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ON EQUICONTINUITY OF FAMILIES OF MAPPINGS BETWEEN RIEMANNIAN SURFACES WITH RESPECT TO PRIME ENDS

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Given a domain of some Riemannian surface, we consider questions related to the possibility of a continuous extension to the boundary of one class of Sobolev mappings. It is proved that such maps have a continuous boundary extension in terms of prime ends, and under some additional restrictions their families are equicontinuous at inner and boundary points of the domain. We have separately considered the cases of homeomorphisms and mappings with branching.

1. Introduction. The problems of extension of mappings of the Sobolev class acting between Riemannian surfaces have been considered in several papers by V. Ryazanov and S. Volkov, see [1] and [2]. In this manuscript, we develop research in this direction, studying here the behavior of mappings in the closure of a given domain. In the first part, we have proved the possibility of continuous extension $\overline{f}\colon \overline{D}_P \to \overline{D_*}$, where D and D_* are domains on Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, and $\overline{D}_P$ is the closure of the domain D with respect to prime ends. On this occasion, we note the publication [2], where slightly similar results were obtained for classes of Sobolev homeomorphisms, and also the paper [3], where similar results were obtained for the case of Riemannian manifolds. Compared to [2], we are dealing with the extension $\overline{f}\colon \overline{D}_P \to \overline{D_*}$ rather than $\overline{f}\colon \overline{D}_P \to \overline{D_*}_P$. In the second part of the paper we show that the families of the mappings mentioned above are equicontinuous in $\overline{D}_P$. In this regard, we point out to classical results of R. Näkki and B. Palka ([4]), as well as results of the first author obtained for metric spaces, see [5].

Quite exhaustively, all definitions concerning Riemannian surfaces, their representations in terms of quotient spaces, as well as the elements of length and area in them, are given in [1], and therefore are omitted. Everywhere below, unless otherwise stated, the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$ are of the hyperbolic type. Further $ds_{\widetilde{h}}$ and $d\widetilde{v}$, $ds_{\widetilde{h_*}}$ and $d\widetilde{v}_*$ denote the length and area elements on Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively. We also use the notation $\widetilde{h}$ for the metric on the surface $\mathbb{S}$, in particular,

$$\widetilde{B}(p_0, r) := \{p \in \mathbb{S}\colon \widetilde{h}(p, p_0) < r\}, \quad \widetilde{S}(p_0, r) := \{p \in \mathbb{S}\colon \widetilde{h}(p, p_0) = r\}$$

are the disk and the circle on $\mathbb{S}$ centered at p_0 and of the radius $r > 0$, respectively. The following definitions refer to Carathéodory [6], cf. [7], [8], [9] and [10]. Recall that a continuous

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mapping $\sigma\colon \mathbb{I}\to\mathbb{S}$, $\mathbb{I}=(0,1)$, is called a *Jordan arc* in $\mathbb{S}$, if $\sigma(t_1)\ne\sigma(t_2)$ for $t_1\ne t_2$. Further we will sometimes use σ for $\sigma(\mathbb{I})$, $\overline{\sigma}$ for $\overline{\sigma(\mathbb{I})}$ and $\partial\sigma$ for $\overline{\sigma(\mathbb{I})}\setminus\sigma(\mathbb{I})$. A *cut* in a domain D is either a Jordan arc $\sigma\colon\mathbb{I}\to D$, ends which lie on ∂D, or a closed Jordan curve in D. The sequence $\sigma_1,\sigma_2,\ldots,\sigma_m,\ldots$ of cuts in D is called a *chain* if:

(i) $\overline{\sigma_i}\cap\overline{\sigma_j}=\varnothing$ for any $i\ne j$, $i,j=1,2,\ldots$;

(ii) σ_m separates D, i.e., $D\setminus\sigma_m$ consists precisely from two components one of which contains σ_{m-1}, and another contains σ_{m+1},

(iii) $\widetilde{h}(\sigma_m)\to\infty$ as $m\to\infty$, $\widetilde{h}(\sigma_m)=\sup\limits_{p_1,p_2\in\sigma_m}\widetilde{h}(p_1,p_2)$.

By the definition, a chain of cuts $\{\sigma_m\}$ defines a chain of domains $d_m\subset D$ such that $\partial\, d_m\cap D\subset\sigma_m$ and $d_1\supset d_2\supset\ldots\supset d_m\supset\ldots$. Two chains of cuts $\{\sigma_m\}$ and $\{\sigma_k'\}$ are called *equivalent,* if for each $m=1,2,\ldots$ the domain d_m contains all the domains d_k' except for a finite number, and for each $k=1,2,\ldots$ the domain d_k' also contains all the domains d_m except for a finite number. *End* of D is the class of equivalent chains of cuts in D.

Let K be a prime end in $D\subset\mathbb{S}$, and $\{\sigma_m\}$ and $\{\sigma_m'\}$ are two chains in K, d_m and d_m' are domains corresponding to σ_m and σ_m'. Then

$$\bigcap_{m=1}^{\infty}\overline{d_m}\subset\bigcap_{m=1}^{\infty}\overline{d_m'}\subset\bigcap_{m=1}^{\infty}\overline{d_m}\ ,$$

and thus

$$\bigcap_{m=1}^{\infty}\overline{d_m}=\bigcap_{m=1}^{\infty}\overline{d_m'}\ ,$$

in other words, the set

$$I(K)=\bigcap_{m=1}^{\infty}\overline{d_m}$$

depends only on K and does not depend on the choice of the chain of cuts $\{\sigma_m\}$. The set $I(K)$ is called the *impression of a prime end* K. Further E_D denotes the set of all prime ends in D, and $\overline{D}_P:=D\cup E_D$ denotes the completion of D by its prime ends. Let us turn $\overline{D}_P$ into the topological space as follows. Firstly, open sets from D are considered open in $\overline{D}_P$, as well. Secondly, the base neighborhood of a prime end $P\subset E_D$ is defined as the union of an arbitrary domain d, included in some chain of cuts of P, with all other prime ends in d. In particular, in the topology mentioned above, a sequence of points $x_n\in D$ converges to an element $P\in E_D$ if and only if, for any domain d_m, belonging to the chain of domains $d_1,d_2,d_3,\ldots$, in P there exists $n_0=n_0(m)$ such that $x_n\in d_m$ for $n\geqslant n_0$.

The *dilatation* of the mapping f at the point z is defined (in local coordinates) by the relation

$$K_f(z)=\frac{|f_z|+|f_{\overline{z}}|}{|f_z|-|f_{\overline{z}}|}\tag{1}$$

for $J_f(z)\ne 0$, $K_f(z)=1$ for $\|f'(z)\|=0$ and $K_f(z)=\infty$ otherwise. It is easy to see that K_f does not depend on local coordinates, since the transition maps from one chart to another are conformal by virtue of the definition of Riemannian surface. Given domains $D,D_*\subset\mathbb{C}$, a mapping $f\colon D\to D_*$ is called a *mapping with a finite distortion,* if $f\in W^{1,1}_{\rm loc}(D)$ and, in addition, there is almost everywhere a finite function $K(z)$ such that $\|f'(z)\|^2\leqslant K(z)\cdot J_f(z)$

for almost all $z \in D$, where $J_f(z)$ denotes the jacobian of f at z. If D, D_* are domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively, then $f\colon D \to D_*$ is called a mapping with finite distortion, if it is so in local coordinates.

As usual, a path γ on the Riemannian surface $\mathbb{S}$ is a continuous mapping $\gamma\colon I \to \mathbb{S}$, where I is a finite segment, an interval or a half-interval of a real axis. Let Γ be a family of paths in $\mathbb{S}$. A Borel function $\rho\colon \mathbb{S} \to [0,\infty]$ is called *admissible* for the family Γ of paths γ, if $\int\limits_{\gamma} \rho(p)\, ds_{\widetilde{v}}(p) \geqslant 1$ for any path $\gamma \in \Gamma$. The latter is briefly written in the form: $\rho \in \operatorname{adm}\Gamma$. A *modulus* of the family Γ is a real-valued function

$$M(\Gamma) := \inf_{\rho\in \operatorname{adm}\Gamma} \int\limits_{\mathbb{S}} \rho^2(p)\, d\widetilde{v}(p)\,.$$

Let E, $F \subset \mathbb{S}$ be arbitrary sets. In the future, everywhere by $\Gamma(E,F,D)$ we denote the family of all paths $\gamma\colon [a,b] \to D$, which join E and F in D, that is, $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a,\, b)$. We say that a boundary ∂G of a domain G is *strongly accessible at a point* $p_0 \in \partial G$ if, for each neighborhood U of p_0 there are compact set $E \subset G$, a neighborhood $V \subset U$ of the same point and a number $\delta > 0$ such that, the relation $M(\Gamma(E,F,G)) \geqslant \delta$ holds for any continuum F intersecting both ∂U and ∂V. We also say that a boundary of ∂G is *strongly accessible* if it is strongly accessible at any of its points.

Assume that $p_0 \in \mathbb{S}$ and that a function $\varphi\colon D \to \mathbb{R}$ is integrable with respect to the measure $\widetilde{v}$ in some neighborhood U of p_0. Following [11, Sect. 2] (see [12, Sec. 6.1, Ch. 6]), we say that a function $\varphi\colon D \to \mathbb{R}$ has a *finite mean oscillation* at the point $p_0 \in D$, write $\varphi \in FMO(p_0)$, if

$$\limsup_{\varepsilon\to 0} \frac{1}{\widetilde{v}(\widetilde{B}(p_0,\varepsilon))} \int\limits_{\widetilde{B}(p_0,\,\varepsilon)} |\varphi(p) - \overline{\varphi}_{\varepsilon}|\, d\widetilde{v}(p) < \infty\,,$$

where $\overline{\varphi}_{\varepsilon} = \frac{1}{\widetilde{v}(\widetilde{B}(p_0,\varepsilon))} \int\limits_{\widetilde{B}(p_0,\varepsilon)} \varphi(p)\, d\widetilde{v}(p)$. The following statement holds.

Theorem 1. *Let D, D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively, having compact closures $\overline{D} \subset \mathbb{S}$ and $\overline{D_*} \subset \mathbb{S}_*$, ∂D has a finite number of components, and ∂D_* is strongly accessible. Let $Q\colon \mathbb{S} \to (0,\infty)$ be a given function measurable with respect to the measure $\widetilde{v}$ on $\mathbb{S}$, $Q(p) \equiv 0$ in $\mathbb{S}\setminus D$. Let $f\colon D \to D_*$ be a homeomorphism of a domain D onto D_* of the class $W^{1,1}_{\rm loc}$ with a finite distortion such that $K_f(p) \leqslant Q(p)$ for almost all $p \in D$. Then f extends to a continuous mapping $f\colon \overline{D}_P \to \overline{D_*}$, $f(\overline{D}_P) = \overline{D_*}$, if one of the following conditions is true: 1) for any $p_0 \in \partial D$ there is $\varepsilon_0 = \varepsilon_0(p_0) > 0$ such that*

$$\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} < \infty\,, \qquad \int\limits_{0}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} = \infty \tag{2}$$

for any $0 < \varepsilon < \varepsilon_0$, where $\|Q\|(t) := \int\limits_{\widetilde{S}(p_0,t)} Q(p)\, ds_{\widetilde{h}}(p)$ denotes the L_1-norm of the function Q over the circle $\widetilde{S}(p_0,t)$,

2) $Q \in FMO(\partial D)$.

The following statement holds.

Proposition 1. *Assume that $\overline{D}$ is a compact set in $\mathbb{S}$, in addition, the domain $D\subset \mathbb{S}$ has at most a finite number of boundary components $\Gamma_1, \Gamma_2,\ldots,\Gamma_n\subset \partial D$. Then:*

1) the space $\overline{D}_P$ is metrizable by some metric $\rho\colon \overline{D}_P\times\overline{D}_P\rightarrow {\mathbb R}$ such that the convergence of an arbitrary sequence $x_n\in D$, $n=1,2,\ldots$, to some prime end $P\in E_D$ is equivalent to the convergence of x_n in one of the spaces $\overline{U_i^*}_P$, see [2, Remark 2];

2) each prime end $P\in E_D$ contains an equivalent chain of cuts σ_m, $m=1,2,\ldots$, lying on the circles $\widetilde{S}(z_0, r_m)$, $r_m\rightarrow 0$ as $m\rightarrow\infty$, see [2, Remark 1];

3) the body $I(P)$ of any prime end $P\subset E_D$ is a continuum in ∂D, in addition, there is one and only one $1\leqslant i\leqslant n$ such that $I(P)\subset \Gamma_i$, see [2, Proposition 1, Remark 1].

Let (X, d) and $(X^{\,\prime}, d^{\,\prime})$ be metric spaces with distances d and $d^{\,\prime}$, respectively. A family $\mathfrak{F}$ of mappings $f\colon X\rightarrow X^{\,\prime}$ is called *equicontinuous at a point* $x_0\in X$, if for any $\varepsilon>0$ there exists $\delta>0$ such that $d^{\,\prime}(f(x), f(x_0))<\varepsilon$ for all $x\in X$ such that $d(x, x_0)<\delta$ and for all $f\in \mathfrak{F}$. A family $\mathfrak{F}$ is *equicontinuous* if $\mathfrak{F}$ is equicontinuous at every point $x_0\in X$. Everywhere below, unless otherwise stated, $(X, d)=(\overline{D}_P, \rho)$ and $(X^{\,\prime}, d^{\,\prime})=(\mathbb{S}_*, \widetilde{h}_*)$, where ρ is one of the metric from Proposition 1.

The next definition can be found, e.g., in [4]. A domain $D\subset \mathbb{S}$ is called a *uniform* if for every $r>0$ there exists $\delta>0$ such that $M(\Gamma(F, F^*, D))\geqslant \delta$ for any continua F and F^* in D, satisfying the conditions $\widetilde{h}(F)\geqslant r$ and $\widetilde{h}(F^*)\geqslant r$. Domains D_i, $i\in I$, are called *equi-uniform* if, for each $r>0$ the above inequality holds for any D_i with the same number δ.

Consider now the following class of mappings. For a given $\delta>0$, $D\subset \mathbb{S}$, a continuum $A\subset D$ and a function $Q\colon D\rightarrow [0, \infty]$ measurable with respect to the measure $\widetilde{h}$, denote $\mathfrak{S}_{Q, \delta, A}(D)$ a family of all homeomorphisms of the Sobolev class with finite distortion $f\colon D\rightarrow \mathbb{S}_*$ for which there exists a continuum $G_f\subset \mathbb{S}_*$ such that $f\colon D\rightarrow \mathbb{S}_*\setminus G_f$ and $\widetilde{h}_*(G_f)=\sup\limits_{x, y\in G_f}\widetilde{h}_*(x, y)\geqslant \delta$, $\widetilde{h}_*(f(A))\geqslant \delta$. The following statement holds.

Theorem 2. *Let D be a domain in $\mathbb{S}$, such that $\overline{D}$ is compact in $\mathbb{S}$, and $Q\colon \mathbb{S}\rightarrow (0, \infty)$ is a function locally integrable in D, $Q(x)\equiv 0$ on $\mathbb{S}\setminus D$. Assume that, for any point $p_0\in \overline{D}$ either the condition (2), or the condition $Q\in FMO(p_0)$ holds. Assume also that ∂D consist of a finite number of components, and $\overline{D_f}=\overline{f(D)}$ is a compact set in $\mathbb{S}_*$ for any $f\in\mathfrak{S}_{Q, \delta, A}(D)$, moreover, the domains D_f and $\mathbb{S}_*$ are equi-uniform over the class $f\in\mathfrak{S}_{Q, \delta, A}(D)$.*

Then each $f\in\mathfrak{S}_{Q, \delta, A}(D)$ has a continuous extension $\overline{f}\colon \overline{D}_P\rightarrow \overline{D_f}$ and, moreover, the family $\mathfrak{S}_{Q, \delta, A}(\overline{D})$ consisting of all extended mappings $\overline{f}\colon \overline{D}_P\rightarrow \overline{D_f}$ is equicontinuous in $\overline{D}_P$.

2. A continuous boundary extension of mappings. Let $Q\colon \mathbb{S}\rightarrow [0, \infty]$ be a function measurable with respect to the measure $\widetilde{v}$, $Q(p)\equiv 0$ for $p\notin D$, $D\subset \mathbb{S}$. We say that $f\colon D\rightarrow \mathbb{S}_*$ is a *ring Q-mapping at* $p_0\in \overline{D}$, if the relation

$$M(f(\Gamma(\widetilde{S}(p_0, r_1),\, \widetilde{S}(p_0, r_2),\, D)))\leqslant \int\limits_{\widetilde{A}} Q(p)\cdot \eta^2(\widetilde{h}(p, p_0))\, d\widetilde{v}(p)\,, \tag{3}$$

holds for some $r_0=r(p_0)>0$, any ring $\widetilde{A}=\widetilde{A}(p_0, r_1, r_2)=\{p\in \mathbb{S}\colon r_1<\widetilde{h}(p, p_0)<r_2\}$ and any $0<r_1<r_2<r_0$, where $\eta\colon (r_1, r_2)\rightarrow [0, \infty]$ is arbitrary nonnegative Lebesgue

measurable function such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \tag{4}$$

We say that f is a ring Q-mapping at $E \subset \overline{D}$, if f is a ring Q-mapping at every point $p_0 \in E$.

Given a mapping $f\colon D \to \mathbb{S}_*$ and a set $E \subset \overline{D} \subset \mathbb{S}$ we set

$$C(f, E) = \{y \in \mathbb{S}_*\colon \exists\, x \in E, x_k \in D\colon x_k \to x, f(x_k) \to y, k \to \infty\}\,.$$

A mapping $f\colon D \to D_*$ is called *discrete* if the pre-image $f^{-1}(y)$ of each point $y \in D_*$ consists only of isolated points. A mapping $f\colon D \to D_*$ is called *open* if the image of any open set $U \subset D$ is an open set in D_*. A mapping f of D onto D_* is called *closed* if $f(E)$ is closed in D_* for any closed set $E \subset D$.

Let $D \subset \mathbb{S}$, $f\colon D \to \mathbb{S}_*$ be an open discrete mapping, $\beta\colon [a,\, b) \to \mathbb{S}_*$ is a path and $x \in f^{-1}(\beta(a))$. A path $\alpha\colon [a,\, c) \to D$ is called a *maximal f-lifting* of β starting at the point x, if (1) $\alpha(a) = x$; (2) $f \circ \alpha = \beta|_{[a,\, c)}$; (3) for any $c < c' \leqslant b$, there is no a path $\alpha'\colon [a,\, c') \to D$ such that $\alpha = \alpha'|_{[a,\, c)}$ and $f \circ \alpha' = \beta|_{[a,\, c')}$. Observe that, maximal f-liftings under open discrete mappings always exist in $\mathbb{S} = \mathbb{S}_* = \mathbb{R}^2$ due to Rickman's theorem, see [13, corollary II.3.3] or [14, Lemma 3.12]. Since Riemannian surfaces are orientable topological manifolds, by [15, Theorem 3.4], cf. [16, Example 1.4(a)] and [17, Lemma 2.1], we obtain the following assertion.

Proposition 2. *Let D, D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively, let $x \in f^{-1}(\beta(a))$, and let $f\colon D \to D_*$ be an open discrete mapping. Then any path $\beta\colon [a,\, b) \to \mathbb{S}_*$ has a maximal f-lifting $\alpha\colon [a,\, c) \to D$ of β starting at the point x.*

The following statement carries the main semantic load related to the main result of this section.

Lemma 1. *Let D, D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively, having compact closures $\overline{D} \subset \mathbb{S}$ and $\overline{D_*} \subset \mathbb{S}_*$. Assume that ∂D has a finite number of components, and ∂D_* is strongly accessible. Let $Q\colon \mathbb{S} \to (0, \infty)$ be a function measurable with respect to the measure $\widetilde{v}$ on $\mathbb{S}$, $Q(p) \equiv 0$ in $\mathbb{S} \setminus D$, furthermore, assume that, for any point $p_0 \in \partial D$ there are $\varepsilon_0 = \varepsilon_0(p_0) > 0$ and a Lebesgue measurable function $\psi\colon (0, \infty) \to (0, \infty)$ such that*

$$I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty \qquad \forall\, \varepsilon \in (0, \varepsilon_0)\,, \tag{5}$$

$I(\varepsilon, \varepsilon_0) > 0$ for sufficiently small $\varepsilon > 0$, and

$$\int\limits_{\varepsilon < \widetilde{h}(p, p_0) < \varepsilon_0} Q(p) \cdot \psi^2(\widetilde{h}(p, p_0))\; d\widetilde{v}(p) \;=\; o\left(I^2(\varepsilon, \varepsilon_0)\right)\,, \quad \varepsilon \to 0\,. \tag{6}$$

Let $f\colon D \to D_*$ be an open discrete and closed ring Q-mapping of D onto D_* at all points of ∂D. Then f is extendable to a continuous mapping $\overline{f}\colon \overline{D}_P \to \overline{D_*}$, $\overline{f}(\overline{D}_P) = \overline{D_*}$.

Proof. Let us first prove that f has a continuous extension $\overline{f}\colon \overline{D}_P \to \overline{D_*}$. We fix $P \in E_D$. Since, by assumption, the space $\overline{D_*}$ is compact, it suffices to establish that the set

$$L = C(f, P) := \left\{ y \in \mathbb{S}_*\colon y = \lim_{m\to\infty} f(p_m), \quad p_m \to P \right\} \quad \text{as} \quad m \to \infty$$

consists of a single point $y_0 \in \overline{D_*}$.

Let us assume the opposite. Then there exist at least two points y_0 and $z_0 \in L$. That is, there are at least two sequences $p_k, p_k^{\prime} \in D$, such that $p_k \to P$ and $p_k^{\prime} \to P$ as $k \to \infty$, and, moreover, $f(p_k) \to y_0$ and $f(p_k^{\prime}) \to z_0$ as $k \to \infty$. By item 2) of Proposition 1 the prime end P contains a chain of cuts σ_k lying on circles S_k centered at some point $p_0 \in \partial D$ and radii $r_k \to 0$, $k \to \infty$. Let d_k be the domains associated with cuts σ_k, $k = 1, 2, \ldots$. Without loss of generality, passing to a subsequence if necessary, we may assume that $p_k, p_k^{\prime} \in d_k$ (see Figure 1 for an illustration). Observe that y_0 and $z_0 \in \partial D_*$, because the mapping f is

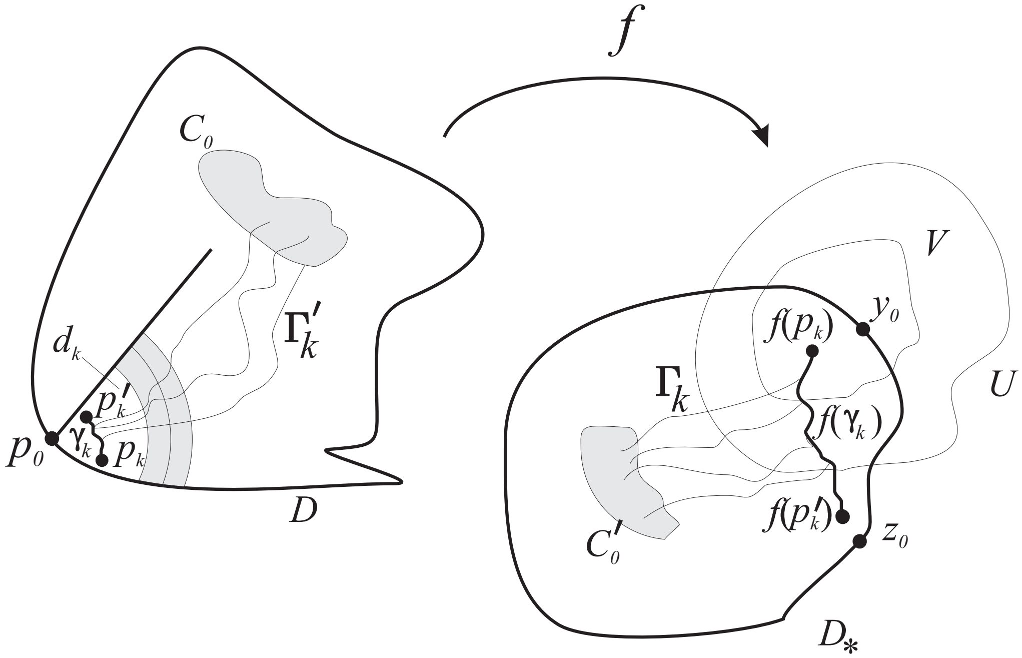

Figure 1: To proof of Lemma 1

closed and hence $C(f, \partial D) \subset \partial D_*$ (see [5, Proposition 2.1]). By the definition of a strongly accessible boundary at the point $y_0 \in \partial D_*$, for any neighborhood U of this point, there are a compact set $C_0^{\prime} \subset D_*$, a neighborhood V of the point y_0, $V \subset U$, and a number $\delta > 0$ such that

$$M(\Gamma(C_0^{\prime}, F, D_*)) \geqslant \delta > 0 \tag{7}$$

for any continuum F, intersecting ∂U and ∂V. Put $C_0 := f^{-1}(C_0^{\prime})$. By [5, Proposition 2.1] $C_0 \cap \partial D = \varnothing$, since f is open, discrete and closed. By item 3) of Proposition 1, we obtain that $I(P) = \bigcap_{m=1}^{\infty} \overline{d_m} \subset \partial D$. Then we may assume that $C_0 \cap \overline{d_k} = \varnothing$ for any $k \in \mathbb{N}$. Join the points p_k and $p_k^{\prime}$ by a path γ_k in d_k. Observe that $f(p_k) \in V$ and $f(p_k^{\prime}) \in D \setminus \overline{U}$ for all sufficiently large $k \in \mathbb{N}$. In this case, due to (7), there is a number $k_0 \in \mathbb{N}$ such that

$$M(\Gamma(C_0^{\prime}, |f(\gamma_k)|, D_*)) \geqslant \delta > 0 \tag{8}$$

for any $k \geqslant k_0 \in \mathbb{N}$. Denote by Γ_k the family of all half-open paths $\beta_k\colon [a,b) \to D'$ such that $\beta(a) \in |f(\gamma_k)|$, $\beta_k(t) \in D'$ for any $t \in [a,b)$ and

$$\lim_{t \to b-0} \beta_k(t) := B_k \in C_0'\,.$$

Denote by $\overline{\Gamma_k}$ the family of all extended paths $\beta_k\colon [a,b] \to D'$, $\beta \in \Gamma_k$. Obviously,

$$M(\overline{\Gamma_k}) = M(\Gamma_k) = M\left(\Gamma\left(C_0', |f(\gamma_k)|, D'\right)\right)\,. \tag{9}$$

We fix $k \in \mathbb{N}$, $k \geqslant k_0$, and denote by Γ_k' the family of all maximal f-liftings $\alpha_k\colon [a,c) \to D$ of the family Γ_k starting at $|\gamma_k|$. Such a family is well defined due to Proposition 2. Note that, any path $\alpha_k \in \Gamma_k'$, $\alpha_k\colon [a,c) \to D$, can not tend to the boundary of D as $t \to c-0$, because $C(f, \partial D) \subset \partial D'$ (see [5, Proposition 2.1]). Then $C(\alpha_k, c) \subset D$.

Let us now show that there exists a limit of $\alpha_k(t)$ as $t \to c-0$. Consider the set

$$G = \left\{x \in \mathbb{S}\colon\ x = \lim_{k \to \infty} \alpha(t_k)\right\}\,, \quad t_k \in [a,\, c)\,, \quad \lim_{k \to \infty} t_k = c\,.$$

Passing to subsequences, we may restrict ourselves by monotone sequences t_k. Let $x \in G$, then by the continuity of f we have that $f\left(\alpha(t_k)\right) \to f(x)$ as $k \to \infty$, where $t_k \in [a,\, c)$, $t_k \to c$ as $k \to \infty$. However, $f\left(\alpha(t_k)\right) = \beta(t_k) \to \beta(c)$ as $k \to \infty$. Then f is constant on G. On the other hand, $\overline{\alpha}$ is a compactum, because $\overline{\alpha}$ is a closed subset of the compact space $\overline{D}$ (see [18, Theorem 2.II.4, § 41]). Then, by the Cantor condition on the compact set $\overline{\alpha}$, due to the monotonicity of the sets $\alpha\left([t_k,\, c)\right)$, we obtain that

$$G = \bigcap_{k=1}^{\infty} \overline{\alpha\left([t_k,\, c)\right)} \neq \varnothing\,,$$

see [18, 1.II.4, § 41]. Then, by [18, Theorem 5.II.5, § 47] the set $\overline{\alpha}$ is connected. Since f is discrete, the set G is one-point. Thus, the path $\alpha\colon [a,\, c) \to D$ can be extended to a closed path $\alpha\colon [a,\, c] \to D$ and $f\left(\alpha(c)\right) = \beta(c)$.

Hence, there exists $\lim\limits_{t \to c-0} \alpha_k(t) = A_k \in D$. Then, by the definition of a maximal lifting, we have that $c = b$. In this case, $\lim\limits_{t \to b-0} \alpha_k(t) := A_k$ and, simultaneously, by the continuity of the mapping f in D,

$$f(A_k) = \lim_{t \to b-0} f(\alpha_k(t)) = \lim_{t \to b-0} \beta_k(t) = B_k \in C_0'\,.$$

Hence, by the definition of the set C_0, we obtain that $A_k \in C_0$.

Hence, $\overline{\alpha} \in \Gamma(|\gamma_k|, C_0, D)$, where $\overline{\alpha}$ denotes the extended path $\overline{\alpha}\colon [a,b] \to D$. Denote by $\overline{\Gamma_k'}$ the family of all such extended paths $\overline{\alpha}\colon [a,b] \to D$, $\alpha \in \Gamma_k'$. Note that $\Gamma(|\gamma_k|, C_0, D) > \Gamma(\sigma_k, C_1, D)$ because σ_k is a cut corresponding to the domain d_k. Now, we apply the definition of a ring Q-mapping in (3) to the family $\Gamma(\sigma_k, C_1, D)$. Let us recall that $\sigma_k \in \widetilde{S}(p_0, r_k)$ for some point $p_0 \in \partial D$ and some sequence $r_k > 0$, $r_k \to 0$ as $k \to \infty$. Without loss of generality, reducing ε_1, if necessary, we may assume that $\widetilde{h}(p_0, C_1) > \varepsilon_0$. In addition, observe that the function

$$\eta_k(t) = \begin{cases} \psi(t)/I(r_k, \varepsilon_0), & t \in (r_k, \varepsilon_0),\\ 0, & t \in \mathbb{R} \setminus (r_k, \varepsilon_0)\,, \end{cases}$$

$I(\varepsilon,\varepsilon_0):=\int_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt$, satisfies the normalization condition (4). Due to the proving above, we obtain that $\overline{\Gamma_k'}\subset\Gamma(|\gamma_k|,C_0,D)$, therefore $M(f(\overline{\Gamma_k'}))\leqslant M(f(\Gamma(|\gamma_k|,C_0,D)))$. Then, by definition of a ring Q-mapping at the boundary point, taking into account conditions (5)–(6), we obtain that

$$M(f(\overline{\Gamma_k'}))\leqslant M(f(\Gamma(|\gamma_k|,C_0,D)))\leqslant M(f(\Gamma(\sigma_k,C_1,D))\leqslant\Delta(k)\,, \tag{10}$$

where $\Delta(k)\to 0$ as $k\to\infty$. However, $\Gamma_k=f(\Gamma_k')$ and $\overline{\Gamma_k}=f(\overline{\Gamma_k'})$. Then it follows from (10) that

$$M(\overline{\Gamma_k})=M\left(f(\overline{\Gamma_k'})\right)\leqslant\Delta(k)\to 0 \tag{11}$$

as $k\to\infty$. However, relation (11) together with equality (9) contradict inequality (8), which proves the lemma. □

A particular case of Lemma 1 is the following most important statement.

Theorem 3. *Let D, D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively, having compact closures $\overline{D}\subset\mathbb{S}$ and $\overline{D_*}\subset\mathbb{S}_*$, while ∂D has finitely many components, and ∂D_* is strongly accessible. Let $Q\colon\mathbb{S}\to(0,\infty)$ be a function measurable with respect to the measure $\widetilde{v}$ on $\mathbb{S}$, $Q(p)\equiv 0$ on $\mathbb{S}\setminus D$, furthermore, suppose that at least one of the following conditions holds:*

1) for any $p_0\in\partial D$, there is $\varepsilon_0=\varepsilon_0(p_0)>0$ such that

$$\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dt}{\|Q\|(t)}<\infty\,,\qquad\int\limits_{0}^{\varepsilon_0}\frac{dt}{\|Q\|(t)}=\infty \tag{12}$$

for any $0<\varepsilon<\varepsilon_0$, where $\|Q\|(r):=\int\limits_{\widetilde{S}(p_0,r)}Q(p)\,ds_{\widetilde{h}}(p)$ denotes the L_1-norm of the function Q over the circle $\widetilde{S}(p_0,r)$,

2) $Q\in FMO(\partial D)$.

Let $f\colon D\to D_*$ be an open discrete and closed ring Q-mapping of D onto D_* at the points of ∂D. Then f extends to a continuous mapping $\overline{f}\colon\overline{D}_P\to\overline{D_*}$, $\overline{f}(\overline{D}_P)=\overline{D_*}$.

Proof. In case 1), when conditions (12) are satisfied, we set $\psi(t)=\frac{1}{\|Q\|(t)}$, where, as usually $\|Q\|(t)=\int_{\widetilde{S}(p_0,t)}Q(p)\,ds_{\widetilde{h}}(p)$. Note that the function ψ satisfies the conditions (5)–(6) of Lemma 1. In particular, (6) holds for sufficiently small $0<\varepsilon<\varepsilon_0$, because

$$\frac{1}{J}=\int\limits_{\widetilde{A}(p_0,\varepsilon,\varepsilon_0)}Q(p)\cdot\eta_0^2(\widetilde{h}(p,p_0))\,d\widetilde{v}(p)\,,$$

$J=J(p_0,\varepsilon,\varepsilon_0):=\int_{\varepsilon}^{\varepsilon_0}\frac{dr}{\|Q\|(r)}$ (this fact can be proved completely by analogy with [12, Lemma 7.4, Ch. 7], cf. [19, Lemma 3.7] or [20, Lemma 4.2], and therefore its proof is omitted). Thus, in case 1) the necessary conclusion follows directly from Lemma 1. In case 2), when $Q\in FMO(\partial D)$, we set $\psi(t):=\frac{1}{(t\,\log\frac{1}{t})}$. Then the fulfillment of the condition (6) of Lemma 1 follows by [21, Lemma 3]. The necessary conclusion follows again from Lemma 1. □

The proof of Theorem 1. follows immediately from the fact that mappings of the Sobolev class with finite a distortion on Riemannian surfaces are ring Q-homeomorphisms for $Q = K_f(p)$, and also by Theorem 3. In particular, homeomorphisms acting between domains of two Riemannian surfaces are obviously discrete; moreover, they are open due to Brouwer's theorem (see [22, Theorem VI 9 and Corollary]), and closed due to general topological considerations (see [23, Theorem 1.VII.1, § 13]). Thus, the desired conclusion follows directly from Theorem 3. $\Box$

3. Equicontinuity of families of homeomorphisms. For a given $\delta > 0$, $D \subset \mathbb{S}$ and a function $Q\colon D \to [0, \infty]$ measurable with respect to the measure $\widetilde{v}$, denote by $\mathfrak{R}_{Q,\delta}(D)$ the family of all ring Q-homeomorphisms $f\colon D \to \mathbb{S}_*$ for which there is a continuum $G_f \subset \mathbb{S}_*$ such that $f\colon D \to \mathbb{S}_* \setminus G_f$ and $\widetilde{h}_*(G_f) = \sup_{x,y\in G_f} \widetilde{h}_*(x, y) \geqslant \delta$. The following assertion holds (see [5, Lemma 5.1]).

Lemma 2. *A family of mappings $\mathfrak{R}_{Q,\delta}(D)$ is equicontinuous in D, if $\mathbb{S}_*$ is a uniform domain, moreover, $Q\colon \mathbb{S} \to (0, \infty)$ is a locally integrable function in D such that conditions (5)–(6) hold at any point $p_0 \in D$.*

Consider the following class of mappings. Given $\delta > 0$, $D \subset \mathbb{S}$, a continuum $A \subset D$ and a function $Q\colon D \to [0, \infty]$ measurable with respect to the measure $\widetilde{v}$, denote by $\mathfrak{F}_{Q,\delta,A}(D)$ the family of all ring Q-homeomorphisms $f\colon D \to \mathbb{S}_* \setminus G_f$ in $\overline{D}$ for which there is a continuum $G_f \subset \mathbb{S}_*$ satisfying the condition $\widetilde{h}_*(G_f) = \sup_{x,y\in G_f} \widetilde{h}_*(x, y) \geqslant \delta$, while $\widetilde{h}_*(f(A)) \geqslant \delta$. An analogue of the following theorem was obtained in [4, Theorem 3.1].

Lemma 3. *Let D be a domain in $\mathbb{S}$ and $Q\colon \mathbb{S} \to (0, \infty)$ be a function measurable with respect to measure $\widetilde{v}$, $Q(p) \equiv 0$ for $p \in \mathbb{S} \setminus D$. Assume that, for any point $p_0 \in \overline{D}$ there are $\varepsilon_0 = \varepsilon_0(p_0) > 0$ and a function $\psi\colon (0, \infty) \to (0, \infty)$ such that*

$$I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty \quad \forall\, \varepsilon \in (0, \varepsilon_0)\,, \tag{13}$$

$I(\varepsilon, \varepsilon_0) > 0$ for sufficiently small $\varepsilon > 0$, and, in addition,

$$\int\limits_{\varepsilon<\widetilde{h}(p,p_0)<\varepsilon_0} Q(p)\cdot \psi^2(\widetilde{h}(p, p_0))\, d\widetilde{v}(p) = o(I^2(\varepsilon, \varepsilon_0))\,, \quad \varepsilon \to 0\,. \tag{14}$$

Let D and $D_f := f(D)$ be domains which have compact closures $\overline{D} \subset \mathbb{S}$ and $\overline{D_f} \subset \mathbb{S}_*$, moreover, ∂D consists of a finite number of components.

Assume that the domains D_f and $\mathbb{S}_*$ are equi-uniform over $f \in \mathfrak{F}_{Q,\delta,A}(D)$. Then each $f \in \mathfrak{F}_{Q,\delta,A}(D)$ has a continuous extension $\overline{f}\colon \overline{D}_P \to \overline{D_f}$. Moreover, the family $\mathfrak{F}_{Q,\delta,A}(\overline{D}_P)$, consisting of all extended mappings $\overline{f}\colon \overline{D}_P \to \overline{D_f}$, is equicontinuous in $\overline{D}_P$.

Proof. Observe that $\partial D_f = \partial f(D)$ is strongly accessible for any $f \in \mathfrak{F}_{Q,\delta,A}(D)$. Indeed, let $x_0 \in \partial D_f$ and let U be an arbitrary neighborhood of the point x_0. Choose $\varepsilon_1 > 0$ such that $V := \widetilde{B}(x_0, \varepsilon_1)$, $\overline{V} \subset U$. Let $\partial U \neq \varnothing$ and $\partial V \neq \varnothing$. Then $\varepsilon_2 := \widetilde{h}_*(\partial U, \partial V) > 0$. Since the domains D_f are equi-uniform and, moreover, $\widetilde{h}_*(F) \geqslant \varepsilon_2$ and $\widetilde{h}_*(G) \geqslant \varepsilon_2$, then we obtain that

$$M(\Gamma(F, G, D_f)) \geqslant \delta$$

for any continua F and G in D_f with $F\cap\partial U\neq\varnothing\neq F\cap\partial V$ and $G\cap\partial U\neq\varnothing\neq G\cap\partial V$, where $\delta>0$ is some number depending only on ε_2. Thus, $\partial D_f=\partial f(D)$ is strongly accessible. By Lemma 1, any $f\in\mathfrak{F}_{Q,\delta,A}(D)$ has a continuous extension $f\colon\overline{D}_P\rightarrow\overline{D_f}$.

Since $\mathfrak{F}_{Q,\delta,A}(D)\subset\mathfrak{R}_{Q,\delta}(D)$, the equicontinuity of the family $\mathfrak{F}_{Q,\delta,A}(D)$ at inner points of D follows by Lemma 2.

It remains to prove the equicontinuity of the family $\mathfrak{F}_{Q,\delta,A}(\overline{D}_P)$ on $E_D:=\overline{D}_P\setminus D$. Assume the contrary, namely, that this family is not equicontinuous at E_D. Then there are $P_0\in E_D$ and a number $a>0$ with the following property: for each $m=1,2,\ldots$ there is an element $p_m\in\overline{D}_P$ and a mapping $\overline{f}_m\in\mathfrak{F}_{Q,\delta,A}(\overline{D}_P)$ such that $\rho(P_0,p_m)<1/m$ and $\widetilde{h}_*(\overline{f}_m(p_m),\overline{f}_m(P_0))\geqslant a$. (Here ρ means the metric in the space $\overline{D}_P$, see Proposition 1). Since $f_m:=\overline{f}_m|_D$ has a continuous extension to the point P_0, we may assume that $p_m\in D$. By the same considerations, there exists a sequence $p_m^{\prime}\in D$, $p_m^{\prime}\rightarrow P_0$ as $m\rightarrow\infty$ such that $\widetilde{h}_*(f_m(p_m^{\prime}),\overline{f}_m(P_0))<1/m$. Thus,

$$\widetilde{h}_*(f_m(p_m),f_m(p_m^{\prime}))\geqslant a/2\qquad\forall\, m\in\mathbb{N}\,. \tag{15}$$

Let d_m, $m=1,2,\ldots$ be a sequence of cuts of D corresponding to the end P_0, and let the cuts α_m, corresponding to d_m, lie on the circles $\widetilde{S}(p_0,r_m)$, $r_m\rightarrow 0$ as $m\rightarrow\infty$. (Such cuts and circles exist by item 2) of Proposition 1). Without loss of generality, passing to a subsequence

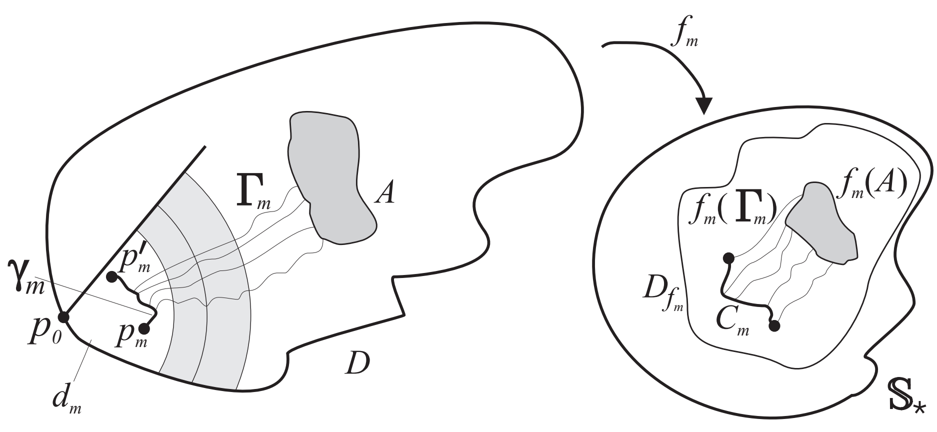

Figure 2: To the proof of Lemma 3

if necessary, we may assume that $p_m,p_m^{\prime}\in d_m$, see Figure 2 for illustrations. Join the points p_m and $p_m^{\prime}$ with a path $\gamma_m\colon[0,1]\rightarrow\mathbb{S}$ such that $\gamma_m(0)=p_m$, $\gamma_m(1)=p_m^{\prime}$ and $\gamma_m(t)\in d_m$ for $t\in(0,1)$. Denote by C_m the locus of the image of the path γ_m under the mapping f_m. From relation (15) it follows that

$$\widetilde{h}_*(C_m)\geqslant a/2\qquad\forall\, m\in\mathbb{N}\,, \tag{16}$$

where $\widetilde{h}_*(C_m)$ denotes the diameter of the set C_m in metric $\widetilde{h}_*$. Without loss of generality, we may assume that the continuum A from definition of the family $\mathfrak{F}_{Q,\delta,A}(D)$ is such that $d_m\cap A=\varnothing$, $m=1,2,\ldots$. Let Γ_m be a family of paths joining $|\gamma_m|$ and A in D. Let $m_1\in\mathbb{N}$ be such that $r_{m_1}<\varepsilon_0$, where $\varepsilon_0>0$ is the number from the condition of the lemma. Let $\alpha\in\Gamma_m$, $\alpha\colon[0,1]\rightarrow D$, $\alpha(0)\in|\gamma_m|$ and $\gamma(1)\in A$. Then $|\alpha|\cap d_{m_1}\neq\varnothing\neq|\alpha|\cap(D\setminus d_{m_1})$ and, therefore, by [18, Theorem 1.I.5, § 46] $|\alpha|\cap\partial d_{m_1}\neq\varnothing$. Since $\partial d_{m_1}\cap D\subset\gamma_{m_1}\subset\widetilde{S}(p_0,r_{m_1})$, there is a point $t_1\in[0,1]$ such that $\alpha(t_1)\in\widetilde{S}(p_0,r_{m_1})$. Without loss of generality, we may assume that $\alpha(t)\in \widetilde{B}(p_0, r_{m_1})$ for any $t\in[0, t_2)$. Let $\alpha_1:=\alpha|_{[0,t_1]}$ and let $m>m_1$. Then by the same considerations $|\alpha_1|\cap\partial d_m\ne\varnothing$. Since $\partial d_m\cap D\subset|\gamma_m|\subset\widetilde{S}(p_0, r_m)$, there is a point $t_2\in[0, t_1]$ such that $\alpha_1(t_2)\in\widetilde{S}(p_0, r_m)$. Without loss of generality, we may assume that $\widetilde{h}_*(\alpha_1(t), p_0)>r_m$ for any $t\in[t_2, t_1]$. Put $\alpha_2:=\alpha_1|_{[t_2,t_1]}$. Thus, we have proved that the path α has a subpath α_2 such that $\alpha_2\in\Gamma(\widetilde{S}(p_0, r_m), \widetilde{S}(p_0, r_{m_1}), \widetilde{A}(p_0, r_m, r_{m_1}))$, $\widetilde{A}(p_0, r_m, r_{m_1})=\{p\in\mathbb{S}\colon r_m<\widetilde{h}(p, p_0)<r_{m_1}\}$. By the definition of a ring Q-mapping at the point p_0 and due to the minority property of the modulus of families of paths (see [24, Theorem 1(c)]) we obtain that

$$M(f_m(\Gamma_m))\leqslant M(f_m(\Gamma(\widetilde{S}(p_0, r_m), \widetilde{S}(p_0, r_{m_1}), \widetilde{A}(p_0, r_m, r_{m_1}))))\leqslant$$
$$\leqslant\int\limits_{\widetilde{A}(p_0,r_m,r_{m_1})} Q(p)\cdot\eta^2(\widetilde{h}(p, p_0))\,d\widetilde{v}(p) \tag{17}$$

for any Lebesgue measurable function $\eta\colon(r_m, r_{m_1})\to[0,\infty]$ such that $\int_{r_m}^{r_{m_1}}\eta(r)\,dr\geqslant 1$. Observe that a function

$$\eta(t)=\begin{cases}\psi(t)/I(r_m, r_{m_1}), & t\in(r_m, r_{m_1}),\\ 0, & t\in\mathbb{R}\setminus(r_m, r_{m_1})\,,\end{cases}$$

satisfies the normalized condition (4) for $r_1:=r_m$, $r_2:=r_{m_1}$, where $I(\varepsilon, \varepsilon_0):=\int_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt$. Then by (13)–(14) and (17), it follows that

$$M(f_m(\Gamma_m))\leqslant\alpha(r_m)\to 0\quad\text{as}\quad m\to\infty\,, \tag{18}$$

where $\alpha(\varepsilon)$ is some nonnegative function tending to zero as $\varepsilon\to 0$, which exists due to relations (13)–(14).

On the other hand, observe that $f_m(\Gamma_m)=\Gamma(C_m, f_m(A), D_{f_m})$, where $\widetilde{h}_*(f_m(A))\geqslant\delta$ for any $m\in\mathbb{N}$ and by the definition of the class $\mathfrak{F}_{Q,\delta,A}(D)$. Taking into account (16) and the definition of an equi-uniform family of domains, we conclude that there exists $\sigma>0$ such that

$$M(f_m(\Gamma_m))=M(\Gamma(C_m, f_m(A), D_{f_m}))\geqslant\sigma\qquad\forall\ m\in\mathbb{N}\,,$$

which contradicts condition (18). The resulting contradiction proves the lemma. □

From Lemma 3, arguing similarly to the proof of Theorem 3, we obtain the following assertion.

Theorem 4. *Let D be a domain in $\mathbb{S}$ and $Q\colon\mathbb{S}\to(0,\infty)$ be a locally integrable function in D, $Q(p)\equiv 0$ on $\mathbb{S}\setminus D$. Assume that at least one of the following conditions is true:*

1) for any point $p_0\in\overline{D}$ there exists $\varepsilon_0=\varepsilon_0(p_0)>0$ such that the conditions (12) hold for all $0<\varepsilon<\varepsilon_0$;
2) $Q\in FMO(\overline{D})$. Let D and $D_f:=f(D)$ be domains which have compact closures $\overline{D}\subset\mathbb{S}$ and $\overline{D_f}\subset\mathbb{S}_*$, moreover, ∂D consists of a finite number of components.

Assume that the domains D_f and $\mathbb{S}_*$ are equi-uniform over $f\in\mathfrak{F}_{Q,\delta,A}(D)$. Then any $f\in\mathfrak{F}_{Q,\delta,A}(D)$ has a continuous extension $\overline{f}\colon\overline{D}_P\to\overline{D_f}$ and, in addition, the family $\mathfrak{F}_{Q,\delta,A}(\overline{D}_P)$ consisting of all extended mappings $\overline{f}\colon\overline{D}_P\to\overline{D_f}$, is equicontinuous in $\overline{D}_P$.

The proof of Theorem 2. follows by Theorem 4 due to the fact that homeomorphisms of the Sobolev class with finite distortion are ring Q-mappings for $Q = K_f(p)$ provided that Q is locally integrable (see [1, Lemma 3.1]). □

4. Equicontinuity of ring Q-mappings with a branching. The main ideas related to the study of mappings on Riemannian surfaces with branching refer to the first author's paper [5, Sec. 6]. Consider the following class of mappings. Let $\delta > 0$, $D \subset \mathbb{S}$ and $Q\colon D \to [0, \infty]$ be a function measurable with respect to the measure $\widetilde{v}$. Denote $\mathfrak{E}_{Q,\delta}(D)$ a family of open, discrete, and closed ring Q-mappings $f\colon D \to \mathbb{S}_*$ satisfying the following conditions: 1) there is a continuum $K_f \subset \mathbb{S}_* \setminus f(D)$ such that $\widetilde{h}_*(K_f) = \sup_{x,y\in K_f} \widetilde{h}_*(x, y) \geqslant \delta$; 2) there is a continuum $A_f \subset f(D)$ such that $\widetilde{h}_*(A_f) \geqslant \delta$ and $\widetilde{h}(f^{-1}(A_f), \partial D) \geqslant \delta$. The following assertion holds.

Lemma 4. *Let $D \subset \mathbb{S}$, let $D'_f := f(D) \subset \mathbb{S}_*$, let $f \in \mathfrak{E}_{Q,\delta}(D)$ and let $Q\colon \mathbb{S} \to (0, \infty)$ be a measurable function with respect to measure $\widetilde{v}$. Assume that $\overline{D}$ and $\overline{D'_f}$ are compact and ∂D has a finite number of components. Moreover, suppose that for any point $p_0 \in \partial D$ there exist $\varepsilon_0 = \varepsilon_0(p_0) > 0$ and a Lebesgue measurable function $\psi\colon (0, \infty) \to (0, \infty)$ such that conditions (5)–(6) are satisfied. If domains $D'_f := f(D)$ and $\mathbb{S}_*$ are equi-uniform over $f \in \mathfrak{E}_{Q,\delta}(D)$, then any mapping $f \in \mathfrak{E}_{Q,\delta}(D)$ has a continuous extension $\overline{f}\colon \overline{D}_P \to \overline{D'_f}$, in addition, the family $\mathfrak{E}_{Q,\delta}(\overline{D}_P)$ of all extended mappings $\overline{f}\colon \overline{D}_P \to \overline{D'_f}$ is equicontinuous at $\overline{D}_P$.*

Proof. Arguing as in the proof of Lemma 3, we obtain that $\partial D'_f = \partial f(D)$ is strongly accessible for any $f \in \mathfrak{E}_{Q,\delta}(D)$. Then, by Lemma 1, the mapping $f \in \mathfrak{E}_{Q,\delta}(D)$ has a continuous extension $\overline{f}\colon \overline{D}_P \to \overline{D'_f}$. Further, by [25, Theorem 7.2.2] the surface $\mathbb{S}$ is locally Ahlfors 2-regular, so that the family $f \in \mathfrak{E}_{Q,\delta}(D)$ is equicontinuous at inner points of D by [5, Lemma 6.1]. It remains to check the equicontinuity of the "extended" family $\mathfrak{E}_{Q,\delta}(\overline{D}_P)$ on $E_D := \overline{D}_P \setminus D$.

Assume the contrary, namely, that there exists $P_0 \in E_D$ such that $\mathfrak{E}_{Q,\delta}(\overline{D}_P)$ is not equicontinuous at P_0. In this case, there are a number $\delta_0 > 0$ and sequences $P_k \in \overline{D}_P$, $f_k \in \mathfrak{E}_{Q,\delta}(\overline{D}_P)$ such that $P_k \to P_0$ as $k \to \infty$ and

$$\widetilde{h}_*(f_k(P_k), f_k(P_0)) \geqslant \delta_0\,. \tag{19}$$

Since f_k has a continuous extension to $\overline{D}_P$, then for a given $k \in \mathbb{N}$ there is an element $x_k \in D$ such that $\rho(x_k, P_k) < 1/k$ and $\widetilde{h}(f_k(x_k), f_k(P_k)) < 1/k$, where ρ means the metric in $\overline{D}_P$, see Proposition 1. In this case, it follows from (19) that

$$\widetilde{h}_*(f_k(x_k), f_k(P_0)) \geqslant \varepsilon_0/2 \quad \forall \quad k = 1, 2, \ldots, . \tag{20}$$

Similarly, there is $x'_k \in D$ such that $x'_k \to P_0$ as $k \to \infty$ and, moreover, $\widetilde{h}_*(f_k(x'_k), f_k(P_0)) < 1/k$, $k = 1, 2, \ldots$. Then from (20) it follows that

$$\widetilde{h}_*(f_k(x_k), f_k(x'_k)) \geqslant \varepsilon_0/4 \quad \forall \quad k = 1, 2, \ldots\,,$$

where $x_k, x'_k \in D$, $x_k \to P_0$ as $k \to \infty$ and $x'_k \to P_0$ as $k \to \infty$ (see Figure 3 for an illustration).

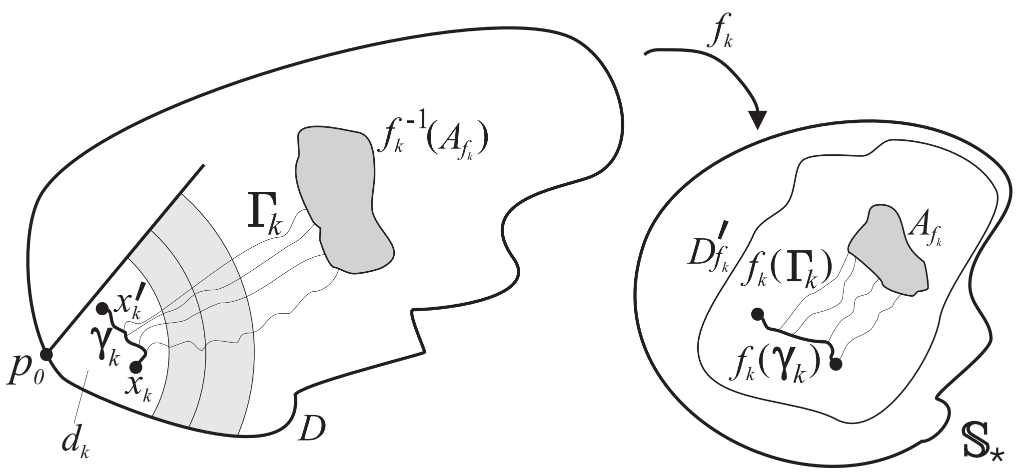

Figure 3: To the proof of Lemma 4

Let α_k, $k=1,2,\ldots$, be the cuts corresponding to the end P_0, and let d_k be a corresponding sequence of domains in D. We may assume that α_k belong to some circles $\widetilde{S}(p_0, r_k)$, where $r_k\rightarrow 0$ as $k\rightarrow\infty$. (Such cuts and circles exist by item 2) of Proposition 1). Without loss of generality, passing to a subsequence, if necessary, we may assume that $x_k, x_k^{\prime}\in d_k$. Join the points x_k and $x_k^{\prime}$ by a path $\alpha_k\colon [0, 1]\rightarrow \mathbb{S}$ such that $\gamma_k(0)=x_k$, $\gamma_k(1)=x_k^{\prime}$ and $\alpha_k(t)\in d_k$ for $t\in (0, 1)$.

Let A_{f_k} be the set corresponding to the mapping $f\in\mathfrak{E}_{Q, \delta}(D)$ from the definition of the class $\mathfrak{E}_{Q, \delta}(D)$. Let us prove that

$$f_k^{-1}(A_{f_k})\subset D\setminus d_k \tag{21}$$

for some sufficiently large $k_0\in {\mathbb N}$ and all $k\geqslant k_0$. Indeed, if the relation (21) does not hold, then there exists a sequence $x_l\in f_k^{-1}(A_{f_{k_l}})\cap d_{k_l}$, $l=1,2,\ldots$. Since by condition of the lemma $\overline{D}$ is a compact set in $\mathbb{S}$, we may consider the sequence x_l converging to some point $x_0\in D$, while $x_0\in \partial D$ by item 3) of Proposition 1. However, the latter contradicts the condition $\widetilde{h}(f^{-1}(A_{f_{k_l}}), \partial D)\geqslant \delta$, included in the definition of the class $\mathfrak{E}_{Q, \delta}(D)$. Thus, relation (21) is proved.

Without loss of generality, we may assume that $r_{k_0}<\varepsilon_0$, where $\varepsilon_0>0$ is a number corresponding to relations (5)–(6). Denote by Γ_k a family of paths joining $|\gamma_k|$ and $f_k^{-1}(A_{f_k})$ in D. Arguing similarly to the proof of Lemma 3 and taking into account that $\partial d_k\subset \widetilde{S}(p_0, r_k)$, we obtain that $\Gamma_k>\Gamma(\widetilde{S}(p_0, r_k), \widetilde{S}(p_0, r_{k_0}), \widetilde{A}(p_0, r_k, r_{k_0})\cap D)$. Thus,

$$M(f_k(\Gamma_k))\leqslant M(f_k(\Gamma(\widetilde{S}(p_0, r_k), \widetilde{S}(p_0, r_{k_0}), \widetilde{A}(p_0, r_k, r_{k_0})\cap D)))\,. \tag{22}$$

Observe that the function

$$\eta(t)=\begin{cases}\psi(t)/I(r_k, r_{k_0}), & t\in (r_k, r_{k_0}),\\ 0, & t\in {\mathbb R}\setminus (r_k, r_{k_0})\,,\end{cases}$$

satisfies the condition (6) for $\varepsilon=r_k$ and $\varepsilon_0=r_{k_0}$, where $I(\varepsilon, \varepsilon_0):=\int_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt$. In this case, by (22) and by the definition of a ring Q-mapping, as well as by the condition (6) we obtain that

$$M(f_k(\Gamma_k))\leqslant \alpha(2^{-k})\rightarrow 0\quad\text{as}\quad k\rightarrow\infty\,, \tag{23}$$

where $\alpha(\varepsilon)$ is some nonnegative function satisfying the condition $\alpha(\varepsilon)\to 0$ as $\varepsilon\to 0$.

On the other hand, consider the family of paths $\Gamma(|f_k(\gamma_k)|, A_{f_k}, D^{\,\prime}_{f_k})$. Since, by the assumption, the domains $D^{\,\prime}_{f_k}:=f_k(D)$ are equi-uniform, there is some $r_0>0$ such that

$$M(\Gamma(|f_k(\gamma_k)|, A_{f_k}, D^{\,\prime}_{f_k}))\geqslant r_0\,,\qquad k=1,2,\ldots\,. \tag{24}$$

Let Γ_k^* be the family of all maximal f_k-liftings of paths in $\Gamma(|f_k(\gamma_k)|, A_{f_k}, D^{\,\prime}_{f_k})$ starting at $|\gamma_k|$, which exists due to Proposition 2. Arguing similarly to the proof of Lemma 1, we may show that $\Gamma_k^*\subset\Gamma_k$. In addition, observe that $f_k(\Gamma_k^*)<\Gamma(|f_k(\gamma_k)|, A_{f_k}, D^{\,\prime}_{f_k})$. Hence, we have that

$$M(\Gamma(|f_k(\gamma_k)|, A_{f_k}, D^{\,\prime}_{f_k}))\leqslant M(f_k(\Gamma_k^*))\leqslant M(f_k(\Gamma_k))\,. \tag{25}$$

However, (24) and (25) contradict relation (23). The resulting contradiction completes the proof. □

Arguing similarly to the proof of Theorem 3 and using the assertion of Lemma 4, we obtain the following statement.

Theorem 5. *Let $D\subset\mathbb{S}$, let $D^{\,\prime}_f:=f(D)\subset\mathbb{S}_*$, let $f\in\mathfrak{E}_{Q,\delta}(D)$ and let $Q\colon\mathbb{S}\to(0,\infty)$ be a measurable function with respect to measure $\widetilde{v}$. Assume that, $\overline{D}$ and $\overline{D^{\,\prime}_f}$ are compact and ∂D has a finite number of components. Assume that at least one of the following conditions holds: 1) for any point $p_0\in\overline{D}$ there is $\varepsilon_0=\varepsilon_0(p_0)>0$ such that conditions (12) hold for any $0<\varepsilon<\varepsilon_0$; 2) $Q\in FMO(\overline{D})$.*

If domains $D^{\,\prime}_f:=f(D)$ and $\mathbb{S}_*$ are equi-uniform over $f\in\mathfrak{E}_{Q,\delta}(D)$, then any mapping $f\in\mathfrak{E}_{Q,\delta}(D)$ has a continuous extension $\overline{f}\colon\overline{D}_P\to\overline{D^{\,\prime}_f}$, in addition, the family $\mathfrak{E}_{Q,\delta}(\overline{D}_P)$ of all extended mappings $\overline{f}\colon\overline{D}_P\to\overline{D^{\,\prime}_f}$ is equicontinuous in $\overline{D}_P$.

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