УДК 519.6

I. K. Argyros, S. M. Shakhno, H. P. Yarmola

EXTENDED SEMILOCAL CONVERGENCE FOR THE NEWTON-KURCHATOV METHOD

I. K. Argyros, S. M. Shakhno, H. P. Yarmola. *Extended semilocal convergence for the Newton-Kurchatov method*, Mat. Stud. **53** (2020), 85–91.

We provide a semilocal analysis of the Newton-Kurchatov method for solving nonlinear equations involving a splitting of an operator. Iterative methods have a limited restricted region in general. A convergence of this method is presented under classical Lipschitz conditions. The novelty of our paper lies in the fact that we obtain weaker sufficient semilocal convergence criteria and tighter error estimates than in earlier works. We find a more precise location than before where the iterates lie resulting to at least as small Lipschitz constants. Moreover, no additional computations are needed than before. Finally, we give results of numerical experiments.

1. Introduction. Many problems in computational mathematics are reduced to solving nonlinear equations in particular systems of nonlinear equations. To solve such problems one often uses the Newton method, which has a quadratic convergence order but requires analytically given derivatives [1, 2, 7, 13, 14]. We can also apply difference methods [6, 9, 10], which in some cases are not inferior to the Newton method, and use only the values of functions.

Recently, much attention has been paid to solving nonlinear equations with a decomposition of operator [3, 4, 5, 8, 11, 12]:

$$H(x) \equiv F(x) + G(x) = 0. \tag{1}$$

Here F, $G\colon D \subseteq E_1 \to E_2$, D is a open convex subset of E_1, E_1 and E_2 are Banach spaces, F is a Fréchet differentiable operator. The operator G is not necessarily differentiable just continuous. Taking into account the properties of operators, it is possible to apply combined methods [3, 11, 12] that show better results than difference methods [5, 8] or other methods.

In this paper, we consider the Newton-Kurchatov method

$$x_{n+1} = x_n - A_n^{-1}F(x_n), \quad A_n = F'(x_n) + G(2x_n - x_{n-1}, x_{n-1}), \quad n = 0, 1, \dots. \tag{2}$$

$G(x, y)$ denotes a first-order divided difference of the operator G at the points x and y. This method was proposed in [12], and studied under various conditions [3, 12].

The main goal of this paper is improving the results obtained in [12]. We use our new technique, and get weaker sufficient convergence criteria and tighter error estimates. Our idea can be used to extend the applicability of other methods in a similar way.

2020 *Mathematics Subject Classification:* 47H17, 65H05, 65J15.
Keywords: nonlinear equation; Newton-Kurchatov method; semilocal convergence; decomposition of operator.
doi:10.30970/ms.53.1.85-91

2. Convergence analysis. Let us denote $U(x_0,\tau)=\{x\in D\colon\ \|x-x_0\|\le\tau\}$ to be a closed ball, $x_0\in D$, $\tau>0$.

Theorem 1. *Let $F\colon D\subseteq E_1\to E_2$ be a Fréchet-differentiable operator, and $G\colon D\subseteq E_1\to E_2$ be a continuous operator. Assume that A_0 is an invertible operator, and the Lipschitz conditions are fulfilled for each $x,y\in D$ with $2x-y\in D$*

$$\|A_0^{-1}(F'(x)-F'(x_0))\|\le 2p_0^0\|x-x_0\|, \tag{3}$$

$$\|A_0^{-1}(G(2x-y,y)-G(2x_0-x_{-1},x_{-1}))\|\le q_0^0(\|2x-y-(2x_0-x_{-1})\|+\|y-x_{-1}\|) \tag{4}$$

and for each $x,y,u,v\in D_0=D\cap U(x_0,r)$, $r=\frac{1-2q_0^0a}{2(p_0^0+2q_0^0)}$,

$$\|A_0^{-1}(F'(x)-F'(y))\|\le 2p_0\|x-y\|, \tag{5}$$

$$\|A_0^{-1}(G(x,y)-G(u,v))\|\le q_0(\|x-u\|+\|y-v\|). \tag{6}$$

Let $a,\ c,\ r_0$ be non-negative numbers such that

$$\|x_0-x_{-1}\|\le a,\quad \|A_0^{-1}F(x_0)\|\le c,\quad c>a, \tag{7}$$

$$r_0\ge\frac{c}{1-\gamma},\quad 2q_0^0a+2p_0^0r_0+4q_0^0r_0<1, \tag{8}$$

$$\gamma=\frac{p_0r_0+2q_0(r_0+a)}{1-2q_0^0a-2p_0^0r_0-4q_0^0r_0},\quad 0\le\gamma<1$$

and $U(x_0,r_0)\subset D$.

Then, for each $n\in\{-1,0,1,2,\ldots\}$ the following assertions hold

$$\|x_n-x_{n+1}\|\le t_n-t_{n+1}, \tag{9}$$

$$\|x_n-x_*\|\le t_n-t_*, \tag{10}$$

where

$$t_{-1}=r_0+a,\quad t_0=r_0,\quad t_1=r_0-c,\quad t_{n+1}-t_{n+2}=\gamma_n(t_n-t_{n+1}),\quad n\ge 0, \tag{11}$$

$$\gamma_n=\frac{p_0(t_n-t_{n+1})+q_0(2t_{n-1}-t_{n+1}-t_n)}{1-2q_0^0a-2p_0^0(t_0-t_{n+1})-2q_0^0(2t_0-t_n-t_{n+1})},$$

$\{t_n\}_{n\ge0}$ is a non-negative, decreasing sequence that converges to some t_* such that $r_0-\frac{c}{1-\gamma}\le t_*<t_0$; sequence $\{x_n\}_{n\ge0}$ is well-defined, remains in $U(x_0,r_0)$ and converges to a solution x_* of equation (1).

Proof. We use mathematical induction to prove the statement of the theorem. First, let us show that, for each $k\ge0$ the following inequalities are satisfied

$$t_{k+1}\ge t_{k+2}\ge r_0-\frac{c}{1-\gamma}\ge 0, \tag{12}$$

$$t_{k+1}-t_{k+2}\le\gamma(t_k-t_{k+1}). \tag{13}$$

Letting $k=0$ in (11), we get

$$t_1-t_2=\frac{p_0(t_0-t_1)+q_0(2t_{-1}-t_1-t_0)}{1-2q_0^0a-2p_0^0(t_0-t_1)-2q_0^0(2t_0-t_0-t_1)}(t_0-t_1)\le\gamma(t_0-t_1),$$

$$t_0\ge t_1,\quad t_1\ge t_2\ge t_1-\gamma(t_0-t_1)\ge r_0-(1+\gamma)c=r_0-\frac{(1-\gamma^2)c}{1-\gamma}\ge r_0-\frac{c}{1-\gamma}\ge 0.$$

Assume that (12) and (13) are true for $k = 0, 1, ..., n-1$. Then, for $k = n$, we obtain

$$t_{n+1} - t_{n+2} = \frac{p_0(t_n - t_{n+1}) + q_0(2t_{n-1} - t_{n+1} - t_n)}{1 - 2q_0^0 a - 2p_0^0(t_0 - t_{n+1}) - 2q_0^0(2t_0 - t_n - t_{n+1})}(t_n - t_{n+1}) \leq$$
$$\leq \frac{p_0 t_n + 2q_0 t_{n-1}}{1 - 2q_0^0 a - 2p_0^0 t_0 - 4q_0^0 t_0}(t_n - t_{n+1}) \leq \gamma(t_n - t_{n+1}),$$
$$t_{n+1} \geq t_{n+2} \geq t_{n+1} - \gamma(t_n - t_{n+1}) \geq r_0 - \frac{1-\gamma^{n+2}}{1-\gamma}c \geq r_0 - \frac{c}{1-\gamma} \geq 0.$$

Thus, the first part of the theorem is proved.

Let us prove that the method (2) is well-defined, and for each $n \geq 0$ $x_n \in U(x_0, r_0)$ and the inequality (9) is satisfied.

Since $t_{-1} - t_0 = a$, $t_0 - t_1 = c$ and conditions (7) are fulfilled then $x_1 \in U(x_0, r_0)$ and (9) is satisfied for $n \in \{-1, 0\}$.

We have by conditions (3) and (4)

$$\|I - A_0^{-1}A_{n+1}\| = \|A_0^{-1}[A_0 - A_{n+1}]\| \leq \|A_0^{-1}[F'(x_0) - F'(x_{n+1})]\| +$$
$$+ \|A_0^{-1}[G(2x_0 - x_{-1}, x_{-1}) - G(2x_{n+1} - x_n, x_n)]\| \leq$$
$$\leq 2p_0^0\|x_0 - x_{n+1}\| + q_0^0(\|2x_{n+1} - x_n - 2x_0 + x_{-1}\| + \|x_n - x_{-1}\|) \leq$$
$$\leq 2p_0^0\|x_0 - x_{n+1}\| + q_0^0(2\|x_0 - x_{n+1}\| + 2\|x_n - x_{-1}\|) \leq$$
$$\leq 2q_0^0 a + 2p_0^0(t_0 - t_{n+1}) + 2q_0^0(2t_0 - t_{n+1} - t_n) \leq 2q_0^0 a + (2p_0^0 + 4q_0^0)r_0 < 1.$$

According to the Banach lemma on inverse operators [1] A_{n+1} is invertible, and

$$\|A_{n+1}^{-1}A_0\| \leq (1 - 2q_0^0 a - 2p_0^0\|x_0 - x_{n+1}\| - 2q_0^0(\|x_0 - x_{n+1}\| + \|x_0 - x_n\|))^{-1}.$$

By the definition of the divided difference and conditions (5), (6), we obtain

$$\|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| =$$
$$= \|A_0^{-1}\left[F(x_{n+1}) + G(x_{n+1}) - F(x_n) - G(x_n) - A_n(x_n - x_{n+1})\right]\| \leq$$
$$\leq \left\|A_0^{-1}\Big[\int_0^1 \{F'(x_{n+1} + \Theta(x_n - x_{n+1})) - F'(x_n)\}d\Theta\Big]\right\|\|x_n - x_{n+1}\| +$$
$$+ \left\|A_0^{-1}\left[G(x_{n+1}, x_n) - G(2x_n - x_{n-1}, x_{n-1})\right]\right\| \|x_n - x_{n+1}\| \leq$$
$$\leq (p_0\|x_n - x_{n+1}\| + q_0(\|x_n - x_{n+1}\| + 2\|x_{n-1} - x_n\|))\|x_n - x_{n+1}\|.$$

By condition (9), we have

$$\|x_{n+1} - x_{n+2}\| = \|A_{n+1}^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \leq$$
$$\leq \|A_{n+1}^{-1}A_0\|\|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \leq$$
$$\leq \frac{p_0\|x_n - x_{n+1}\| + q_0(\|x_n - x_{n+1}\| + 2\|x_{n-1} - x_n\|)}{1 - 2q_0^0 a - 2p_0^0\|x_0 - x_{n+1}\| - 2q_0^0(\|x_0 - x_{n+1}\| + \|x_0 - x_n\|)}\|x_n - x_{n+1}\| \leq$$
$$\leq \frac{\big[p_0(t_n - t_{n+1}) + q_0(2t_{n-1} - t_{n+1} - t_n)\big](t_n - t_{n+1})}{1 - 2q_0^0 a - 2p_0^0(t_0 - t_{n+1}) - 2q_0^0(2t_0 - t_n - t_{n+1})} = t_{n+1} - t_{n+2}.$$

That is method (2) is well-defined for each $n \geq 0$. Hence, it follows that

$$\|x_n - x_k\| \leq t_n - t_k, \quad -1 \leq n \leq k. \tag{14}$$

It follows $\{x_n\}_{n\geq 0}$ is a fundamental sequence, so it converges to some $x_* \in U(x_0, r_0)$. Inequality (10) is obtained from (14) for $k \to \infty$. Let us show that x_* solves the equation $F(x) + G(x) = 0$. Indeed, we get in turn that

$$\|A_0^{-1}F(x_{n+1})\| \leq (p_0\|x_n - x_{n+1}\| + q_0(\|x_n - x_{n+1}\| + 2\|x_{n-1} - x_n\|))\|x_n - x_{n+1}\| \to 0,$$

for $n \to \infty$. Hence, we conclude $F(x_*) = 0$. □

Corollary 1. *The convergence order of method (2) is no less than* $\frac{1+\sqrt{5}}{2}$.

Proof. In view of $t_n - t_{n+1} < t_{n-1} - t_n$, and (11), we obtain

$$\begin{aligned} t_{n+1} - t_{n+2} &= \frac{p_0(t_n - t_{n+1}) + q_0(t_n - t_{n+1} + 2(t_{n-1} - t_n))}{1 - 2q_0^0 a - 2p_0^0(t_0 - t_{n+1}) - 2q_0^0(2t_0 - t_n - t_{n+1})}(t_n - t_{n+1}) < \\ &< \frac{p_0(t_{n-1} - t_n) + 3q_0(t_{n-1} - t_n)}{1 - 2q_0^0 a - 2p_0^0(t_0 - t_{n+1}) - 2q_0^0(2t_0 - t_n - t_{n+1})}(t_n - t_{n+1}) = \\ &= \frac{p_0 + 3q_0}{1 - 2q_0^0 a - 2p_0^0(t_0 - t_{n+1}) - 2q_0^0(2t_0 - t_n - t_{n+1})}(t_{n-1} - t_n)(t_n - t_{n+1}) \leq \\ &\leq \frac{p_0 + 3q_0}{1 - 2q_0^0 a - 2p_0^0 t_0 - 4q_0^0 t_0}(t_{n-1} - t_n)(t_n - t_{n+1}). \end{aligned}$$

Hence, it follows that the order of convergence of the sequence $\{t_n\}_{n\geq 0}$ is no less than $\frac{1+\sqrt{5}}{2}$, and, according (10), the sequence $\{x_n\}_{n\geq 0}$ converges with the same order. □

Remark 1. The conditions used in [12], and corresponding to (5), (6), respectively are: for all $x, y, u, v \in D$

$$\|A_0^{-1}(F'(x) - F'(y))\| \leq 2p_0^1\|x - y\|, \tag{15}$$
$$\|A_0^{-1}(G(x, y) - G(u, v))\| \leq q_0^1(\|x - u\| + \|y - v\|). \tag{16}$$

But $D_0 \subseteq D$, so we get

$$p_0^0 \leq p_0^1, \quad q_0^0 \leq q_0^1, \quad p_0 \leq p_0^1, \quad q_0 \leq q_0^1,$$
$$r_0^1 \geq \frac{c}{1 - \gamma^1}, \quad 2q_0^1 a + 2p_0^1 r_0^1 + 4q_0^1 r_0^1 < 1, \quad r_0 \geq \frac{c}{1 - \gamma}, \quad 2q_0^0 a + 2p_0^0 r_0 + 4q_0^0 r_0 < 1,$$

but not necessarily vice versa unless, if

$$q_0^1 = q_0^0,\ p_0^1 = p_0^0,\ r_0^1 = r_0,\ \gamma^1 = \gamma.$$

The corresponding majorizing sequence in [12] is given by

$$s_{-1} = r_0^1 + a, \quad s_0 = r_0^1,\ s_1 = r_0^1 - c, \quad s_{n+1} - s_{n+2} = \gamma_n^1(s_n - s_{n+1}),\ n \geq 0, \tag{17}$$
$$\gamma_n^1 = \frac{p_0^1(s_n - s_{n+1}) + q_0^1(2s_{n-1} - s_{n+1} - s_n)}{1 - 2q_0^1 a - 2p_0^1(s_0 - s_{n+1}) - 2q_0^1(2s_0 - s_n - s_{n+1})}.$$

Then, a simple inductive argument shows that

$$\gamma \leq \gamma^1 \Rightarrow r_0 \leq r_0^1, \quad t_n \leq s_n, \quad t_* \leq s_* = \lim_{n\to\infty} s_n, \tag{18}$$
$$\gamma \geq \gamma^1 \Rightarrow r_0 \geq r_0^1, \quad s_n \leq t_n, \quad s_* \leq t_* = \lim_{n\to\infty} t_n. \tag{19}$$

Hence, we have obtained: weaker sufficient semilocal convergence criteria and tighter estimates on $\|x_n - x_{n+1}\|$. These improvements do not involve additional hypotheses, since in practice the computation of the old constants p_0^1, q_0^1 requires the computation of the new constants p_0^0, p_0, q_0^0, q_0 as special cases. Hence, we extended the applicability of method (2). This technique can be used to extend the applicability of other methods too along the same lines [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

3. Numerical results. In this section, we test the old and the new convergence criteria.

Let us define function $F + G\colon \mathbb{R} \to \mathbb{R}$, where

$$F(x) = e^{x-0.5} + x^3 - 1.3, \; G(x) = 0.2x|x^2 - 2|.$$

The exact solution of $F(x) + G(x) = 0$ is $x_* = 0.5$. Let $D = (0, 1)$. Then, we have

$$F'(x) = e^{x-0.5} + 3x^2,$$
$$G(x,y) = \frac{0.2x(2-x^2) - 0.2y(2-y^2)}{x-y} = 0.2(2 - x^2 - xy - y^2).$$
$$A_0 = e^{x_0-0.5} + 3x_0^2 + 0.2(2 - (2x_0 - x_{-1})^2 - (2x_0 - x_{-1})x_{-1} - x_{-1}^2),$$
$$|A_0^{-1}(F'(x) - F'(y))| \le \frac{e^{0.5} + 3|x+y|}{|A_0|}|x-y|,$$
$$|A_0^{-1}(G(x,y) - G(u,v))| = \frac{0.2}{|A_0|}|(u+x+y)(u-x) + (v+y+u)(v-y)|.$$

Let $x_0 = 0.55$, $x_{-1} = 0.551$. Then, we get $a = 0.001$, $c \approx 0.047937$, $p_0^0 \approx 1.446471$, $q_0^0 \approx 0.234146$, $r \approx 0.261006$,

$$D_0 = D \cap U(x_0, r) = (0.28899,\ 0.81101),$$

$p_0 \approx 1.496084$, $q_0 \approx 0.223493$, $p_0^1 \approx 1.756493$, $q_0^1 \approx 0.275574$. By solving inequalities $r_0 \ge \frac{c}{1-\gamma}$ and $r_0^1 \ge \frac{c}{1-\gamma^1}$, we get

$$r_0 \in [0.055607,\ 0.149268], \quad r_0^1 \in [0.059051,\ 0.117195].$$

Table 1: Results for $\varepsilon = 10^{-15}$

n	$\lvert x_{n-1} - x_n\rvert$	$t_{n-1} - t_n$	$s_{n-1} - s_n$
1	0.047937	0.047937	0.047937
2	0.002060	0.004739	0.005836
3	3.1428e-06	0.000175	0.000296
4	7.0617e-12	5.3161e-07	1.5074e-06
5	0	5.2828e-11	3.3420e-10

Let $r_0 = 0.149268$ and $r_0^1 = 0.117195$. Then $\gamma = 0.678856$ and $\gamma^1 = 0.590966$. In Table 1 there are absolute values of the corrections at each iteration for the sequences $\{x_n\}_{n\ge0}$, $\{t_n\}_{n\ge0}$ and $\{s_n\}_{n\ge0}$. The obtained results show that the estimate (9) and the similar one from [12] are fulfilled. Moreover, the new estimates on $|x_{n-1} - x_n|$ are more accurate because $t_{n-1} - t_n \le s_{n-1} - s_n$. Table 2 shows values of the majorizing sequences $\{t_n\}_{n\ge0}$, $\{s_n\}_{n\ge0}$

Table 2: Results for $\varepsilon = 10^{-15}$

n	t_n	s_n	γ_{n-2}	γ^1_{n-2}
-1	0.150268	0.118195		
0	0.149268	0.117195		
1	0.101332	0.069258		
2	0.096593	0.063422	0.098852	0.121748
3	0.096418	0.063126	0.036968	0.050736
4	0.096417	0.063124	0.003035	0.005091
5	0.096417	0.063124	9.9375e-05	0.000222
6	0.096417	0.063124	2.9821e-07	1.1088e-06

and values of the constants γ_n and γ^1_n. Although $\gamma \geq \gamma^1$ but $\gamma_n \leq \gamma^1_n$. Therefore, (19) is confirmed, and we have obtained weaker sufficient semilocal convergence criteria and tighter estimates on $|x_n - x_{n+1}|$.

4. Conclusions. We investigated the semilocal convergence of Newton-Kurchatov method under classical Lipschitz conditions. We use our technique, which weakens the sufficient convergence criteria and gives tighter error estimates. Extending the applicability of iterative methods is very important in computational disciplines, since this leads to handling problems not possible before, more initial points and fewer iterations to reach the solution. This approach can be used to study other methods for solving nonlinear equations.

REFERENCES

1. I.K. Argyros, Á. A. Magreñán, A contemporary study of iterative methods, Elsevier (Academic Press), New York, NY, USA, 2018.
2. I.K. Argyros, Á. A. Magreñán, Iterative Methods and Their Dynamics with Applications: A Contemporary Study, CRC Press, 2017.
3. I.K. Argyros, S.M. Shakhno, *Extended Local Convergence for the Combined Newton-Kurchatov Method Under the Generalized Lipschitz Conditions*, Mathematics, **7(2), 207** (2019).
4. X. Chen, *On the convergence of Broyden-like methods for nonlinear equations with nondifferentiable terms*, Ann. Inst. Statist. Math., **42** (1990), 387–401.
5. M.A. Hernandez, M.J. Rubio, *The secant method for nondifferentiable operators*, Appl. Math. Lett., **15** (2002), 395–399.
6. V.A. Kurchatov, *On a method of linear interpolation for the solution of functional equations*, Dokl. Akad. Nauk SSSR. Ser. Mathematics. Physics., **198** (1971), 524–526 (in Russian). Translation in: Soviet Math. Dokl., **12** (1974), 835–838.
7. J.M. Ortega, W.C. Rheinboldt, Iterative solution of nonlinear equations in several variables, Academic Press, New York, 1970.
8. H. Ren, I.K. Argyros, *A new semilocal convergence theorem for a fast iterative method with nondifferentiable operators*, J. Appl. Math. Comp., **34** (2010), 39–46.
9. S.M. Shakhno, *On a Kurchatov's method of linear interpolation for solving nonlinear equations*, PAMM: Proceedings in Applied Mathematics and Mechanics, **4** (2004), 650–651.
10. S.M. Shakhno, *On the difference method with quadratic convergence for solving nonlinear operator equations*, Mat. Stud., **26** (2006), 105–110. (in Ukrainian)

11. S.M. Shakhno, I.V. Mel'nyk, H.P. Yarmola, *Analysis of the convergence of a combined method for the solution of nonlinear equations*, Journal of Mathematical Sciences, **201** (2014), 32–43.
12. S.M. Shakhno, H.P. Yarmola, *Two-point method for solving nonlinear equations with nondifferentiable operator*, Mat. Stud., **36** (2011), 213–220. (in Ukrainian)
13. J.F. Traub, Iterative methods for the solution of equations, Prentice Hall, Englewood Cliffs, 1964.
14. X. Wang, *Convergence of Newton's method and uniqueness of the solution of equations in Banach space*, IMA Journal of Numerical Analysis, **20** (2000), 123–134.

Department of Mathematics, Cameron University, Lawton, USA
iargyros@cameron.edu

Department of Theory of Optimal Processes,
Ivan Franko National University of Lviv, Lviv, Ukraine
stepan.shakhno@lnu.edu.ua

Department of Computational Mathematics,
Ivan Franko National University of Lviv, Lviv, Ukraine
halyna.yarmola@lnu.edu.ua

Received 19.11.2019
Revised 27.01.2020