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THE LEAST DIMONOID CONGRUENCES ON RELATIVELY FREE TRIOIDS

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When Loday and Ronco studied ternary planar trees, they introduced types of algebras, called trioids and trialgebras. A trioid is a nonempty set equipped with three binary associative operations satisfying additional eight axioms relating these operations, while a trialgebra is just a linear analog of a trioid. If all operations of a trioid (trialgebra) coincide, we obtain the notion of a semigroup (associative algebra), and if two concrete operations of a trioid (trialgebra) coincide, we obtain the notion of a dimonoid (dialgebra) and so, trioids (trialgebras) are a generalization of semigroups (associative algebras) and dimonoids (dialgebras). Trioids and trialgebras have close relationships with the Hopf algebras, the Leibniz 3-algebras, the Rota-Baxter operators, and the post-Jordan algebras. Originally, these structures arose in algebraic topology. One of the most useful concepts in algebra is the free object. Every variety contains free algebras and free objects in any variety of algebras are important in the study of that variety. Loday and Ronco constructed the free trioid of rank 1 and the free trialgebra. Recently, the free trioid of an arbitrary rank, the free commutative trioid, the free n-nilpotent trioid, the free rectangular triband, the free left n-trinilpotent trioid and the free abelian trioid were constructed and the least dimonoid congruences as well as the least semigroup congruence on the first four free algebras were characterized. However, just mentioned congruences on free left (right) n-trinilpotent trioids and free abelian trioids were not considered. In this paper, we characterize the least dimonoid congruences and the least semigroup congruence on free left (right) n-trinilpotent trioids and free abelian trioids.

1. Introduction and preliminaries. Trioids and their linear analogs, trialgebras [8], are generalizations of semigroups and associative algebras, respectively. They have close relationships with dimonoids and dialgebras [7], the Hopf algebras [9], the Leibniz 3-algebras [3], the Rota-Baxter operators [4], and the post-Jordan algebras [1]. Originally, these structures arose in algebraic topology. We recall that a trioid [8] is a nonempty set T equipped with three binary associative operations $\dashv$, $\vdash$, and $\perp$ satisfying the following axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{T1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{T2}$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{T3}$$

$$(x \dashv y) \dashv z = x \dashv (y \perp z), \tag{T4}$$

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$$(x \perp y) \dashv z = x \perp (y \dashv z), \tag{$T5$}$$

$$(x \dashv y) \perp z = x \perp (y \vdash z), \tag{$T6$}$$

$$(x \vdash y) \perp z = x \vdash (y \perp z), \tag{$T7$}$$

$$(x \perp y) \vdash z = x \vdash (y \vdash z) \tag{$T8$}$$

for all $x, y, z \in T$. For some examples of trioids and further details, see [8, 12].

A construction of free algebras in the variety of trioids was thoroughly studied in the papers [8, 12, 16, 20, 21, 23, 24]. In particular, free trioids of rank 1 and an arbitrary rank can be found in [8] and [12], respectively. The constructions of the free n-nilpotent trioid, the free rectangular triband, the free commutative trioid, the free left (right) n-trinilpotent trioid and the free abelian trioid were given in [20], [21], [12], [16] and [23], respectively. One should observe that congruences play an important role in investigation of different universal algebras. The least certain congruences on relatively free trioids have been investigated lately. In particular, the problem of the characterization of the least dimonoid congruences and the least semigroup congruence on free commutative trioids, free trioids, free rectangular tribands and free n-nilpotent trioids was solved in [19] and [15], respectively. However, the least dimonoid congruences and the least semigroup congruence on free left (right) n-trinilpotent trioids as well as free abelian trioids were not considered. The main purpose of the present paper is to characterize these congruences.

This paper is a continuation of [15, 19]. In Section 2, the least dimonoid congruence and the least semigroup congruence on the free left n-trinilpotent trioid are characterized (Theorem 3). Some descriptions of the least dimonoid congruence and the least semigroup congruence on the free right n-trinilpotent trioid are obtained in a dual way. Section 3 is devoted to the description of the least dimonoid congruence and the least semigroup congruence on the free abelian trioid (Theorem 6).

Now we present various notions and notations used in the paper. A dimonoid [7] is a nonempty set T equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T1)$–$(T3)$. We remark that a dialgebra (see, e.g., [2, 6, 7, 10]) is just a linear analog of a dimonoid. For extensive information on dimonoids, see [7, 17]. If $T = (T, \dashv, \vdash)$ is a dimonoid, then the trioid $(T, \dashv, \vdash, \dashv)$ (respectively, $(T, \dashv, \vdash, \vdash)$) is denoted by $(T)^{\dashv}$ (respectively, $(T)^{\vdash}$). If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that two operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and it is a dimonoid, we say that ρ is a dimonoid congruence [12]. A dimonoid congruence ρ on a trioid $(T, \dashv, \vdash, \perp)$ is called a $d^{\perp}_{\dashv}$-congruence (respectively, $d^{\perp}_{\vdash}$-congruence) [12] if the operations $\dashv$ and $\perp$ (respectively, $\vdash$ and $\perp$) of $(T, \dashv, \vdash, \perp)/\rho$ coincide. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that all operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide, we say that ρ is a semigroup congruence. If $f \colon T_1 \to T_2$ is a homomorphism of trioids, then the kernel of f is denoted by Δ_f.

2. The least dimonoid congruence on free left n-trinilpotent trioids. In this section, we characterize the least dimonoid congruence and the least semigroup congruence on the free left n-trinilpotent trioid.

Following Schein [11], a semigroup S is called a left (right) nilpotent semigroup of rank m if the product of any m elements from this semigroup gives a left (right) zero. The class of all left nilpotent semigroups of rank m is characterized by the identity $g_1 g_2 \ldots g_m g_{m+1} = g_1 g_2 \ldots g_m$. The least such m is called the left nilpotency index of a semigroup S [18]. As usual, we denote the set of all positive integers by $\mathbb{N}$. Following [18], for $k \in \mathbb{N}$, a left nilpotent semigroup of left nilpotency index $\leq k$ is called a left k-nilpotent semigroup. Right

k-nilpotent semigroups are defined dually. The class of all left (right) k-nilpotent semigroups forms a subvariety of the variety of semigroups. A semigroup which is free in the variety of left (right) k-nilpotent semigroups is called a free left (right) k-nilpotent semigroup. Recently, analogs of a left (right) nilpotent semigroup of rank m were introduced in the varieties of dimonoids [17], doppelsemigroups [13], trioids [16], and n-tuple semigroups [18].

Now we present the free left n-nilpotent semigroup [18].

Let X be an arbitrary nonempty set, let $F[X]$ be the free semigroup on X, and $w \in F[X]$. The length of w is denoted by l_w. Fix $n \in \mathbb{N}$. Following [16], if $l_w \geq n$, by $\overset{n}{\overrightarrow{w}}$ we denote the initial subword with the length n of w, and if $l_w < n$, we put $\overset{n}{\overrightarrow{w}} = w$. Let

$$U_n = \{w \in F[X] \,|\, l_w \leq n\}.$$

A binary operation $\cdot$ is defined on U_n by the rule

$$w_1 \cdot w_2 = \overset{n}{\overrightarrow{w_1 w_2}}$$

for all $w_1, w_2 \in U_n$. With respect to this operation, U_n is a semigroup generated by X. It is denoted by $FLNS_n(X)$.

Lemma 1 ([18], Lemma 3.2). *$FLNS_n(X)$ is the free left n-nilpotent semigroup.*

Further we recall how to extend the construction of $FLNS_n(X)$ to the case of dimonoids [17].

By Ω we denote the signature of a dimonoid. Let $x_1, \ldots, x_n$ be individual variables. By $T(x_1, \ldots, x_n)$ we will denote the set of all terms of the signature Ω having the form $x_1 \circ_1 \ldots \circ_{n-1} x_n$ with parenthesizing, where $\circ_1, \ldots, \circ_{n-1} \in \Omega$. A dimonoid $(D, \dashv, \vdash)$ is called left dinilpotent if for some $n \in \mathbb{N}$, any $x \in D$ and any $t(x_1, \ldots, x_n) \in T(x_1, \ldots, x_n)$ the following identities hold:

$$t(x_1, \ldots, x_n) \dashv x = t(x_1, \ldots, x_n), \quad t(x_1, \ldots, x_n) \vdash x = x_1 \vdash \ldots \vdash x_n.$$

The least such n is called the left dinilpotency index of $(D, \dashv, \vdash)$. For $k \in \mathbb{N}$, a left dinilpotent dimonoid of left dinilpotency index $\leq k$ is said to be left k-dinilpotent. Right k-dinilpotent dimonoids are defined dually [17]. The class of all left (right) n-dinilpotent dimonoids forms a subvariety of the variety of dimonoids. A dimonoid which is free in the variety of left (right) n-dinilpotent dimonoids is called a free left (right) n-dinilpotent dimonoid.

Following [16], for any $m, n \in \mathbb{N}$, let $\overset{n}{\overrightarrow{m}} = \begin{cases} m, & m \leq n, \\ n, & m > n. \end{cases}$ Define operations $\dashv$ and $\vdash$ on

$$F_n = \{(w, m) \in F[X] \times \mathbb{N} \,|\, m \leq l_w \leq n\}$$

by

$$(w_1, m_1) \dashv (w_2, m_2) = (\overset{n}{\overrightarrow{w_1 w_2}}, m_1), \tag{2.1}$$

$$(w_1, m_1) \vdash (w_2, m_2) = (\overset{n}{\overrightarrow{w_1 w_2}}, \overset{n}{\overrightarrow{l_{w_1} + m_2}}) \tag{2.2}$$

for all $(w_1, m_1), (w_2, m_2) \in F_n$. The algebra $(F_n, \dashv, \vdash)$ is denoted by $FD_n^l(X)$.

Theorem 1 ([17], Theorem 3.4). *$FD_n^l(X)$ is the free left n-dinilpotent dimonoid.*

Note that the operations $\dashv$ and $\vdash$ were firstly defined in [17] by a slightly cumbersome form, and then they were reformulated in [14] by a more concise form (2.1), (2.2).

Recall the construction of the free left n-trinilpotent trioid [16].

By $\Omega^{'}$ we denote the signature of a trioid. Let $a_1, \ldots, a_n$ be individual variables. By $P\,(a_1, \ldots, a_n)$ we will denote the set of all terms of the signature $\Omega^{'}$ having the form $a_1 \circ_1 \ldots \circ_{n-1} a_n$ with parenthesizing, where $\circ_1, \ldots, \circ_{n-1} \in \Omega^{'}$. A trioid $(T, \dashv, \vdash, \perp)$ is called left trinilpotent if for some $n \in \mathbb{N}$, any $a \in T$ and any $p\,(a_1, \ldots, a_n) \in P(a_1, \ldots, a_n)$ the following identities hold:

$$p\,(a_1, \ldots, a_n) * a = p\,(a_1, \ldots, a_n)\,, \tag{2.3}$$

$$p\,(a_1, \ldots, a_n) \vdash a = a_1 \vdash \ldots \vdash a_n, \tag{2.4}$$

where $* \in \{\dashv, \perp\}$. The least such n is called the left trinilpotency index of $(T, \dashv, \vdash, \perp)$. For $k \in \mathbb{N}$, a left trinilpotent trioid of left trinilpotency index $\leq k$ is said to be left k-trinilpotent. Right k-trinilpotent trioids are defined dually. The class of all n-nilpotent trioids was defined and studied in [20]. The class of all left (right) n-trinilpotent trioids forms a subvariety of the variety of trioids. A trioid which is free in the variety of left (right) n-trinilpotent trioids is called a free left (right) n-trinilpotent trioid.

Let $n, k \in \mathbb{N}$ and $L \subseteq \{1, 2, \ldots, n\}$. We will regard $L + k = \{m + k \mid m \in L\}$. It is clear that $\varnothing + k = \varnothing$. For $L \neq \varnothing$, we put $L^{k,n} = \{m \in L \mid k + m \leq n\}$, and denote the least (greatest) number of L by L_{min} (L_{max}). Obviously, $L^{k,n} = \varnothing$ if $k + m > n$ for all $m \in L$.

Define operations $\dashv, \vdash$, and $\perp$ on

$$V_n = \{(w, L) \mid w \in F[X],\, l_w \leq n,\, L \subseteq \{1, 2, \ldots, l_w\},\, L \neq \varnothing\}$$

by

$$(w, L) \dashv (u, R) = (\overset{n}{\overrightarrow{wu}}, L),$$

$$(w, L) \vdash (u, R) = \begin{cases} (\overset{n}{\overrightarrow{wu}}, \{n\}), & n < l_w + R_{min}, \\ (\overset{n}{\overrightarrow{wu}}, R^{l_w,n} + l_w) & \text{otherwise}, \end{cases}$$

$$(w, L) \perp (u, R) = (\overset{n}{\overrightarrow{wu}}, L \cup (R^{l_w,n} + l_w))$$

for all $(w, L), (u, R) \in V_n$. The algebra $(V_n, \dashv, \vdash, \perp)$ is denoted by $FT^l_n(X)$.

Theorem 2 ([16], Theorem 3.1)**.** *$FT^l_n(X)$ is the free left n-trinilpotent trioid.*

The main result of this section is the following theorem.

Theorem 3. *Let $FT^l_n(X)$ be the free left n-trinilpotent trioid, $(w, L), (u, R) \in FT^l_n(X)$, $FD^l_n(X)$ be the free left n-dinilpotent dimonoid, and $FLNS_n(X)$ be the free left n-nilpotent semigroup.*

(i) Define a relation $\widetilde{\mu^{\perp}_{\dashv}}$ on $FT^l_n(X)$ by $(w, L)\widetilde{\mu^{\perp}_{\dashv}}(u, R)$ if and only if $w = u,\ L_{min} = R_{min}$. Then $\widetilde{\mu^{\perp}_{\dashv}}$ is the least $d^{\perp}_{\dashv}$-congruence on $FT^l_n(X)$.

(ii) Define a relation $\widetilde{\mu}$ on $FT^l_n(X)$ by $(w, L)\widetilde{\mu}(u, R)$ if and only if $w = u$. Then $\widetilde{\mu}$ is the least semigroup congruence on $FT^l_n(X)$.

Proof. (i) Define a map $\mu_{\dashv}^{\perp}\colon FT_n^l(X) \to (FD_n^l(X))^{\dashv}$ by $(w, L)\mu_{\dashv}^{\perp} = (w, L_{min})$.

We aim to show that $\mu_{\dashv}^{\perp}$ is an epimorphism. We have

$$\begin{gathered}((w,L) \dashv (u,R))\mu_{\dashv}^{\perp} = (\overset{n}{\overrightarrow{wu}}, L)\mu_{\dashv}^{\perp} = (\overset{n}{\overrightarrow{wu}}, L_{min}) = \\ = (w, L_{min}) \dashv (u, R_{min}) = (w,L)\mu_{\dashv}^{\perp} \dashv (u,R)\mu_{\dashv}^{\perp}, \\ ((w,L) \perp (u,R))\mu_{\dashv}^{\perp} = (\overset{n}{\overrightarrow{wu}}, L \cup (R^{l_w,n} + l_w))\mu_{\dashv}^{\perp} = (\overset{n}{\overrightarrow{wu}}, L_{min}) = \\ = (w, L_{min}) \dashv (u, R_{min}) = (w,L)\mu_{\dashv}^{\perp} \dashv (u,R)\mu_{\dashv}^{\perp}\end{gathered}$$

since $L \subseteq \{1, 2, \ldots, l_w\}$ implies $(L \cup (R^{l_w,n} + l_w))_{min} = L_{min}$. Besides, $l_w + R_{min} \leq n$ yields $(R^{l_w,n})_{min} = R_{min}$, and we get

$$\begin{aligned}((w,L) \vdash (u,R))\mu_{\dashv}^{\perp} &= \left(\begin{cases}(\overset{n}{\overrightarrow{wu}}, \{n\}), & n < l_w + R_{min}; \\ (\overset{n}{\overrightarrow{wu}}, R^{l_w,n} + l_w) & \text{otherwise,}\end{cases}\right)\mu_{\dashv}^{\perp} = \\ &= \begin{cases}(\overset{n}{\overrightarrow{wu}}, n), & n < l_w + R_{min}; \\ (\overset{n}{\overrightarrow{wu}}, (R^{l_w,n})_{min} + l_w) & \text{otherwise,}\end{cases} = \\ &= \begin{cases}(\overset{n}{\overrightarrow{wu}}, n), & n < l_w + R_{min}; \\ (\overset{n}{\overrightarrow{wu}}, R_{min} + l_w) & \text{otherwise,}\end{cases} = (\overset{n}{\overrightarrow{wu}}, \overset{n}{\overrightarrow{l_w + R_{min}}}) = \\ &= (w, L_{min}) \vdash (u, R_{min}) = (w,L)\mu_{\dashv}^{\perp} \vdash (u,R)\mu_{\dashv}^{\perp}.\end{aligned}$$

Thus, $\mu_{\dashv}^{\perp}$ is a homomorphism. It is a surjection as $(w, \{m\})\mu_{\dashv}^{\perp} = (w, m)$ for any $(w, m) \in FD_n^l(X)$. Since by Theorem 1, $FD_n^l(X)$ is the free left n-dinilpotent dimonoid, $(FD_n^l(X))^{\dashv}$ is the trioid which is free in the variety of left n-trinilpotent trioids with $\dashv = \perp$. It means that $\Delta_{\mu_{\dashv}^{\perp}}$ is the least $d_{\dashv}^{\perp}$-congruence on $FT_n^l(X)$. From the definition of $\mu_{\dashv}^{\perp}$ it follows that $\Delta_{\mu_{\dashv}^{\perp}} = \widetilde{\mu_{\dashv}^{\perp}}$.

(ii) Define a map $\mu\colon FT_n^l(X) \to FLNS_n(X)$ by $(w, L)\mu = w$. It is easy to see that

$$((w,L) * (u,R))\mu = (\overset{n}{\overrightarrow{wu}}, H_*)\mu = \overset{n}{\overrightarrow{wu}} = w \cdot u = (w,L)\mu \cdot (u,R)\mu$$

for some $H_* \subseteq \{1, 2, ..., l_{\overset{n}{\overrightarrow{wu}}}\}$ and all $* \in \{\dashv, \vdash, \perp\}$.

It means that μ is a surjective homomorphism. Since $FLNS_n(X)$ is the free left n-nilpotent semigroup, Δ_μ is the least semigroup congruence on $FT_n^l(X)$. By definition of μ, one has $\Delta_\mu = \widetilde{\mu}$.

□

Remark 1. Let $\alpha\colon FT_n^l(X) \to (FD_n^l(X))^{\vdash}$ be a map defined by $(w, L)\alpha = (w, L_{max})$. This map is dual to the map $\mu_{\dashv}^{\perp}$ but α is not a homomorphism in general. Indeed, for $(x_1x_2x_3, \{3\})$, $(x_4x_5x_6x_7, \{1, 4\}) \in FT_5^l(X)$, where $x_i \in X, 1 \leq i \leq 7$, we have

$$\begin{gathered}((x_1x_2x_3, \{3\}) \perp (x_4x_5x_6x_7, \{1,4\}))\alpha = (x_1x_2x_3x_4x_5, \{3,4\})\alpha = \\ = (x_1x_2x_3x_4x_5, 4) \neq (x_1x_2x_3x_4x_5, 5) = (x_1x_2x_3, 3) \vdash (x_4x_5x_6x_7, 4) = \\ = (x_1x_2x_3, \{3\})\alpha \vdash (x_4x_5x_6x_7, \{1,4\})\alpha.\end{gathered}$$

Remark 2. By Theorem 1, $FD^l_n(X)$ is the free left n-dinilpotent dimonoid but $(FD^l_n(X))^{\vdash}$ is not a left n-trinilpotent trioid since it does not satisfy the identity (2.3) with $* = \perp$. Indeed, in this case, $p(a_1, \ldots, a_n) \neq a_1 \vdash \ldots \vdash a_n$ in general. For example,

$$(x_1, 1) \dashv \ldots \dashv (x_n, 1) = (x_1 \ldots x_n, 1) \neq (x_1 \ldots x_n, n) = (x_1, 1) \vdash \ldots \vdash (x_n, n)$$

for $x_i \in X$, $1 \leq i \leq n$, and $n > 1$. It means that we cannot construct the least $d^{\perp}_{\vdash}$-congruence on $FT^l_n(X)$ with the help of a homomorphism from $FT^l_n(X)$ to $(FD^l_n(X))^{\vdash}$.

Remark 3. In order to characterize the least dimonoid congruence and the least semigroup congruence on the free right n-trinilpotent trioid we use the duality principle.

3. The least dimonoid congruences on free abelian trioids In this section, we characterize the least dimonoid congruences and the least semigroup congruence on the free abelian trioid. We will use notations of Section 2.

We begin with the construction of the free abelian dimonoid [22].

A dimonoid $(D, \dashv, \vdash)$ is called abelian if $x \dashv y = y \vdash x$ for all $x, y \in D$. A dimonoid which is free in the variety of abelian dimonoids is called a free abelian dimonoid.

Let $FCm(X)$ be the free commutative monoid on X. Put

$$FAd(X) = X \times FCm(X)$$

and define operations $\dashv$ and $\vdash$ on $FAd(X)$ as follows:

$$(x, v) \dashv (y, q) = (x, vyq), \ \ (x, v) \vdash (y, q) = (y, xvq).$$

Theorem 4 ([22], Theorem 1). *The algebra $(FAd(X), \dashv, \vdash)$ is the free abelian dimonoid.*

The dimonoid $(FAd(X), \dashv, \vdash)$ will be denoted by $FAd[X]$.

Recall the construction of the free abelian trioid [23].

A trioid $(T, \dashv, \vdash, \perp)$ is called abelian if $x \dashv y = y \vdash x$ for all $x, y \in T$. Note that abelian trioids with a commutative operation $\perp$ were considered in [5]. A trioid which is free in the variety of abelian trioids is called a free abelian trioid.

Suppose that $FAt(X) = F[X] \times FCm(X)$ and define binary operations $\dashv$, $\vdash$, and $\perp$ on $FAt(X)$ by

$$(u, v) \dashv (p, q) = (u, vpq), \quad (u, v) \vdash (p, q) = (p, quv),$$
$$(u, v) \perp (p, q) = (up, vq).$$

Theorem 5 ([23], Theorem 1). *The algebra $(FAt(X), \dashv, \vdash, \perp)$ is the free abelian trioid.*

The trioid $(FAt(X), \dashv, \vdash, \perp)$ will be denoted by $FAt[X]$.

For clarity, we denote the multiplication in $FCm(X)$ by $*$. By θ we denote the identity of $FCm(X)$ and of the free monoid on X, that is the empty word. For every $w \in F[X]$, we denote the first (respectively, last) letter of w by $w^{(0)}$ (respectively, $w^{(1)}$) and the word obtained from w by deleting $w^{(0)}$ (respectively, $w^{(1)}$) by $(w]$ (respectively, $[w)$). We will regard $(w] = [w) = \theta$ if $w \in X$.

It is straightforward to prove the following lemma.

Lemma 2. *In the free monoid on X, for any $p, u \in F[X]$,*

$$p^{(0)}(p] = p, \quad [p)p^{(1)} = p, \quad (up)^{(0)} = u^{(0)}, \quad (up)^{(1)} = p^{(1)}, \quad (up] = (u]p, \quad [up) = u[p).$$

Now we are ready to formulate the main result of this section.

Theorem 6. *Let $FAt[X]$ be the free abelian trioid, $(u,v),(p,q) \in FAt[X]$. Let $FAd[X]$ be the free abelian dimonoid, and let $FC(X)$ be the free commutative semigroup on X.*

(i) Define a relation $\widetilde{\phi_{\dashv}^{\perp}}$ on $FAt[X]$ by $(u,v)\widetilde{\phi_{\dashv}^{\perp}}(p,q)$ if and only if $u^{(0)}=p^{(0)},\ (u]*v=(p]*q$. Then $\widetilde{\phi_{\dashv}^{\perp}}$ is the least $d_{\dashv}^{\perp}$-congruence on $FAt[X]$.

(ii) Define a relation $\widetilde{\phi_{\vdash}^{\perp}}$ on $FAt[X]$ by $(u,v)\widetilde{\phi_{\vdash}^{\perp}}(p,q)$ if and only if $u^{(1)}=p^{(1)},\ [u)*v=[p)*q$. Then $\widetilde{\phi_{\vdash}^{\perp}}$ is the least $d_{\vdash}^{\perp}$-congruence on $FAt[X]$.

(iii) Define a relation $\widetilde{\phi}$ on $FAt[X]$ by $(u,v)\widetilde{\phi}(p,q)$ if and only if $u*v=p*q$. Then $\widetilde{\phi}$ is the least semigroup congruence on $FAt[X]$.

Proof. (i) Define a map $\phi_{\dashv}^{\perp}\colon FAt[X]\to(FAd[X])^{\dashv}$ by $(u,v)\phi_{\dashv}^{\perp}=(u^{(0)},(u]*v)$.

Our aim is to show that $\phi_{\dashv}^{\perp}$ is an epimorphism. Using Lemma 2, we have

$$((u,v)\dashv(p,q))\phi_{\dashv}^{\perp}=(u,vpq)\phi_{\dashv}^{\perp}=(u^{(0)},(u]*vpq)=$$
$$=(u^{(0)},(u]*v(p^{(0)}(p])q)=(u^{(0)},(u]*v*p^{(0)}*(p]*q)=$$
$$=(u^{(0)},(u]*v)\dashv(p^{(0)},(p]*q)=(u,v)\phi_{\dashv}^{\perp}\dashv(p,q)\phi_{\dashv}^{\perp},$$
$$((u,v)\vdash(p,q))\phi_{\dashv}^{\perp}=(p,quv)\phi_{\dashv}^{\perp}=(p^{(0)},(p]*quv)=$$
$$=(p^{(0)},(p]*q(u^{(0)}(u])v)=(p^{(0)},(p]*q*u^{(0)}*(u]*v)=$$
$$=(u^{(0)},(u]*v)\vdash(p^{(0)},(p]*q)=(u,v)\phi_{\dashv}^{\perp}\vdash(p,q)\phi_{\dashv}^{\perp},$$
$$((u,v)\perp(p,q))\phi_{\dashv}^{\perp}=(up,vq)\phi_{\dashv}^{\perp}=((up)^{(0)},(up]*vq)=$$
$$=(u^{(0)},(u]p*vq)=(u^{(0)},(u]*v(p^{(0)}(p])q)=$$
$$=(u^{(0)},(u]*v*p^{(0)}*(p]*q)=(u^{(0)},(u]*v)\dashv(p^{(0)},(p]*q)=(u,v)\phi_{\dashv}^{\perp}\dashv(p,q)\phi_{\dashv}^{\perp}.$$

Thus, $\phi_{\dashv}^{\perp}$ is a homomorphism. Since $(x,v)\phi_{\dashv}^{\perp}=(x,v)$ for any $(x,v)\in FAd[X]$, the map $\phi_{\dashv}^{\perp}$ is surjective. By Theorem 4, $FAd[X]$ is the free abelian dimonoid, hence $(FAd[X])^{\dashv}$ is the free trioid in the variety of abelian trioids with $\dashv=\perp$. It means that $\Delta_{\phi_{\dashv}^{\perp}}$ is the least $d_{\dashv}^{\perp}$-congruence on $FAt[X]$. From the definition of $\phi_{\dashv}^{\perp}$ it follows that $\Delta_{\phi_{\dashv}^{\perp}}=\widetilde{\phi_{\dashv}^{\perp}}$.

(ii) Define a map $\phi_{\vdash}^{\perp}\colon FAt[X]\to(FAd[X])^{\vdash}$ by $(u,v)\phi_{\vdash}^{\perp}=(u^{(1)},[u)*v)$.

Prove that $\phi_{\vdash}^{\perp}$ is an epimorphism. Applying Lemma 2, we get

$$((u,v)\dashv(p,q))\phi_{\vdash}^{\perp}=(u,vpq)\phi_{\vdash}^{\perp}=(u^{(1)},[u)*vpq)=$$
$$=(u^{(1)},[u)*v([p)p^{(1)})q)=(u^{(1)},[u)*v*p^{(1)}*[p)*q)=$$
$$=(u^{(1)},[u)*v)\dashv(p^{(1)},[p)*q)=(u,v)\phi_{\vdash}^{\perp}\dashv(p,q)\phi_{\vdash}^{\perp},$$
$$((u,v)\vdash(p,q))\phi_{\vdash}^{\perp}=(p,quv)\phi_{\vdash}^{\perp}=(p^{(1)},[p)*quv)=$$
$$=(p^{(1)},[p)*q([u)u^{(1)})v)=(p^{(1)},[p)*q*u^{(1)}*[u)*v)=$$
$$=(u^{(1)},[u)*v)\vdash(p^{(1)},[p)*q)=(u,v)\phi_{\vdash}^{\perp}\vdash(p,q)\phi_{\vdash}^{\perp},$$
$$((u,v)\perp(p,q))\phi_{\vdash}^{\perp}=(up,vq)\phi_{\vdash}^{\perp}=((up)^{(1)},[up)*vq)=(p^{(1)},u[p)*vq)=$$
$$=(p^{(1)},[u)u^{(1)}[p)*vq)=(p^{(1)},[p)*q*u^{(1)}*[u)*v)=$$
$$=(u^{(1)},[u)*v)\vdash(p^{(1)},[p)*q)=(u,v)\phi_{\vdash}^{\perp}\vdash(p,q)\phi_{\vdash}^{\perp}.$$

Consequently, $\phi_{\vdash}^{\perp}$ is a homomorphism. It is surjective, the proof of this fact is the same as in the proof of statement (i). Since by Theorem 4 $FAd[X]$ is the free abelian dimonoid,

$(FAd[X])^{\vdash}$ is the trioid which is free in the variety of abelian trioids with $\vdash = \perp$. Hence, $\Delta_{\phi_{\vdash}^{\perp}}$ is the least $d_{\vdash}^{\perp}$-congruence on $FAt[X]$. From the construction of $\phi_{\vdash}^{\perp}$ it follows that $\Delta_{\phi_{\vdash}^{\perp}} = \widetilde{\phi_{\vdash}^{\perp}}$.

(iii) Define a map $\phi\colon FAt[X] \to FC(X)$ by $(u, v)\phi = u * v$. We can see that

$$((u, v) \dashv (p, q))\phi = (u, vpq)\phi = u * vpq = (u * v)(p * q) = (u, v)\phi\ (p, q)\phi,$$
$$((u, v) \vdash (p, q))\phi = (p, quv)\phi = p * quv = (u * v)(p * q) = (u, v)\phi\ (p, q)\phi,$$
$$((u, v) \perp (p, q))\phi = (up, vq)\phi = up * vq = (u * v)(p * q) = (u, v)\phi\ (p, q)\phi.$$

Since $(v^{(0)}, (v])\phi = v$ for any $v \in FC(X)$, the map ϕ is surjective. It means that ϕ is a surjective homomorphism. Since $FC(X)$ is the free commutative semigroup, Δ_ϕ is the least semigroup congruence on $FAt[X]$. By the construction of ϕ, we have $\Delta_\phi = \widetilde{\phi}$.

□

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