}{(9h^2+\gamma_2)(4h^2+\gamma_2)} > 0 \quad (25)$$

provided $|\gamma_0|(4h^2+\gamma_2) < |\gamma_1|(|a_2| - |\gamma_1|)$. Therefore, from (24) it follows that

$$((\lambda_3+h)^2+\gamma_2) f_3 + \sum_{k=4}^{\infty}((\lambda_k+h)^2+\gamma_2) f_k \exp\{s(\lambda_k+h)\} + \gamma_1 f_2 \exp\{s(\lambda_2+2h)\} + $$
$$+\gamma_1 \sum_{k=3}^{\infty} f_k \exp\{s(\lambda_k+2h)\} + \gamma_0 f_1 \exp\{s(\lambda_1+3h)\}, +\gamma_0 \sum_{k=2}^{\infty} f_k \exp\{s(\lambda_k+3h)\} = 0, \quad (26)$$

whence $((\lambda_3+h)^2+\gamma_2) f_3 \exp\{s(\lambda_3+h)\}(1+o(1)) = -(\gamma_1 f_2 + \gamma_0 f_1)e^{4hs}$ as $\sigma \to -\infty$ and, thus, $\lambda_3 = 3h$ and

$$f_3 = -\frac{\gamma_1 f_2 + \gamma_0 f_1}{16h^2+\gamma_2} = \frac{|\gamma_1| f_2 + |\gamma_0| f_1}{16h^2+\gamma_2} > 0$$

Continuing this process, we will come to the formulas

$$\lambda_k = kh, \quad f_k = -\frac{\gamma_1 f_{k-1} + \gamma_0 f_{k-2}}{(k+1)^2 h^2 + \gamma_2} = \frac{|\gamma_1| f_{k-1} + |\gamma_0| f_{k-2}}{(k+1)^2 h^2 + \gamma_2} > 0 \quad (k \ge 3). \quad (27)$$

We remark that the condition $|\gamma_0|(4h^2+\gamma_2) < |\gamma_1|(|a_2| - |\gamma_1|)$ holds if and if $|a_2| > |\gamma_0|(4h^2+\gamma_2)/|\gamma_1| + |\gamma_1|$.

Thus, the following statement is true.

Lemma 1. *Let* $\gamma_0 < 0$, $\gamma_1 < 0$, $\gamma_2 < 0$, $a_1 = h^2 + \gamma_2 > 0$, $a_2 < 0$ *and* $|a_2| > |\gamma_0|(4h^2+\gamma_2)/|\gamma_1| + |\gamma_1|$. *Then differential equation* (20) *has a solution*

$$F(s) = e^{sh} - \sum_{k=1}^{\infty} f_k \exp\{s(k+1)h\}, \tag{28}$$

where the coefficients f_k *are determined by formulas* (23), (25) *and* (27).

Using Lemma 1, we prove the following theorem.

Theorem 6. *Let the parameters* a_1, a_2, γ_0, γ_1 *and* γ_2 *satisfy the assumptions of Lemma 1. Then differential equation* (20) *has a solution* (28) *such that:*

1) if

$$\frac{(l+1)(|a_2|-|\gamma_1|)}{4h^2+\gamma_2} - \frac{(2l+1)|\gamma_0|}{9h^2+\gamma_2} \le \\ \le (h-\alpha)\left(1 - \frac{(2l+1)|\gamma_1|}{(l+1)(9h^2+\gamma_2)} - \frac{(3l+1)|\gamma_0|}{(l+1)(16h^2+\gamma_2)}\right) \tag{29}$$

then $F \in \mathfrak{D}\mathfrak{F}_h(l,\alpha)$;

2) if

$$h\left(\frac{2(l+1)(|a_2|-|\gamma_1|)}{4h^2+\gamma_2} - \frac{3(2l+1)|\gamma_0|}{9h^2+\gamma_2}\right) \le \\ \le (h-\alpha)\left(1 - \frac{3(2l+1)|\gamma_1|}{2(l+1)(9h^2+\gamma_2)} - \frac{4(3l+1)|\gamma_0|}{2(l+1)(16h^2+\gamma_2)}\right) \tag{30}$$

then $F \in \mathfrak{D}\mathfrak{G}_h(l,\alpha)$;

3) Dirichlet series (28) *is entire and* $\ln\ln M(\sigma,F) = (1+o(1))h\sigma$ *as* $0 \le \sigma \to +\infty$, *where* $M_F(\sigma) = \sup\{|F(\sigma+it)|\colon t \in \mathbb{R}\}$.

Proof. For function (27) in view of (27), (25) and (23) we have

$$\sum_{k=1}^{\infty}(h+l\lambda_k)f_k = \sum_{k=1}^{\infty}(lk+1)hf_k = (l+1)hf_1 + (2l+1)hf_2 + \sum_{k=3}^{\infty}(lk+1)hf_k =$$

$$= (l+1)hf_1 + (2l+1)hf_2 + \sum_{k=3}^{\infty}(lk+1)h\frac{|\gamma_1|f_{k-1}+|\gamma_0|f_{k-2}}{(k+1)^2h^2+\gamma_2}f_k =$$

$$= (l+1)hf_1 + (2l+1)hf_2 + \sum_{k=3}^{\infty}(lk+1)h\frac{|\gamma_1|f_{k-1}}{(k+1)^2h^2+\gamma_2} + \sum_{k=3}^{\infty}(lk+1)h\frac{|\gamma_0|f_{k-2}}{(k+1)^2h^2+\gamma_2} =$$

$$= (l+1)hf_1 + (2l+1)hf_2 + \sum_{k=2}^{\infty}(lk+l+1)h\frac{|\gamma_1|f_k}{(k+2)^2h^2+\gamma_2} +$$

$$+\sum_{k=1}^{\infty}(lk+2l+1)h\frac{|\gamma_0|f_k}{(k+3)^2h^2+\gamma_2} = (l+1)hf_1 + (2l+1)hf_2 - (2l+1)h\frac{|\gamma_1|f_1}{9h^2+\gamma_2} +$$

$$+h\sum_{k=1}^{\infty}\left(\frac{(lk+l+1)|\gamma_1|f_k}{(k+2)^2h^2+\gamma_2} + \frac{(lk+2l+1)|\gamma_0|f_k}{(k+3)^2h^2+\gamma_2}\right) \le$$

$$\leq h\left(\frac{(l+1)(|a_2|-|\gamma_1|)}{4h^2+\gamma_2}-\frac{(2l+1)|\gamma_0|}{9h^2+\gamma_2}\right)+$$
$$+\sum_{k=1}^{\infty}\left(\frac{(2l+1)|\gamma_1|}{(l+1)(9h^2+\gamma_2)}+\frac{(3l+1)|\gamma_0|}{(l+1)(16h^2+\gamma_2)}\right)(lk+1)hf_k. \qquad (31)$$

From (29) it follows that

$$\frac{(2l+1)|\gamma_1|}{(l+1)(9h^2+\gamma_2)}+\frac{(3l+1)|\gamma_0|}{(l+1)(16h^2+\gamma_2)}<1$$

and, thus, (31) implies

$$\left(1-\frac{(2l+1)|\gamma_1|}{(l+1)(9h^2+\gamma_2)}-\frac{(3l+1)|\gamma_0|}{(l+1)(16h^2+\gamma_2)}\right)\sum_{k=1}^{\infty}(lk+1)hf_k\leq$$
$$\leq h\left(\frac{(l+1)(|a_2|-|\gamma_1|)}{4h^2+\gamma_2}-\frac{(2l+1)|\gamma_0|}{9h^2+\gamma_2}\right).$$

In view of condition (29) from hence we get

$$\sum_{k=1}^{\infty}(lk+1)hf_k\leq\frac{h\left(\frac{(l+1)(|a_2|-|\gamma_1|)}{4h^2+\gamma_2}-\frac{(2l+1)|\gamma_0|}{9h^2+\gamma_2}\right)}{h\left(1-\frac{(2l+1)|\gamma_1|}{(l+1)(9h^2+\gamma_2)}-\frac{(3l+1)|\gamma_0|}{(l+1)(16h^2+\gamma_2)}\right)}\leq h-\alpha$$

and by Theorem 1 $F\in\mathfrak{D}\mathfrak{F}_h(l,\alpha)$. The first part of Theorem 6 is proved.

Similarly,

$$\sum_{k=1}^{\infty}(h+l\lambda_k)(h+\lambda_k)f_k=\sum_{k=1}^{\infty}(lk+1)(k+1)h^2f_k=$$
$$=2(l+1)h^2f_1+3(2l+1)h^2f_2+\sum_{k=3}^{\infty}(lk+1)(k+1)h^2f_k=2(l+1)h^2f_1+$$
$$+3(2l+1)h^2f_2+\sum_{k=3}^{\infty}(lk+1)(k+1)h^2\left(\frac{|\gamma_1|f_{k-1}}{(k+1)^2h^2+\gamma_2}+\frac{|\gamma_0|f_{k-2}}{(k+1)^2h^2+\gamma_2}\right)=$$
$$=(2(l+1)f_1+3(2l+1)f_2)h^2+h^2\sum_{k=3}^{\infty}\frac{(lk+1)(k+1)|\gamma_1|}{(k+1)^2h^2+\gamma_2}f_{k-1}+$$
$$+h^2\sum_{k=3}^{\infty}\frac{(lk+1)(k+1)|\gamma_0|}{(k+1)^2h^2+\gamma_2}f_{k-2}=(2(l+1)f_1+3(2l+1)f_2)h^2-h^2\frac{3(2l+1))|\gamma_1|}{9h^2+\gamma_2}f_1+$$
$$+h^2\sum_{k=1}^{\infty}\frac{((k+1)l+1)(k+2)|\gamma_1|}{(k+2)^2h^2+\gamma_2}f_k+h^2\sum_{k=1}^{\infty}\frac{((k+2)l+1)(k+3)|\gamma_0|}{(k+3)^2h^2+\gamma_2}f_k=$$
$$=h^2\left(2(l+1)f_1+3(2l+1)\frac{\gamma_0-\gamma_1f_1}{9h^2+\gamma_2}-\frac{3(2l+1))|\gamma_1|}{9h^2+\gamma_2}f_1\right)+$$
$$+\sum_{k=1}^{\infty}\left(\frac{((k+1)l+1)(k+2)|\gamma_1|}{(lk+1)(k+1)((k+2)^2h^2+\gamma_2)}+\frac{((k+2)l+1)(k+3)|\gamma_0|}{(lk+1)(k+1)((k+3)^2h^2+\gamma_2)}\right)\times$$

$$\times(lk+1)(k+1)h^2 f_k \le h^2\left(\frac{2(l+1)(|a_2|-|\gamma_1|)}{4h^2+\gamma_2}-\frac{3(2l+1)|\gamma_0|}{9h^2+\gamma_2}\right)+$$

$$+\sum_{k=1}^{\infty}\left(\frac{3(2l+1)|\gamma_1|}{2(l+1)(9h^2+\gamma_2)}+\frac{4(3l+1)|\gamma_0|}{2(l+1)(16h^2+\gamma_2)}\right)(lk+1)(k+1)h^2 f_k.$$

Since (30) implies

$$\frac{3(2l+1)|\gamma_1|}{2(l+1)(9h^2+\gamma_2)}+\frac{4(3l+1)|\gamma_0|}{2(l+1)(16h^2+\gamma_2)}<1,$$

as above we have

$$\left(1-\frac{3(2l+1)|\gamma_1|}{2(l+1)(9h^2+\gamma_2)}-\frac{4(3l+1)|\gamma_0|}{2(l+1)(16h^2+\gamma_2)}\right)\sum_{k=1}^{\infty}(lk+1)(k+1)h^2 f_k \le$$

$$\le h^2\left(\frac{2(l+1)(|a_2|-|\gamma_1|)}{4h^2+\gamma_2}-\frac{3(2l+1)|\gamma_0|}{9h^2+\gamma_2}\right)$$

and, thus, in view of (30)

$$\sum_{k=1}^{\infty}(lk+1)(k+1)h^2 f_k \le \frac{h^2\left(\frac{2(l+1)(|a_2|-|\gamma_1|)}{4h^2+\gamma_2}-\frac{3(2l+1)|\gamma_0|}{9h^2+\gamma_2}\right)}{\left(1-\frac{3(2l+1)|\gamma_1|}{2(l+1)(9h^2+\gamma_2)}-\frac{4(3l+1)|\gamma_0|}{2(l+1)(16h^2+\gamma_2)}\right)} \le h(h-\alpha)$$

and by Corollary1 $F\in\mathfrak{D}\mathfrak{G}_h(l,\alpha)$. The second part of Theorem 6 is proved.

Finally, since for every $\sigma\in\mathbb{R}$ there exits $k_0=k_0(\sigma)\ge 3$ such that

$$\frac{|\gamma_1|e^{\sigma h}}{(k+1)^2h^2+\gamma_2}+\frac{|\gamma_0|e^{\sigma h}}{(k+3)^2h^2+\gamma_2}\le\frac{1}{2}\quad (k\ge k_0),$$

we have as above

$$\sum_{k=k_0}^{\infty} f_k\exp\{\sigma(k+1)h\}=\sum_{k=k_0}^{\infty}\left(\frac{|\gamma_1|f_{k-1}+|\gamma_0|f_{k-2}}{(k+1)^2h^2+\gamma_2}\right)\exp\{\sigma(k+1)h\}=$$

$$=\sum_{k=k_0-1}^{\infty}\frac{|\gamma_1|f_k e^{\sigma(k+2)h}}{(k+2)^2h^2+\gamma_2}+\sum_{k=k_0-2}^{\infty}\frac{|\gamma_0|f_k e^{\sigma(k+3)h}}{(k+3)^2h^2+\gamma_2}=\frac{|\gamma_1|f_{k_0}e^{\sigma(k_0+2)h}}{(k_0+2)^2h^2+\gamma_2}+\frac{|\gamma_0|f_{k_0}e^{\sigma(k_0+3)h}}{(k_0+3)^2h^2+\gamma_2}+$$

$$+\frac{|\gamma_0|f_{k_0-1}e^{\sigma(k_0+2)h}}{(k_0+2)^2h^2+\gamma_2}+\sum_{k=k_0}^{\infty}\left(\frac{|\gamma_1|e^{\sigma h}}{(k+1)^2h^2+\gamma_2}+\frac{|\gamma_0|e^{\sigma h}}{(k+3)^2h^2+\gamma_2}\right)f_k\exp\{\sigma(k+1)h\}\le$$

$$\le\frac{|\gamma_1|f_{k_0}e^{\sigma(k_0+2)h}}{(k_0+2)^2h^2+\gamma_2}+\frac{|\gamma_0|f_{k_0}e^{\sigma(k_0+3)h}}{(k_0+3)^2h^2+\gamma_2}+\frac{|\gamma_0|f_{k_0-1}e^{\sigma(k_0+2)h}}{(k_0+2)^2h^2+\gamma_2}+\frac{1}{2}\sum_{k=k_0}^{\infty}f_k\exp\{\sigma(k+1)h\},$$

whence it follows that $\sum_{k=1}^{\infty}f_k\exp\{\sigma(k+1)h\}<+\infty$, i. e. Dirichlet series (28) is absolutely convergent in $\mathbb{C}$ and, thus, entire. □

For the proof of the equality $\ln\ln M(\sigma,F)=(1+o(1))h\sigma$ as $0\le\sigma\to+\infty$ we use the following lemma [34, p. 123] (see also [35]).

Lemma 2. *If $\sum_{k=1}^{\infty}\frac{1}{k\lambda_k}<+\infty$ then for an entire Dirichlet series $F(s)=\sum_{k=0}^{\infty}a_k\exp s\lambda_k$*

$$F''(s)=\lambda_\nu^2(F(s)+o(M(\sigma,F))),\quad \nu=\nu(\sigma,F), \tag{32}$$

outside some set $C=\bigcup_{n\ge1}[\sigma_n',\ \sigma_n'')$ of finite measure, where $\nu(\sigma,F)=\max\{k\colon |a_k|\exp\{\sigma\lambda_k\}=\mu(\sigma,F)\}$ is the central index and $\mu(\sigma,F)=\max\{|a_k|\exp\{\sigma\lambda_k\}\colon k\ge0\}$ is the maximal term.

Substituting (32) in (20) we get

$$\lambda_\nu^2(F(s)+o(M(\sigma,F)))+(\gamma_0e^{2hs}+\gamma_1e^{hs}+\gamma_2)F(s)=a_1e^{hs}+a_2e^{2hs},$$

and if $s=\sigma+it_0$ is such that $|F(s)|=(1+o(1))M(\sigma,F)$ then $\lambda_\nu^2F(s)=(1+o(1)|\gamma_0|e^{2hs}F(s)$, i. e. $\lambda_{\nu(\sigma)}=(1+o(1)\sqrt{|\gamma_0|}e^{h\sigma}$ as $0\le\sigma\to+\infty$ outside C. If $\sigma\in C$, that is $\sigma_n'\le\sigma<\sigma_n''$, then

$$\lambda_{\nu(\sigma)}\le\lambda_{\nu(\sigma_n'')}=(1+o(1)\sqrt{|\gamma_0|}e^{h\sigma_n''}=(1+o(1)\sqrt{|\gamma_0|}e^{h\sigma_n'}e^{h(\sigma_n''-\sigma_n')}\le(1+o(1)\sqrt{|\gamma_0|}e^{h\sigma}e^{hK},$$

where K is the measure of C. Similarly,

$$\lambda_{\nu(\sigma)}\ge\lambda_{\nu(\sigma_n')}=(1+o(1)\sqrt{|\gamma_0|}e^{h\sigma_n'}=(1+o(1)\sqrt{|\gamma_0|}e^{h\sigma_n''}e^{h(\sigma_n'-\sigma_n'')}\ge(1+o(1)\sqrt{|\gamma_0|}e^{h\sigma}e^{-hK},$$

Therefore, $\lambda_{\nu(\sigma)}\asymp e^{h\sigma}$ and, since [34, p.17] $\ln\,\mu(\sigma)=\ln\,\mu(\sigma)+\int_{\sigma_0}^{\sigma}\lambda_{\nu(x)}dx$ for $-\infty<\sigma_0\le$ $\le\sigma<+\infty$, we get $\ln\,\mu(\sigma)\asymp e^{h\sigma}$. Finally, since $\ln\,k=o((k+1)h)$ as $k\to\infty$, we have [34, p. 22] $\mu(\sigma)\le M(\sigma)\le\mu(\sigma+\varepsilon)$ for every $\varepsilon>0$ and all $\sigma\ge\sigma_0(\varepsilon)$ and, thus, $\ln\,M(\sigma)\asymp e^{h\sigma}$. Hence it follows that $\ln\,\ln\,M(\sigma,F)=(1+o(1))h\sigma$ as $0\le\sigma\to+\infty$. The proof of Theorem 6 is complete.

Remark. Now suppose that $a_1=a_2=0$, i. e. function (4) satisfies (19). Then, as above, $\gamma_2=-h^2$, $\lambda_1=h$ and $f_1=\frac{\gamma_1}{3h^2}$, $\lambda_2=2h$ and $f_2=\frac{\gamma_0-\gamma_1f_1}{8h^2}$, $\lambda_3=3h$ and $f_3=-\frac{\gamma_1f_2+\gamma_0f_1}{15h^2}$.

Let us show that these equalities contradict to the condition $f_k>0$ for all $k\ge1$. Indeed, $f_1>0$ if and only if $\gamma_1>0$, and $f_2>0$ if and only if $\gamma_0-\gamma_1f_1>0$, whence $\gamma_0>\gamma_1f_1>0$. Then $\gamma_1f_2+\gamma_0f_1>0$ and $f_3<0$.

Thus, *differential equation* (19) *has no solution in the class* $\mathfrak{D}_0$.

REFERENCES

1. Lee S.K., Owa S., Srivastava H.M. *Basic properties and characterization of a certain class of analytic functions with negative coefficients*// Utilitas Math. – 1989. – V.36. – P. 121–128.
2. Aouf M.K., Darwish H.E. *Basic properties and characterization of a certain class of analytic functions with negative coefficients, II*// Utilitas Math. – 1994. – V.46. – P. 167–177.
3. Aouf M.K. *A subclass of analytic p–valent functions with negative coefficients, I*// Utilitas Math. – 1994. – V.46. – P. 219–231.
4. Altintas O. *A subclass of analytic functions with negative coefficients*// Hacettepe Bull. Natur. Sci. Engrg. – 1990.– V.19. – P. 15–24.
5. Altintas O. *On a subclass of certain starlike functions with negative coefficients*// Math. Japon. – 1991.– V.36. – P. 489–495.
6. Aouf M.K., Srivastava H.M. Certain families of analytic functions with negative coefficients// DMS-669-IR. – June 1994. – 49 p.
7. Aouf M.K., Hossen H.M., Srivastava H.M. *A certain subclass of analytic p-valent functions with negative coefficients*// Demonstratio Mathematica. – 1998. – V.51, №3. – P. 595–608.
8. Hadamard J. *Théorème sur le series entières*// Acta math. – 1899. – Bd.22. – S. 55–63.
9. Hadamard J. *La serie de Taylor et son prolongement analitique* // Scientia phys.– math. – 1901. – №12. – P. 43–62.
10. Bieberbach L. Analytische Fortzetzung – Berlin, 1955.
11. Korobeinik Yu.F., Mavrodi N.N. *Singular points of the Hadamard composition*// Ukr. Math. Journ. – 1990. – V.42, №12. – P. 1711–1713. (in Russian); Engl. transl.: Ukr. Math. Journ. – 1990. – V.42, №12. – P. 1545–1547.

12. Zalzman L. *Hadamard product of shlicht functions*// Proc. Amer. Math. Soc. – 1968. – V.19, №3. – P. 544–548.
13. Mogra M.L. *Hadamard product of certain meromorphic univalent functions*// J. Math. Anal. Appl. – 1991. – V.157. – P. 10–16.
14. Choi J.H., Kim Y.C., Owa S. *Generalizations of Hadamard products of functions with negative coefficients*// J. Math. Anal. Appl. – 1996. – V.199. – P. 495–501.
15. Aouf M.K., Silverman H. *Generalizations of Hadamard products of meromorphic univalent functions with positive coefficients*// Demonstratio Mathematica. – 2008. – V.51, №2. – P. 381–388.
16. Liu J., Srivastava P. *Hadamard products of certain classes of p–valent starlike functions*// RACSM. – 2019. – V.113. – P. 2001–205.
17. Ruscheweyh S. *Neighborhoods of univalent functions*// Proc. Amer. Math. Soc. – 1981. – V.81, №4. – P. 521–527.
18. Sheremeta M.M. *Pseudostarlike Dirichlet series of the order α and the type β*// Mat. Stud. – 2020. – V.54, №1. – P. 23–31.
19. Goodman A.W. *Univalent functions and nonanalytic curves*// Proc. Amer. Math. Soc. –1957. – V.8. – P. 598–601.
20. Fournier R. *A note on neighborhoods of univalent functions*// Proc. Amer. Math. Soc. –1983. – V.87, №1. – P. 117–121.
21. Silverman H. *Neighborhoods of a class of analytic functions*// Far East J. Math. Sci. – 1995. – V.3, №2. – P. 165–169.
22. Altintas O., *Neighborhoods of certain analytic functions with negative coefficients*// Internat. J. Math. and Math. Sci. – 1996. – V.13, №4. – P. 210–219.
23. Altintas O., Ozkan O., Srivastava H.M. *Neighborhoods of a class of analytic functions with negative coefficients*// Applied Math. Lettr. – 2000. – V.13. – P. 63–67.
24. Frasin B.A., Daras M. *Integral means and neighborhoods for analytic functions with negative coefficients*// Soochow Journal Math. – 2004. – V.30, №2. – P. 217–223.
25. Murugusundaramoorthy G., Srivastava H.M. *Neighborhoods of certain classes of analytic functions of complex order*// J. Inequal. Pure and Appl. Math. – 2004. – V.5, №2. – Article 24.
26. Pascu M.N., Pascu N.R. *Neighborhoods of univalent functions*// Bull. Amer. Math. Soc. – 2011. – V.83. – P. 510–219.
27. Shah S.M. *Univalence of a function f and its successive derivatives when f satisfies a differential equation, II*// J. Math. Anal. and Appl. – 1989. – V.142. – P. 422–430.
28. Sheremeta Z.M. *Close-to-convexity of entire solutions of a differential equation*// Mat. methods and fiz.-mech. polya. – 1999. – V.42, №3. – P. 31–35. (in Ukrainian)
29. Sheremeta Z.M. *On properties of entire solutions of a differential equation*// Diff. Uravnyeniya. – 2000. – V.36, №8. – P. 1–6. (in Russian)
30. Sheremeta Z.M. *On entire solutions of a differential equation*// Mat. Stud. – 2000. – V.14, №1. – P. 54–58.
31. Sheremeta Z.M., Sheremeta M.M. *Close-to-convexity of entire solutions of a differential equation*// Diff. uravnyeniya. – 2002. – V.38, №4. – P. 435–440. (in Russian)
32. Holovata O.M., Mulyava O.M., Sheremeta M.M. *Pseudostarlike, pseudoconvex and close-to-pseudoconvex Dirichlet series satisfying differential equations with exponential coefficients*// Mat. Method. and Fiz.-Mech. Polya. – 2018 – V.61, №1. – P. 57–70.
33. Sheremeta M.M. Geometric properties of analytic solution of differential equations – Publisher I. E. Chyzhykov. – 2019. – 164 p.
34. Sheremeta M.M. Entire Dirichlet series. – K.:ISDO, 1993. (in Ukrainian)
35. Sheremeta M.M. *On the derivative of an entire Dirichlet series*// Math. USSR Sbornik. – 1990. – V.65, №1. – P.133–139.

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