K. Al-Zoubi, M. Al-Azaizeh

ON GRADED $WAG2$-ABSORBING SUBMODULE

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Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce the concept of graded $WAG2$-absorbing submodule. A number of results concerning of these classes of graded submodules and their homogeneous components are given.

Let $N = \bigoplus_{h\in G} N_h$ be a graded submodule of M and $h \in G$. We say that N_h is a h-$WAG2$-absorbing submodule of the R_e-module M_h if $N_h \neq M_h$; and whenever $r_e, s_e \in R_e$ and $m_h \in M_h$ with $0 \neq r_e s_e m_h \in N_h$, then either $r_e^i m_h \in N_h$ or $s_e^j m_h \in N_h$ or $(r_e s_e)^k \in (N_h :_{R_e} M_h)$ for some $i, j, k \in \mathbb{N}$. We say that N is a graded $WAG2$-absorbing submodule of M if $N \neq M$; and whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N$, then either $r_g^i m_\lambda \in N$ or $s_h^j m_\lambda \in N$ or $(r_g s_h)^k \in (N :_R M)$ for some $i, j, k \in \mathbb{N}$. In particular, the following assertions have been proved:

Let R be a G-graded ring, M a graded cyclic R-module with $Gr((0 :_R M)) = 0$ and N a graded submodule of M. If N is a graded $WAG2$-absorbing submodule of M, then $Gr((N :_R M))$ is a graded $WAG2$-absorbing ideal of R (Theorem 4).

Let R_1 and R_2 be a G-graded rings. Let $R = R_1 \bigoplus R_2$ be a G-graded ring and $M = M_1 \bigoplus M_2$ a graded R-module. Let N_1, N_2 be a proper graded submodule of M_1, M_2 respectively. If $N = N_1 \bigoplus N_2$ is a graded $WAG2$-absorbing submodule of M, then N_1 and N_2 are graded weakly primary submodule of R_1-module M_1, R_2-module M_2, respectively. Moreover, If $N_2 \neq 0$ ($N_1 \neq 0$), then N_1 is a graded weak primary submodule of R_1-module M_1 (N_2 is a graded weak primary submodule of R_2-module M_2) (Theorem 7).

1. Introduction and preliminaries. Throughout this paper all rings are commutative with identity and all modules are unitary.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [10–13] for these basic properties and more information on graded rings and modules.

Let G be a multiplicative group and e denote the identity element of G. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_α of R indexed by the elements $\alpha \in G$ such that $R = \bigoplus_{\alpha\in G} R_\alpha$ and $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all $\alpha, \beta \in G$.

The elements of R_α are called homogeneous of degree α and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \bigcup_{\alpha\in G} R_\alpha$. If $r \in R$, then r can be written uniquely as $\sum_{\alpha\in G} r_\alpha$, where r_α is called a homogeneous component of r in R_α. Moreover, R_e is a subring of R and $1 \in R_e$. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \bigoplus_{\alpha\in G}(I \cap R_\alpha) := \bigoplus_{\alpha\in G} I_\alpha$. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. A left R-module M is said to be a *graded R-module* (or *G-graded R-module*) if there exists a

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family of additive subgroups $\{M_\alpha\}_{\alpha\in G}$ of M such that $M = \bigoplus_{\alpha\in G} M_\alpha$ and $R_\alpha M_\beta \subseteq M_{\alpha\beta}$ for all $\alpha, \beta \in G$. Also if an element of M belongs to $\bigcup_{\alpha\in G} M_\alpha = h(M)$, then it is called homogeneous. Note that M_α is an R_e-module for every $\alpha \in G$. So, if $I = \bigoplus_{\alpha\in G} I_\alpha$ is a graded ideal of R, then I_α is an R_e-module for every $\alpha \in G$. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. A submodule N of M is said to be a *graded submodule of* M if $N = \bigoplus_{\alpha\in G}(N \cap M_\alpha) := \bigoplus_{\alpha\in G} N_\alpha$. In this case, N_α is called the α-component of N. Moreover, M/N becomes a G-graded R-module with α-component $(M/N)_\alpha := (M_\alpha + N)/N$ for $\alpha \in G$. Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then $(N :_R M)$ is defined as $(N :_R M) = \{r \in R : rM \subseteq N\}$. It is shown in [7] that if N is a graded submodule of M, then $(N :_R M)$ is a graded ideal of R. The *graded radical* of a graded ideal I of R, denoted by $Gr(I)$, is the set of all $x = \sum_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$ (see [16]). A proper graded submodule N of a graded R-module M is said to be a *graded weak primary submodule of* M if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_g s_h m_\lambda \in N$, then $r_g^i m_\lambda \in N$ or $s_h^j m_\lambda \in N$ for some $i, j \in \mathbb{N}$.

The concept of graded weakly 2-absorbing ideal was introduced and studied in [2, 14]. Recall from [2] that a proper graded ideal I of R is said to be a *graded weakly 2-absorbing ideal of* R if whenever $r_g, s_h, t_\lambda \in h(R)$ with $0 \neq r_g s_h t_\lambda \in I$, then either $r_g s_h \in I$ or $r_g t_\lambda \in I$ or $s_h t_\lambda \in I$.

The concept of graded weakly 2-absorbing submodule was introduced by Al-Zoubi and Abu-Dawwas in [1] and studied by Al-Zoubi and Al-Azaizeh in [4]. Recall from [1] that a proper graded submodule N of a graded R-module M is said to be *a graded weakly 2-absorbing submodule of* M if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N$, then either $r_g s_h \in (N :_R M)$ or $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$.

The concept of graded weakly primary ideal was introduced and studied by Atani [8]. Recall from [8] that a proper graded ideal P of a graded ring R is said to be *a graded weakly primary ideal* if whenever $r_g, s_h \in h(R)$ with $0 \neq r_g s_h \in P$, then either $r_g \in P$ or $s_h \in Gr(P)$.

The concept of graded weakly primary submodule was introduced and studied [3, 5]. Recall from [3] that a proper graded submodule N of a graded R-module M is said to be *a graded weakly primary submodule* if whenever $r_g \in h(R)$ and $m_h \in h(M)$ with $0 \neq r_g m_h \in N$, then either $m_h \in N$ or $r_g \in Gr((N :_R M))$.

The concept of $WAG2$-absorbing submodule was introduced and studied in [6, 9]. Recall from [9] that a proper submodule N of R-module M is called *$WAG2$-absorbing*, if for $r, s \in R$ and $m \in M$ with $0 \neq abm \in N$, either $rs \in \sqrt{(N : M)}$ or $r^i m \in N$ or $s^j m \in N$ for some positive integer i and j.

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. Here, we introduce the concept of graded $WAG2$-absorbing submodule. A number of results concerning of these classes of graded submodules and their homogeneous components are given.

2. Results.

Definition 1. Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring and I a graded ideal of R. We say that I is a *graded $WAG2$-absorbing ideal of* R, if $I \neq R$ and whenever $r_g, s_h, t_\lambda \in h(R)$ with $0 \neq r_g s_h t_\lambda \in I$, then either $r_g^i t_\lambda \in I$ or $s_h^j t_\lambda \in I$ or $(r_g s_h)^k \in I$ for some $i, j, k \in \mathbb{N}$.

Definition 2. Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $h \in G$.

(i) We say that N_h is a $WAG2$*-absorbing submodule of the* R_e*-module* M_h if $N_h \neq M_h$; and whenever $r_e, s_e \in R_e$ and $m_h \in M_h$ with $0 \neq r_e s_e m_h \in N_h$, then either $r_e^i m_h \in N_h$ or $s_e^j m_h \in N_h$ or $(r_e s_e)^k \in (N_h :_{R_e} M_h)$ for some i, j, $k \in \mathbb{N}$.
(ii) We say that N is a $WAG2$*-absorbing submodule of* M if $N \neq M$; and whenever r_g, $s_h \in R$ and $m_\lambda \in M$ with $0 \neq r_g s_h m_\lambda \in N$, then either $r_g^i m_\lambda \in N$ or $s_h^j m_\lambda \in N$ or $(r_g s_h)^k \in (N :_R M)$ for some i, j, $k \in \mathbb{N}$.

Theorem 1. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If N is a graded $WAG2$-absorbing submodule of M, then N_h is an h-$WAG2$-absorbing submodule of the R_e-module M_h for every $h \in G$ with $N_h \neq M_h$.*

Proof. Suppose that N is a graded $WAG2$-absorbing submodule of M. Let $h \in G$ with $N_h \neq M_h$. Assume that $0 \neq r_e s_e m_h \in N_h \subseteq N$ where $r_e, s_e \in R_e \subseteq h(R)$ and $m_h \in M_h \subseteq h(M)$. Since N is a graded $WAG2$-absorbing submodule of M, we have either $r_e^i m_h \in N$ or $s_e^j m_h \in N$ or $(r_e s_e)^k \in (N :_R M)$ for some i, j, $k \in \mathbb{N}$. Since $M_h \subseteq M$ and $N_h = N \cap M_h$, we conclude that either $r_e^i m_h \in N_h$ or $s_e^j m_h \in N_h$ or $(r_e s_e)^k \in (N_h :_R M_h)$. Thus N_h is h-$WAG2$-absorbing submodule of the R_e*-module* M_h. □

Theorem 2. *Let R be a G-graded ring and M a graded R-module. Let N and K be two graded submodules of M and $K \subseteq N$. If N is a graded $WAG2$-absorbing submodule of M, then N/K is a graded $WAG2$-absorbing submodule of M/K.*

Proof. Assume that N is a graded $WAG2$-absorbing submodule of M, then N/K is a proper graded submodule of M/K. Let r_g, $s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $0 \neq r_g s_h (m_\lambda + K) = r_g s_h m_\lambda + K \in N/K$, hence $r_g s_h m_\lambda \notin K$, it follows that $0 \neq r_g s_h m_\lambda \in N$. Then either $r_g^i m_\lambda \in N$ or $s_h^j m_\lambda \in N$ or $(r_g s_h)^k \in (N :_R M)$ for some i, j, $k \in \mathbb{N}$ as N is a graded $WAG2$-absorbing submodule of M. Hence, either $r_g^i (m_\lambda + K) = r_g^i m_\lambda + K \in N/K$ or $s_h^j (m_\lambda + K) = s_h^j m_\lambda + K \in N/K$ or $(r_g s_h)^k \in (N/K :_R M/K)$ for some i, j, $k \in \mathbb{N}$. Therefore, N/K is a graded $WAG2$-absorbing submodule of M/K. □

Let R be a G-graded ring and M a graded R-module. Recall that M is called gr-faithful if $r_g M = 0$ implies $r_g = 0$ for $r_g \in h(R)$ (see [15].) Also, M is called graded cyclic if $M = m_\lambda R$ for some $m_\lambda \in h(M)$ (see [13]).

Theorem 3. *Let R be a G-graded ring, M a graded cyclic R-module and N a graded submodule of M.*
(i) If $(N :_R M)$ is a graded $WAG2$-absorbing ideal of R, then N is a graded $WAG2$-absorbing submodule of M.
(ii) If M is a gr-faithful and N is a graded $WAG2$-absorbing submodule of M, then $(N :_R M)$ is a graded $WAG2$-absorbing ideal of R.

Proof. Let $k_\alpha \in h(M)$ be such that $M = k_\alpha R$.

(i) Suppose that $(N :_R M)$ is a graded $WAG2$-absorbing ideal of R. Let r_g, $s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $0 \neq r_g s_h m_\lambda \in N$. Then there exists $t_\beta \in h(R)$ such that $m_\lambda = t_\beta k_\alpha$, it follows that $0 \neq r_g s_h t_\beta k_\alpha \in N$. Hence $0 \neq r_g s_h t_\beta \in (N :_R k_\alpha)$. Then either $r_g^i t_\beta \in (N :_R k_\alpha)$ or $s_h^j t_\beta \in (N :_R k_\alpha)$ or $(r_g s_h)^k \in (N :_R k_\alpha)$ for some i, j, $k \in \mathbb{N}$ as $(N :_R M)$ is a graded $WAG2$-absorbing ideal of R. This yields that either $r_g^i m_\lambda \in N$ or $s_h^j m_\lambda \in N$ or $(r_g s_h)^k \in (N :_R k_\alpha) = (N :_R M)$. Therefore, N is a graded $WAG2$*-absorbing submodule of* M.

(ii) Suppose that M is a gr-faithful and N is a graded $WAG2$-absorbing submodule of M. Let $r_g, s_h, t_\lambda \in h(R)$ be such that $0 \neq r_g s_h t_\lambda \in (N :_R M) = (N : k_\alpha)$. Then $0 \neq r_g s_h t_\lambda k_\alpha \in N$ for otherwise $r_g s_h t_\lambda k_\alpha = 0$ implies that $r_g s_h t_\lambda \in (0 :_R k_\alpha) = (0 :_R M) = 0$ a contradiction. Since N is a graded $WAG2$-*absorbing submodule of* M, we get either $r_g^i t_\lambda k_\alpha \in N$ or $s_h^j t_\lambda k_\alpha \in N$ or $(r_g s_h)^k \in (N :_R M) = (N :_R k_\alpha)$ for some $i, j, k \in \mathbb{N}$. This implies that either $r_g^i t_\lambda \in (N :_R k_\alpha) = (N :_R M)$ or $s_h^j t_\lambda \in (N :_R k_\alpha) = (N :_R M)$ or $(r_g s_h)^k \in (N :_R M)$. Therefore $(N :_R M)$ is a graded $WAG2$-absorbing ideal of R. □

Theorem 4. *Let R be a G-graded ring, M a graded cyclic R-module with $Gr((0 :_R M)) = 0$ and N a graded submodule of M. If N is a graded $WAG2$-absorbing submodule of M, then $Gr((N :_R M))$ is a graded $WAG2$-absorbing ideal of R.*

Proof. Let $m_\lambda \in h(M)$ be such that $M = m_\lambda R$. Assume that N is a graded $WAG2$-absorbing submodule of M. Let $r_g, s_h, t_\lambda \in h(R)$ such that $0 \neq r_g s_h t_\lambda \in Gr((N :_R M))$ and $r_g s_h \notin Gr((N :_R M))$. It follows that $(r_g s_h t_\lambda)^i M \subseteq N$ for some $i \in \mathbb{N}$. Since $Gr((0 : M)) = 0$ and $0 \neq r_g s_h t_\lambda$, we get $0 \neq (r_g s_h t_\lambda)^i M \subseteq N$. This yields that $0 \neq (r_g s_h t_\lambda)^i m_\lambda = r_g^i s_h^i (t_\lambda^i m_\lambda) \in N$. Then either $(r_g^i)^n (t_\lambda^i m_\lambda) = (r_g^n t_\lambda)^i m_\lambda \in N$ or $(s_h^i)^k (t_\lambda^i m_\lambda) = (s_h^k t_\lambda)^i m_\lambda \in N$ for some $n, k \in \mathbb{N}$ as N is a graded $WAG2$-*absorbing submodule of* M. This implies that either $(r_g^n t_\lambda)^i M \subseteq N$ or $(s_h^k t_\lambda)^i M \subseteq N$ for some $n, k \in \mathbb{N}$. Thus either $r_g^n t_\lambda \in Gr((N :_R M))$ or $s_h^k t_\lambda \in Gr((N :_R M))$ for some $n, k \in \mathbb{N}$. Therefore, $Gr((N :_R M))$ is a graded $WAG2$-absorbing ideal of R. □

Definition 3. Let R be a G-graded ring, M a graded R-module and $h \in G$ such that N_h is a h-$WAG2$-absorbing submodule of the R_e-module M_h. We say that (r_e, s_e, m_h) is an *h-AG-triple-zero of* N_h where $r_e, s_e \in R_e$ and $m_h \in M_h$, if $r_e s_e m_h = 0$, $r_e^i m_h \notin N_h$, $s_e^j m_h \notin N_h$ and $(r_e s_e)^k \notin (N_h :_{R_e} M_h)$ for all $i, j, k \in N$.

Theorem 5. *Let R be a G-graded ring, M a graded R-module, $N = \bigoplus_{h\in G} N_h$ a graded submodule of M and $h \in G$ such that N_h is a h-$WAG2$-absorbing submodule of the R_e-module M_h. If (r_e, s_e, m_h) is a h-AG-triple-zero of N_h where $r_e, s_e \in R_e$ and $m_h \in M_h$. Then the following hold: (i) $r_e s_e N_h = 0$; (ii) $r_e(N_h :_{R_e} M_h)m_h = s_e(N_h :_{R_e} M_h)m_h = 0$; (iii) $r_e(N_h :_{R_e} M_h)N_h = s_e(N_h :_{R_e} M_h)N_h = 0$; (iv) $(N_h :_{R_e} M_h)^2 m_h = 0$; (v) $(N_h :_{R_e} M_h)^2 N_h = 0$.*

Proof. Assume that (r_e, s_e, m_h) is an h-AG-triple-zero of N_h where $r_e, s_e \in R_e$ and $m_h \in M_h$.

(i) Suppose that $r_e s_e N_h \neq 0$. Then there exist $n_h \in N_h$ such that $r_e s_e n_h \neq 0$. Then $r_e s_e(n_h + m_h) = r_e s_e n_h + r_e s_e m_h \neq 0$. Since N_h is h-$WAG2$-absorbing submodule of the R_e-module M_h, $0 \neq r_e s_e(n_h + m_h) \in N_h$ and $(r_e s_e)^k \notin (N_h :_{R_e} M_h)$ for all $k \in \mathbb{N}$, we conclude that either $r_e^i(n_h + m_h) \in N_h$ or $s_e^j(n_h + m_h) \in N_h$ for some $i, j \in \mathbb{N}$. This yields that either $r_e^i m_h \in N_h$ or $s_e^j m_h \in N_h$. Which is a contradiction. Hence $r_e s_e N_h = 0$.

(ii) Suppose that $r_e(N_h :_{R_e} M_h)m_h \neq 0$. Then there exist $l_e \in (N_h :_{R_e} M_h)$ such that $r_e l_e m_h \neq 0$. Then $0 \neq r_e l_e m_h = r_e(l_e + s_e)m_h = r_e l_e m_h + r_e s_e m_h \in N_h$. Since N_h is an h-$WAG2$-absorbing submodule of the R_e-module M_h and $r_e^i m_h \notin N_h$ for all $i \in \mathbb{N}$, then either $(l_e + s_e)^j m_h \in N_h$ or $(r_e(l_e + s_e))^k \in (N_h :_{R_e} M_h)$ for some j , $k \in \mathbb{N}$. By the Binomial theorem, we get either $s_e^j m_h \in N_h$ or $(r_e s_e)^k \in (N_h :_{R_e} M_h)$ for some $j, k \in \mathbb{N}$, which is a contradiction. Hence $r_e(N_h :_{R_e} M_h)m_h = 0$. With a same argument, we can show $s_e(N_h :_{R_e} M_h)m_h = 0$.

(iii) Suppose that $r_e(N_h :_{R_e} M_h)N_h \neq 0$. Then there exist $n_h \in N_h$ and $l_e \in (N_h :_{R_e} M_h)$ such that $r_e l_e n_h \neq 0$. By (i) and (ii), we get $r_e l_e m_h = r_e s_e n_h = 0$. Hence $0 \neq r_e l_e n_h = r_e(l_e + s_e)(n_h + m_h) \in N_h$. Since N_h is an h-$WAG2$-absorbing submodule of the R_e-module M_h, then either $r_e^i(n_h + m_h) \in N_h$ or $(l_e + s_e)^j(n_h + m_h) \in N_h$ or $(r_e(l_e + s_e))^k \in$

$(N_h :_{R_e} M_h)$ for some i , j, $k \in \mathbb{N}$. By the Binomial theorem, we have either $r_e^i m_h \in N_h$ or $s_e^j m_h \in N_h$ or $(r_e s_e)^k \in (N_h :_{R_e} M_h)$ for some i , j, $k \in \mathbb{N}$. Which is a contradiction. Hence $r_e(N_h :_{R_e} M_h)N_h = 0$. With a same argument, we can show $s_e(N_h :_{R_e} M_h)N_h = 0$.

(iv) Suppose that $(N_h :_{R_e} M_h)^2 m_h \neq 0$. Then there exist t_e, $l_e \in (N_h :_{R_e} M_h)$ such that $t_e l_e m_h \neq 0$. We obtain from (ii) $0 \neq t_e l_e m_h = (r_e + t_e)(s_e + l_e)m_h \in N_h$. Then either $(r_e + t_e)^i m_h \in N_h$ or $(s_e + l_e)^j m_h \in N_h$ or $((r_e + t_e)(s_e + l_e))^k \in (N_h :_{R_e} M_h)$ for some i , j, $k \in \mathbb{N}$ as N_h is a h-$WAG2$-absorbing submodule of the R_e-module M_h. By the Binomial theorem, we have either $r_e^i m_h \in N_h$ or $s_e^j m_h \in N_h$ or $(r_e s_e)^k \in (N_h :_{R_e} M_h)$ for some i , j, $k \in \mathbb{N}$, which is a contradiction. Hence $(N_h :_{R_e} M_h)^2 m_h = 0$.

(v) Suppose that $(N_h :_{R_e} M_h)^2 N_h \neq 0$. Then there exist t_e, $l_e \in (N_h :_{R_e} M_h)$ and $n_h \in N_h$ with $t_e l_e n_h \neq 0$. It follows from (i)-(iv) that $0 \neq t_e l_e n_h = (t_e + r_e)(l_e + s_e)(n_h + m_h) \in N_h$. Since N_h is aa h-$WAG2$-absorbing submodule of the R_e-module M_h, we have either $(t_e + r_e)^i(n_h + m_h) \in N_h$ or $(l_e + s_e)^j(n_h + m_h) \in N_h$ or $((t_e + r_e)(l_e + s_e))^k \in (N_h :_{R_e} M_h)$ for some i , j, $k \in \mathbb{N}$. By the Binomial theorem, we have either $r_e^i m_h \in N_h$ or $s_e^j m_h \in N_h$ or $(r_e s_e)^k \in (N_h :_{R_e} M_h)$ for some i , j, $k \in \mathbb{N}$, which is a contradiction. Hence $(N_h :_{R_e} M_h)^2 N_h = 0$. □

Lemma 1. *Let R be a G-graded ring and M a graded R-module. Let $N = \bigoplus_{h\in G} N_h$ be a graded submodule of M and $h \in G$ such that N_h is a h-$WAG2$-absorbing submodule of the R_e-module M_h. Suppose that $r_e s_e K \subseteq N_h$ for some r_e, $s_e \in R_e$ and some cyclic submodule K of M_h. If $0 \neq 2r_e s_e K$, then $r_e^i K \subseteq N_h$ or $s_e^j K \subseteq N_h$ or $(r_e s_e)^k \in (N_h :_{R_e} M_h)$ for some $i, j, k \in \mathbb{N}$.*

Proof. Suppose that $0 \neq 2r_e s_e K$ and $(r_e s_e)^n \notin (N_h :_{R_e} M_h)$ for all $n \in \mathbb{N}$. We want to show that $r_e^i K \subseteq N_h$ or $s_e^j K \subseteq N_h$ for some i , $j \in \mathbb{N}$. Let $K =< k_h >$. If $0 \neq r_e s_e k_h \in N_h$, then either $r_e^i k_h \in N_h$ or $s_e^j k_h \in N_h$ for some i , $j \in \mathbb{N}$ since N_h is an h-$WAG2$-absorbing submodule of the R_e-module M_h. Suppose that $r_e s_e k_h = 0$. Since $0 \neq 2r_e s_e K$, there exist $k'_h \in K$ such that $0 \neq 2r_e s_e k'_h$ and so $0 \neq r_e s_e k'_h \in N_h$. As N_h is an h-$WAG2$-absorbing submodule, then either $r_e^{t_1} k'_h \in N_h$ or $s_e^{t_2} k'_h \in N_h$ for some t_1 , $t_2 \in \mathbb{N}$. Let $l_h = k_h + k'_h$. Then $0 \neq r_e s_e l_h \in N_h$ and so either $r_e^{n1} l_h \in N_h$ or $s_e^{n2} l_h \in N_h$ for some n_1 , $n_2 \in \mathbb{N}$. Now, we consider three cases:

Case 1. Assume that $r_e^{m1} k'_h \in N_h$ and $s_e^{m2} k'_h \in N_h$ for some $m_1, m_2 \in \mathbb{N}$. Let $i = \max\{n_1, m_1\}$ and $j = \max\{n_2, m_2\}$. Then either $r_e^i k_h = r_e^i(l_h - k'_h) \in N_h$ or $s_e^j k_h \in N_h$.

Case 2. Assume that $r_e^{m1} k'_h \in N_h$ and $s_e^{m2} k'_h \notin N_h$ for some $m_1 \in \mathbb{N}$ and for all $m_2 \in \mathbb{N}$. Suppose on the contrary that, $r_e^i k_h \notin N_h$ for all $i \in \mathbb{N}$. This yields that $r_e^t l_h \notin N_h$ for all $t \in \mathbb{N}$. Hence, $r_e^i(l_h + k'_h) \notin N_h$ and $s_e^j(l_h + k'_h) \notin N_h$ for all i , $j \in \mathbb{N}$. As N_h is an h-$WAG2$-absorbing submodule and $(r_e s_e)^n \notin (N_h :_{R_e} M_h)$ for all $n \in \mathbb{N}$, then, $r_e s_e(l_h + k'_h) = 0$, i.e, $0 = r_e s_e(k_h + k'_h + k'_h) = 2r_e s_e k'_h + r_e s_e k_h = 2r_e s_e k'_h$, which is a contradiction. Hence, $r_e^i k_h \in N_h$ for some $i \in \mathbb{N}$.

Case 3. Assume that $r_e^{m1} k'_h \notin N_h$ and $s_e^{m2} k'_h \in N_h$ for all $m_1 \in \mathbb{N}$ and for some $m_2 \in \mathbb{N}$. Then in similar manner to the proof of case 2, we can show that $s_e^j k_h \notin N_h$ for some $j \in \mathbb{N}$. □

Theorem 6. *Let R be a G-graded ring and M a graded R-module. Let $N = \bigoplus_{h\in G} N_h$ be a graded submodule of M and $h \in G$ such that N_h is a h-$WAG2$-absorbing submodule of the R_e-module M_h. If $r_e IK \subseteq N_h$ and $0 \neq 4r_e IK$ for some $r_e \in R_e$, some cyclic submodule K of M_h and for some ideal I of R_e, then either $r_e \in Gr((N_h :_{R_e} K))$ or $I \subseteq Gr((N_h :_{R_e} K))$ or $r_e I \subseteq Gr((N_h :_{R_e} M_h))$.*

Proof. Assume that $r_e IK \subseteq N_h$, $0 \neq 4r_e IK$ and $r_e I \nsubseteq Gr((N_h :_{R_e} M_h))$ for some $r_e \in R_e$, some cyclic submodule K of M_h and for some ideal I of R_e. We have to show that either

$r_e \in Gr((N_h :_{R_e} K))$ or $I \subseteq Gr((N_h :_{R_e} K))$. By [3, Lemma 3], there exist $t_e \in I$ such that $0 \neq 4r_et_eK$ and $r_et_e \notin Gr((N_h :_{R_e} M_h))$, so $0 \neq 2r_et_eK$. Since $0 \neq 2r_et_eK$ and $r_et_eK \subseteq N_h$, then by Lemma 1, we have either $r_e^iK \subseteq N_h$ or $t_e^jK \subseteq N_h$ for some i , $j \in \mathbb{N}$. Thus, either $r_e \in Gr((N_h :_{R_e} K))$ or $t_e \in Gr((N_h :_{R_e} K))$. If $r_e \in Gr((N_h :_{R_e} K))$, then we are done. Assume that $r_e \notin Gr((N_h :_{R_e} K))$. Then $t_e \in Gr((N_h :_{R_e} K))$. Let $i_e \in I$. Assume that $0 \neq 2r_ei_eK$. Since $r_ei_eK \subseteq N_h$, by Lemma 1, we get either $i_e^nK \subseteq N_h$ or $(r_ei_e)^m \in (N_h :_{R_e} M_h)$ for some n, $m \in \mathbb{N}$. Thus $I \subseteq Gr((N_h :_{R_e} K))$ or $r_eI \subseteq Gr((N_h :_{R_e} M_h))$. Now, let $2r_ei_eK = 0$. This yields that $0 \neq 2r_e(t_e + i_e)K$ and $r_e(t_e + i_e)K \subseteq N_h$. By Lemma 1, we get either $(t_e + i_e)^nK \subseteq N_h$ or $(r_e(t_e + i_e))^m \in (N_h :_{R_e} M_h)$ for some n, $m \in \mathbb{N}$. If $(t_e + i_e)^nK \subseteq N_h$ for some $n \in \mathbb{N}$, then $t_e + i_e \in Gr((N_h :_{R_e} K))$ and so $i_e \in Gr((N_h :_{R_e} K))$ since $t_e \in Gr((N_h :_{R_e} K))$. If $(r_e(t_e + i_e))^m \in (N_h :_{R_e} M_h)$ for some $m \in \mathbb{N}$ and $(t_e + i_e) \notin Gr((N_h :_{R_e} K))$. Then $r_e(t_e + i_e) \in Gr((N_h :_{R_e} M_h))$ and $2r_e(t_e + i_e + t_e)K = 4r_et_eK \neq 0$ and $r_e(t_e + i_e + t_e)K \subseteq N_h$. As $r_et_e \notin Gr((N_h :_{R_e} M_h))$ and $r_e(t_e + i_e) \in Gr((N_h :_{R_e} M_h))$, we get $r_e(t_e + i_e + t_e) \notin Gr((N_h :_{R_e} M_h))$. By Lemma 1, we have either $r_e^nK \subseteq N_h$ or $(t_e + i_e + t_e)^mK \subseteq N_h$ for some n, $m \in \mathbb{N}$. Then either $r_e \in Gr((N_h :_{R_e} K))$ or $t_e + i_e + t_e \in Gr((N_h :_{R_e} K))$. Since $t_e + i_e \notin Gr((N_h :_{R_e} K))$ and $t_e \in Gr((N_h :_{R_e} K))$, then $t_e + i_e + t_e \notin Gr((N_h :_{R_e} K))$. Thus, $r_e \in Gr((N_h :_{R_e} K))$, which is a contradiction. Thus, $(t_e + i_e) \in Gr((N_h :_{R_e} K))$ and so $i_e \in Gr((N_h :_{R_e} K))$. Hence $I \subseteq Gr((N_h :_{R_e} K))$. □

Let R_i be a graded commutative ring with identity and M_i be a graded R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a graded R-module and each graded submodule C of M is of the form $C = C_1 \times C_2$ for some graded submodules C_1 of M_1 and C_2 of M_2 (see [13].)

Theorem 7. *Let R_1 and R_2 be a G-graded rings. Let $R = R_1 \bigoplus R_2$ be a G-graded ring and $M = M_1 \bigoplus M_2$ a graded R-module. Let N_1, N_2 be a proper graded submodule of M_1, M_2 respectively. If $N = N_1 \bigoplus N_2$ is a graded $WAG2$-absorbing submodule of M, then N_1 and N_2 are graded weakly primary submodule of R_1-module M_1, R_2-module M_2, respectively. Moreover, If $N_2 \neq 0$ ($N_1 \neq 0$), then N_1 is a graded weak primary submodule of R_1-module M_1 (N_2 is a graded weak primary submodule of R_2-module M_2).*

Proof. Let $N = N_1 \bigoplus N_2$ be a graded $WAG2$-absorbing submodule of M. Want to show N_1 and N_2 are weakly graded primary submodule of M_1, M_2, respectively. Let $0 \neq rm_1 \in N_1$ where $r \in h(R_1)$ and $m_1 \in h(M_1)$. Let $m_2 \in h(M_2)/N_2$. Then, $(0, 0) \neq (1, 0)(r, 1)(m_1, m_2) \in N_1 \bigoplus N_2 = N$. As N is a graded $WAG2$-absorbing submodule of M, then either $(1, 0)(r, 1) = (r, 0) \in Gr((N :_R M))$ or $(1, 0)^i(m_1, m_2) = (m_1, 0) \in N$ or $(r, 1)^j(m_1, m_2) = (r^jm_1, m_2) \in N$ for some i, $j \in \mathbb{N}$. So, either $(r, 0) \in Gr((N :_R M))$ or $(m_1, 0) \in N$ or $(r^jm_1, m_2) \in N$ for some $j \in \mathbb{N}$. If $(r, 0) \in Gr((N :_R M))$, then $(r, 0)^kM \subseteq N$ for some $k \in \mathbb{N}$. Which implies that $r^kM_1 \subseteq N_1$ and hence $r \in Gr((N_1 :_{R_1} M_1))$. If $(m_1, 0) \in N$, then $m_1 \in N_1$. If $(r^jm_1, m_2) \in N$ for some $j \in \mathbb{N}$, then $m_2 \in N_2$, which is a contradiction since $m_2 \in h(M_2)/N_2$. Hence, either $r \in Gr((N_1 :_{R_1} M_1))$ or $m_1 \in N_1$. Thus N_1 is a graded weakly primary submodule of M_1. Similarly, we can show that N_2 is a graded weakly primary submodule of M_2. Now, let $N_2 \neq 0$. We want to show that N_1 is a graded weak primary submodule of R_1-module M_1. Let r, $s \in h(R_1)$ and $m_1 \in h(M_1)$ with $rsm_1 \in N_1$. Let $0 \neq m_2 \in N_2$. Then, $(0, 0) \neq (r, 1)(s, 1)(m_1, m_2) \in N_1 \bigoplus N_2 = N$. As N is graded WAG2-absorbing submodule of M, then either $(r, 1)(s, 1) = (rs, 1) \in Gr((N :_R M))$ or $(r, 1)^j(m_1, m_2) = (r^jm_1, m_2) \in N$ or $(s, 0)^k(m_1, m_2) = (s^km_1, m_2) \in N$ for some j, $k \in \mathbb{N}$. If $(rs, 1) \in Gr((N :_R M))$, then

$(rs,1)^iM \subseteq N$ for some $i \in \mathbb{N}$. This implies that $((rs)^i,1)(M_1 \bigoplus M_2) \subseteq N_1 \bigoplus N_2$ and so $M_2 \subseteq N_2$, which is a contradiction. If $(r^jm_1, m_2) \in N$ for some $j \in \mathbb{N}$, then $r^jm_1 \in N_1$. If $(s^km_1, m_2) \in N$ for some $k \in \mathbb{N}$, then $s^km_1 \in N_1$. Thus either $r^jm_1 \in N_1$ or $s^km_1 \in N_1$ for some j, $k \in \mathbb{N}$. Hence N_1 is a graded weak primary submodule of R_1-module M_1. Similarly, If $N_1 \neq 0$, then we can show that N_2 is a graded weak primary submodule of R_2-module M_2. □

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Department of Mathematics and Statistics, Jordan University of Science and Technology, Jordan
kfzoubi@just.edu.jo
Department of Mathematics, University of Jordan
Amman, Jordan
maalazaizeh15@sci.just.edu.jo

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