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EQUIAFFINE IMMERSIONS OF CODIMENSION TWO WITH FLAT CONNECTION AND ONE-DIMENSIONAL WEINGARTEN MAPPING

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In the paper we study equiaffine immersions $f\colon (M^n,\nabla)\to\mathbb{R}^{n+2}$ with flat connection ∇ and one-dimensional Weingarten mapping. For such immersions there are two types of the transversal distribution equiaffine frame. We give a parametrization of a submanifold with the given properties for both types of equiaffine frame. The main result of the paper is contained in Theorems 1, 2 and Corollary 1: Let $f\colon (M^n,\nabla)\to(\mathbb{R}^{n+2},D)$ be an affine immersion with pointwise codimension 2, equiaffine structure, flat connection ∇, one-dimensional Weingarten mapping then there exists three types of its parametrization:

(i) $\vec r = g(u^1,\dots,u^n)\vec a_1 + \int\vec\varphi(u^1)du^1 + \sum\limits_{i=2}^{n} u^i\vec a_i$;

(ii) $\vec r = (g(u^2,\dots,u^n)+u^1)\vec a + \int v(u^1)\vec\eta(u^1)du^1 + \sum\limits_{i=2}^{n} u^i\int\lambda_i(u^1)\vec\eta(u^1)du^1$;

(iii) $\vec r = (g(u^2,\dots,u^n)+u^1)\vec\rho(u^1) + \int(v(u^1)-u^1)\dfrac{d\vec\rho(u^1)}{du^1}du^1 + \sum\limits_{i=2}^{n} u^i\int\lambda_i(u^1)\dfrac{d\vec\rho(u^1)}{du^1}du^1$.

1. Introduction. We consider the affine immersions by K. Nomizu, T. Sasaki [3] in the case of codimension two. Let (M^n,∇) be an affine n-dimensional manifold with affine connection ∇ and $(\mathbb{R}^{n+2},D)$ the standard (arithmetic) affine space with flat connection D. We shall denote by $\mathfrak{X}(M^n)$ the set of all smooth tangent vector fields on M^n. According to [3, p. 29], a differentiable immersion $f\colon (M^n,\nabla)\to(\mathbb{R}^{n+2},D)$ is said to be affine if there exists a two-dimensional transversal differentiable distribution Q along f such that at each point $x\in M^n$ for all $X,\ Y\in\mathfrak{X}(M^n)$ the following decomposition

$$(D_X f_*(Y))_x = (f_*(\nabla_X Y))_x + h_x(X,\ Y),\quad h_x(X,Y)\in Q_x,$$

is held. This decomposition defines the *affine fundamental form* $h(X,Y)$. It is known [3, 7] that the rank of the affine fundamental form is independent of the choice of transversal distribution, and it is called the *pointwise codimension* of an affine immersion.

For arbitrary transversal vector field ξ, the decomposition

$$D_X\xi = -f_*(S_\xi X) + \nabla^\perp_X\xi,$$

defines the *shape operator* S_ξ and the *transversal connection* $\nabla^\perp$.

The *Weingarten mapping* $S_x\colon Q_x\times T_x(M^n)\to T_x(M^n)$ is defined [5] as follows: $(\xi,X)\mapsto S_\xi X$ at every point $x\in M^n$.

For a transversal frame $\{\xi_1,\ \xi_2\}$ we have the affine analogues of Gauss and Weingarten decompositions, namely

$$D_X f_*(Y) = f_*(\nabla_X Y) + h^\alpha(X,\ Y)\xi_\alpha, \tag{1}$$

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$$D_X\xi_\alpha = -f_*(S_\alpha X) + \tau_\alpha^\beta(X)\xi_\beta, \tag{2}$$

where h^α are components of the affine fundamental form, S_α are shape operators, τ_α^β are forms of transversal connection (with respect to $\xi_1,\ \xi_2$).

For an affine immersion $f\colon (M^n,\nabla)\to\mathbb{R}^{n+2}$ with a transversal frame $\{\xi_1,\ \xi_2\}$, the *induced volume element* θ on M^n is defined [4, 3, 1] as follows

$$\theta(X_1,\dots,X_n) = |f_*(X_1),\dots,f_*(X_n),\xi_1,\xi_2|.$$

The transversal distribution Q with frame $\{\xi_1,\ \xi_2\}$ is called *equiaffine*, if $\nabla_X\theta = 0$ for all $X\in T_x(M^n), x\in M^n$. For two-codimension affine immersion this condition is equivalent [4] to

$$\tau_1^1(X) + \tau_2^2(X) \equiv 0. \tag{3}$$

With an equiaffine transversal distribution Q we have an *equiaffine structure* $(\nabla,\ \theta)$ on M^n.

We will consider an affine immersion $f\colon (M^n,\nabla)\to\mathbb{R}^{n+2}$ with pointwise codimension two, flat connection ∇ and equiaffine transversal distribution. Two-codimensional affine surfaces with different additional properties have been studied by many authors. Flat affine surfaces in $\mathbb{R}^4$ with flat normal connection were studied in [1]. The description of a parallel affine immersions $(M^n,\nabla)\to\mathbb{R}^{n+2}$ with flat connection in dependence on the rank of the Weingarten mapping were given in [5].

Let us remind that in general case (codimension k) the kernel and the image of the Weingarten mapping is defined by

$$\ker S = \bigcap_{\alpha=1}^{k}\ker S_\alpha,\ \ \mathrm{im}\ S = \bigcup_{\alpha=1}^{k}\mathrm{im}\ S_\alpha.$$

We say that a Weingarten mapping is p-dimensional if $\mathrm{rank}S := \dim\mathrm{im}S = p$. It was proved [7] that for the immersion $f\colon (M^n,\nabla)\to\mathbb{R}^{n+k}$ (for $k<n$) with maximal pointwise codimension and flat connection ∇ the following relations hold true:

$$1)\dim\ker S\ge n-k;\qquad 2)\ker h\subseteq\ker S;\qquad 3)\dim\mathrm{im}\ S\le k;$$
$$4)\ \text{if}\ \dim\mathrm{im}\ S = k,\ \text{then}\ \dim\ker S = n-k\ \text{and}\ \ker h = \ker S.$$

It was also proved [7] that the distribution $\mathcal{S}$ of the kernels of Weingarten mapping is integrable on M^n and there exists a transversal distribution which is stationary along the leaves of the foliation $\mathcal{FS}$.

Since in the case of codimension two we have $\dim\mathrm{im}\ S\le 2$, $\dim\ker S\ge n-2$, we have only three possible values for the dimension of im S, namely $0,\ 1,\ 2$. The most studied are affine immersions with zero and two-dimensional Weingarten mapping.

Examples of affine immersions with flat connection and one-dimensional Weingarten mapping were given in [7].

The description of the parallel affine immersions $M^n\to\mathbb{R}^{n+2}$ with flat connection in dependence on the rank of the Weingarten mapping was given in [5].

It is well known that an affine immersion with zero Weingarten mapping ($S\equiv 0$) has a flat connection and it is affinely equivalent to the graph of certain smooth map $F\colon M^n\to\mathbb{R}^2$ (see for example [2, 3, 7]), i.e. $f\colon (u^1,\dots,\ u^n)\mapsto(u^1,\dots,\ u^n,\ f^1(u^1,\dots,\ u^n),\ f^2(u^1,\dots,\ u^n))$. Obviously, any graph immersion is equiaffine.

According to the properties which were discussed in [7], in case $\dim \operatorname{im} S = 2$ we obtain $\ker h = \ker S$ and $\dim \ker h = n-2$. Therefore such a submanifold is strongly $(n-2)$-parabolic one or, equivalently, a submanifold of rank two (by the rank of Gaussian (Grassmann) mapping) [6]. Due to the coincidence of the kernels of S and h, the distribution $\mathcal{S} = \ker h$ is not only integrable, but the leaves are totally geodesic in $\mathbb{R}^{n+2}$. A rank-two submanifold is a ruled submanifold with $(n-2)$-dimensional rulings over a two-dimensional base. In the case this submanifold is a cylinder, its connection is determined by the connection of the cylinder base. In the general case the problem on its connection remains open.

We obtain a parametrization of a submanifold with one-dimensional Weingarten mapping and given properties. Such a submanifold is a peculiar "mix" of a graph and a ruled submanifold.

The main result. *Let $f\colon (M^n, \nabla) \to (\mathbb{R}^{n+2}, D)$ be an affine immersion with pointwise codimension 2, equiaffine structure, flat connection ∇, one-dimensional Weingarten mapping. Then there exist three types of its parametrization:*

(i) $\vec{r} = g(u^1, \dots, u^n)\vec{a}_1 + \int \vec{\varphi}(u^1)du^1 + \sum_{i=2}^{n} u^i \vec{a}_i;$

(ii) $\vec{r} = (g(u^2, \dots, u^n) + u^1)\vec{a} + \int v(u^1)\vec{\eta}(u^1)du^1 + \sum_{i=2}^{n} u^i \int \lambda_i(u^1)\vec{\eta}(u^1)du^1;$

(iii) $\vec{r} = (g(u^2, \dots, u^n) + u^1)\vec{\rho}(u^1) + \int (v(u^1) - u^1)\frac{d\vec{\rho}(u^1)}{du^1}du^1 + \sum_{i=2}^{n} u^i \int \lambda_i(u^1)\frac{d\vec{\rho}(u^1)}{du^1}du^1.$

For more details see Theorem 1, Theorem 2, and Corollary 1.

Submanifold (i) is a "mix" of a graph and a cylinder with $(n-2)$-dimensional rulings over a flat curve; submanifold (ii) is a "mix" of a graph and a ruled submanifold with $(n-2)$-dimensional rulings; submanifold (iii) is a "mix" of a "graph over a curve" and a ruled submanifold with $(n-2)$-dimensional rulings.

2. Preliminaries. Let $f\colon (M^n, \nabla) \to (\mathbb{R}^{n+2}, D)$ be an affine immersion. The basic equations of the affine immersions are well-known (see [2], [3], [7]):

$$R(X,\ Y)Z = h^\alpha(Y,\ Z)S_\alpha X - h^\alpha(X,\ Z)S_\alpha Y; \tag{4}$$

$$(\nabla_X h^\alpha)(Y,\ Z) + \tau^\alpha_\beta(X)h^\beta(Y,\ Z) = (\nabla_Y h^\alpha)(X,\ Z) + \tau^\alpha_\beta(Y)h^\beta(X,\ Z); \tag{5}$$

$$(\nabla_X S_\alpha)Y - \tau^\beta_\alpha(X)S_\beta Y = (\nabla_Y S_\alpha)X - \tau^\beta_\alpha(Y)S_\beta X; \tag{6}$$

$$\begin{gathered} h^\beta(X,\ S_\alpha Y) - h^\beta(Y,\ S_\alpha X) = \\ = X(\tau^\beta_\alpha(Y)) + \tau^\beta_\gamma(X)\tau^\gamma_\alpha(Y) - Y(\tau^\beta_\alpha(X)) - \tau^\beta_\gamma(Y)\tau^\gamma_\alpha(X) - \tau^\beta_\alpha([X,\ Y]). \end{gathered} \tag{7}$$

If we change the transversal frame by $\bar{\xi}_\alpha = \Phi^\beta_\alpha \xi_\beta$ where $\Phi = [\Phi^\beta_\alpha]_{2\times 2}$ is a nondegenerate matrix with smooth entries, then ∇ does not change while the other main characteristics of the affine immersion change as follows (see, for example, [7]):

$$\bar{h}^\alpha(X,\ Y) = [\Phi^{-1}]^\alpha_\beta h^\beta(X,\ Y), \tag{8}$$

$$\bar{S}_\alpha X = \Phi^\beta_\alpha S_\beta X, \tag{9}$$

$$\bar{\tau}^\beta_\alpha(X) = [\Phi^{-1}]^\beta_\gamma\{\tau^\gamma_\delta(X)\Phi^\delta_\alpha + X(\Phi^\gamma_\alpha)\}. \tag{10}$$

It is well known and easily verified that in case of two equiaffine frames

$$\det \Phi = \text{const}. \tag{11}$$

We prove two supporting lemmas.

Lemma 1. *Let $f\colon (M^n,\nabla)\to(\mathbb{R}^{n+2},D)$ be an affine immersion with the pointwise codimension 2, flat connection ∇, equiaffine structure, and one-dimensional Weingarten mapping. There exist a parametrization of immersion and a transversal frame such that:*

1) $\Gamma_{ij}^k=0\ \forall k\neq 1,\ \Gamma_{11}^1=0,\ \Gamma_{1j}^1=0$;

2) $h^1=\begin{pmatrix} h_{11}^1 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{pmatrix},\ h^2=\begin{pmatrix} 0 & h_{12}^2 & \dots & h_{1n}^2\\ h_{12}^2 & h_{22}^2 & \dots & h_{2n}^2\\ \vdots & \vdots & \ddots & \vdots\\ h_{1n}^2 & h_{2n}^2 & \dots & h_{nn}^2\end{pmatrix}$;

3) $S_1e_1=U,\ S_1e_j=0,\ j=\overline{2,n},\ S_2\equiv 0$;

4) $\tau_1^1(e_i)=\tau_2^1(e_i)=\tau_2^2(e_i)=0$ *for* $i=\overline{2,n},\ \tau_1^1(e_1)=-\tau_2^2(e_1)$.

Proof. It was proved [5] that if $f\colon (M^n,\nabla)\to(\mathbb{R}^{n+2},D)$ is an affine immersion with the pointwise codimension 2, flat connection ∇, $\dim\operatorname{im} S=1$ and $\dim\ker h<n-2$, then there exist a parametrization of immersion and a transversal frame such that

a) $\Gamma_{ij}^k=0\ \forall k\neq 1,\ \Gamma_{11}^1=0,\ \Gamma_{1j}^1=0$;

b) $h^1=\begin{pmatrix} 1 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{pmatrix},\ h^2=\begin{pmatrix} 0 & h_{12}^2 & \dots & h_{1n}^2\\ h_{12}^2 & h_{22}^2 & \dots & h_{2n}^2\\ \vdots & \vdots & \ddots & \vdots\\ h_{1n}^2 & h_{2n}^2 & \dots & h_{nn}^2\end{pmatrix}$;

c) $S_1e_1=U,\ S_1e_j=0,\ j=\overline{2,n},\ S_2\equiv 0$;

d) $\tau_2^1(e_i)=\tau_2^2(e_i)=0$ for $i=\overline{2,n}$.

Suppose, in addition, that the affine immersion under consideration has the equiaffine transversal distribution. In the proof of a), b) and c) the transversal distribution remains equiaffine. The exception is d) since the proof in [5] does not involve (11).

From the Codazzi equations (6) for S_2 we get that $\tau_2^1(e_i)=0,\ i=\overline{2,n}$.

Consider the transversal distribution frame transformation with the matrix

$$\Phi=\begin{pmatrix}\frac{1}{\phi} & 0\\ 0 & \phi\end{pmatrix},\ \text{where } \phi\neq 0.$$

Then (8) and (9) take the forms $\tilde h^1=\phi h^1,\ \tilde h^2=\frac{1}{\phi}h^2,\ \tilde S_1=\frac{1}{\phi},\ \tilde S_2\equiv 0$.

Using these equalities and a), b), and c) of [5] we obtain items 1), 2), and 3). Let us prove 4). The forms of transversal connection change by (10) as

$$\tilde\tau_2^1(X)=\phi^2\tau_2^1(X),\quad \tilde\tau_1^1(X)=\tau_1^1(X)-\frac{1}{\phi}X(\phi),\quad \tilde\tau_2^2(X)=\tau_2^2(X)+\frac{1}{\phi}X(\phi).$$

Evidently, with transformation under consideration $\tilde\tau_2^1(e_i)=0,\ i=\overline{2,n}$. We want to find a function ϕ such that

$$\tilde\tau_2^2(e_i)=0,\quad i=\overline{2,n}. \tag{12}$$

It leads to the system $\frac{\partial}{\partial u^i}\ln\phi=-\tau_2^2(e_i),\ i=\overline{2,n}$. From the Ricci equations (7) for $\alpha=2,\ \beta=2,\ X=e_i,\ Y=e_j,\ i,j=\overline{2,n}$, namely

$$e_i(\tau_2^2(e_j))+\tau_\gamma^2(e_i)\tau_2^\gamma(e_j)-e_j(\tau_2^2(e_i))-\tau_\gamma^2(e_j)\tau_2^\gamma(e_i)=0,$$

and equalities $\tau_2^1(e_i)=0,\ i=\overline{2,n}$ we obtain the integrability condition for the system of differential equations $\frac{\partial}{\partial u^i}\tau_2^2(e_j)=\frac{\partial}{\partial u^j}\tau_2^2(e_i),\ i,j=\overline{2,n}$. Therefore $\phi=\exp\left(-\int\tau_2^2(e_2)du^2\right)$. Thus we get (12). From the condition $\tau_2^2(X)=-\tau_1^1(X)$ it follows $\tilde\tau_1^1(e_i)=0$ for $i=\overline{2,n}$ and $\tilde\tau_1^1(e_1)=-\tilde\tau_2^2(e_1)$. □

Lemma 2. *Let $f\colon (M^n,\nabla)\to(\mathbb{R}^{n+2},D)$ be an affine immersion with the pointwise codimension 2, flat connection ∇, equiaffine structure, and one-dimensional Weingarten mapping. There exists the parametrization of immersion such that the equiaffine transversal frame depends only on one variable and can be one of two types:*

A) *one of the frame vectors is constant;*

B) *none of the frame vectors are constant.*

Proof. We choose the tangent and transversal frames as in Lemma 1. From the Ricci equations (7) on tangent frame vectors we obtain

$$\text{for } \alpha=1,\ \beta=1 \quad \tau_2^1(e_1)\tau_1^2(e_j)-\frac{\partial\tau_1^1(e_1)}{\partial u^j}=0,\ j=\overline{2,n}; \tag{13}$$

$$\text{for } \alpha=1,\ \beta=2 \quad \frac{\partial\tau_1^2(e_i)}{\partial u^j}=\frac{\partial\tau_1^2(e_j)}{\partial u^i},\quad i,j=\overline{2,n}; \tag{14}$$

$$\text{for } \alpha=2,\ \beta=1 \quad \frac{\partial\tau_2^1(e_1)}{\partial u^j}=0,\ j=\overline{2,n}; \tag{15}$$

$$\text{for } \alpha=2,\ \beta=2 \quad \frac{\partial\tau_2^2(e_1)}{\partial u^j}+\tau_1^2(e_j)\tau_2^1(e_1)=0,\ j=\overline{2,n}. \tag{16}$$

A. We have a coordinate system as in Lemma 1. Consider the case $\tau_2^1\equiv 0$. Ricci equations (13)–(16) on tangent frame vectors are

$$\begin{aligned}
&\text{for } \alpha=1,\ \beta=1 && \frac{\partial\tau_1^1(e_1)}{\partial u^i}=0,\quad i=\overline{2,n};\\
&\text{for } \alpha=1,\ \beta=2 && \frac{\partial\tau_1^2(e_i)}{\partial u^j}=\frac{\partial\tau_1^2(e_j)}{\partial u^j},\quad i,j=\overline{2,n};\\
&\text{for } \alpha=2,\ \beta=2 && \frac{\partial\tau_2^2(e_1)}{\partial u^i}=0,\quad i=\overline{2,n}.
\end{aligned} \tag{17}$$

We change the transversal frame $\tilde{\xi}_\alpha=\Phi_\alpha^\beta\xi_\beta$ with the matrix $\Phi=\begin{pmatrix}\lambda(u_1) & 0\\ \mu & 1/\lambda(u_1)\end{pmatrix}$, where $\lambda(u_1)\neq 0$. Since we have $\Phi^{-1}=\begin{pmatrix}1/\lambda(u_1) & 0\\ -\mu & \lambda(u_1)\end{pmatrix}$ and $S_2\equiv 0$, using (8), (9) we get

$$\tilde{h}^1=\frac{1}{\lambda}h^1,\ \tilde{h}^2=-\mu h^1+\lambda h^2,\ \tilde{S}_1=\lambda S_1,\ \tilde{S}_2=0. \tag{18}$$

The forms of the transversal connection (10) on the tangent frame vectors $e_i,\ i=\overline{1,n}$, are

$$\begin{gathered}
\tilde{\tau}_1^1(e_i)=\tau_1^1(e_i)+\frac{\mu}{\lambda}\tau_2^1(e_i)+\frac{1}{\lambda}\frac{d\lambda}{du^i},\\
\tilde{\tau}_1^2(e_i)=-\lambda\mu\tau_1^1(e_i)-\mu^2\tau_2^1(e_i)-\mu\frac{\partial\lambda}{\partial u^i}+\lambda^2\tau_1^2(e_i)+\lambda\mu\tau_2^2(e_i)+\lambda\frac{\partial\mu}{\partial u^i},\\
\tilde{\tau}_2^1(e_i)=\frac{1}{\lambda^2}\tau_2^1(e_i),\\
\tilde{\tau}_2^2(e_i)=\tau_2^2(e_i)-\frac{\mu}{\lambda}\tau_2^1(e_i)+\lambda\frac{\partial}{\partial u^i}\left(\frac{1}{\lambda}\right).
\end{gathered}$$

From 4) in Lemma 1, $\tau_2^1(e_1)=0$, and $\lambda(u^1)$ we have

$$\tilde{\tau}_1^1(e_i)=\tilde{\tau}_2^1(e_i)=\tilde{\tau}_2^2(e_i)=0 \text{ for } i=\overline{2,n},\ \tilde{\tau}_2^1(e_1)=0.$$

From (17) it follows that the functions $\tau_1^1(e_1)$, $\tau_2^2(e_1)$ depend on u^1 only. Put $\lambda(u^1) = \exp(\int \tau_2^2(e_1)du^1)$. Since $\tau_1^1(e_1) = -\tau_2^2(e_1)$ and

$$\tilde{\tau}_1^1(e_1) = \tau_1^1(e_1) + \frac{1}{\lambda}\frac{d\lambda}{du^1},\ \tilde{\tau}_2^2(e_1) = \tau_2^2(e_1) - \frac{1}{\lambda}\frac{d\lambda}{du^1},$$

we conclude $\tilde{\tau}_1^1(e_1) = \tilde{\tau}_2^2(e_1) = 0$.

Using the properties of the transversal connection forms we have

$$\tilde{\tau}_1^2(e_i) = \lambda(u^1)\frac{\partial\mu}{\partial u^i} + \lambda^2(u^1)\tau_1^2(e_i), \quad i = \overline{2,n}$$

with chosen function $\lambda(u^1)$. Now we want to find a function $\mu(u^1, \dots, u^n)$ such that $\tilde{\tau}_1^2(e_i) = 0$, $i = \overline{2,n}$. It leads us to a system of differential equations $\frac{\partial\mu}{\partial u^i} = -\frac{1}{\lambda(u^1)}\tau_1^2(e_i)$, $i = \overline{2,n}$. This system is integrable because we have $\frac{\partial}{\partial u^j}\tau_1^2(e_i) = \frac{\partial}{\partial u^i}\tau_1^2(e_j)$ for all $i, j = \overline{2,n}$ from (17).

Thus, with new transversal frame we have the only nonzero form of the transversal connection $\tilde{\tau}_1^2(e_1)$. In this case the Weingarten decompositions are

$$\frac{\partial\tilde{\xi}_1}{\partial u^i} = 0, \quad \frac{\partial\tilde{\xi}_2}{\partial u^1} = 0,\ \frac{\partial\tilde{\xi}_2}{\partial u^i} = 0,\ i = \overline{1,n}.$$

Therefore, $\tilde{\xi}_1$ depends on u^1 only, $\tilde{\xi}_2$ is a constant vector field.

B. Consider the case $\tau_2^1(e_1) \neq 0$. The system (15) implies that the function $\tau_2^1(e_1)$ depends only on the variable u^1, denote $\tau_2^1(e_1) = c(u^1)$.

From (14) we conclude that there exist functions $p(u^2, \dots, u^n)$, $t_i(u^1)$, $i = \overline{1,n}$, such that

$$\tau_1^2(e_j) = t_1(u^1)\frac{\partial p}{\partial u^j} + t_j(u^1), \quad j = \overline{2,n}.$$

The system (13) implies

$$\frac{\partial\tau_1^1(e_1)}{\partial u^j} = \tau_2^1(e_1)\tau_1^2(e_j) = c(u^1)\Big(t_1(u^1)\frac{\partial p}{\partial u^j} + t_j(u^1)\Big),\ j = \overline{2,n};$$

$$\tau_1^1(e_1) = c(u^1)\Big(t_1(u^1)p(u^2, \dots, u^n) + \sum_{j=2}^{n} t_j(u^1)u^j + \mu(u^1)\Big)$$

Using the properties of the shape operators (item 3 in Lemma 1) and the forms of transversal connection we write the Weingarten decompositions for ξ_2 as $\frac{\partial}{\partial u^j}\xi_2 = 0$, $j = \overline{2,n}$. Therefore, the vector field ξ_2 depends on u^1 only, i.e. $\xi_2 = \xi_2(u^1)$. The Weingarten decompositions for ξ_1 are

$$\frac{\partial\xi_1}{\partial u^i} = \Big(t_1(u^1)\frac{\partial p}{\partial u^i} + t_i(u^1)\Big)\xi_2(u^1), \quad i = \overline{2,n}.$$

Then

$$\xi_1 = \Big(t_1(u^1)p(u^2, \dots, u^n) + \sum_{i=2}^{n} t_i(u^1)u^i + \mu(u^1)\Big)\xi_2(u^1) + \zeta(u^1).$$

Denote $t_1(u^1)p(u^2, \dots, u^n) + \sum_{i=2}^{n} t_i(u^1)u^i + \mu(u^1) = q(u^1, \dots, u^n)$ and change the transversal frame by $\bar{\xi}_1 = \xi_1 - q \cdot \xi_2$, $\bar{\xi}_2 = \xi_2$.

The transformation matrix and its inverse are

$$\Phi = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}, \ \Phi^{-1} = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}.$$

In this case the main characteristics of the affine immersion change by (8), (9) as

$$\bar{h}^1 = h^1, \ \bar{h}^2 = qh^1 + h^2, \ \bar{S}_1 = S_1, \ \bar{S}_2 = S_2 \equiv 0.$$

The forms of the transversal connection (10) on the tangent frame vectors $e_i, \ i = \overline{1,n}$ are

$$\bar{\tau}_1^1(e_i) = \tau_1^1(e_i) - q\tau_2^1(e_i), \quad \bar{\tau}_1^2(e_i) = q\tau_1^1(e_i) + q^2\tau_2^1(e_i) + \tau_1^2(e_i) - q\tau_2^2(e_i) - \frac{\partial q}{\partial u^i},$$
$$\bar{\tau}_2^1(e_i) = \tau_2^1(e_i), \quad \bar{\tau}_2^2(e_i) = q\tau_2^1(e_i) + \tau_2^2(e_i).$$

Using Lemma 1, $\tau_2^1(e_1) = c(u^1)$, $\tau_1^1(e_1) = c(u^1)q$, and $\tau_1^2(e_i) = \frac{\partial}{\partial u^i}q$ we conclude that the main characteristics of the affine immersion get some new properties

$$\bar{h}_{11}^2 = q\bar{h}_{11}^1; \quad \bar{\tau}_1^2(e_i) = 0, \quad i = \overline{2,n}; \quad \bar{\tau}_1^1(e_1) = \bar{\tau}_2^2(e_1) = 0. \tag{19}$$

The new vectors $\bar{\xi}_1(u^1), \bar{\xi}_2(u^1)$ depend on u^1 only and $\frac{\partial}{\partial u^1}\bar{\xi}_2 = c(u^1)\bar{\xi}_1$. Denote $\bar{\xi}_2(u^1) = \vec{\eta}(u^1)$, then

$$\bar{\xi}_1 = \frac{1}{c(u^1)}\frac{d\vec{\eta}(u^1)}{du^1}, \quad \bar{\xi}_2 = \vec{\eta}(u^1). \tag{20}$$

□

The equiaffine transversal frames from Lemma 2 will be referred to as the frames of type **A** and type **B**.

3. Parametrization in case of type A.

Theorem 1. *Let $f\colon (M^n, \nabla) \to (\mathbb{R}^{n+2}, D)$ be an affine immersion with pointwise codimension 2, equiaffine structure, flat connection ∇, one-dimensional Weingarten mapping, and one of the transversal frame vectors is constant. Then there exists a parametrization ($\vec{a}_i = \overrightarrow{\mathrm{const}}$)*

$$\vec{r} = g(u^1, \ldots, u^n)\vec{a}_1 + \int \vec{\varphi}(u^1)du^1 + \sum_{i=2}^{n} u^i\vec{a}_i; \tag{21}$$

The transversal frame is $\xi_1 = \frac{1}{h(u^1)}\frac{d}{du^1}\vec{\varphi}(u^1)$, $\xi_2 = \vec{a}_1$, where $\vec{\varphi}(u^1)$ is a flat curve, $h(u^1) = \exp(\int k_1(u^1)du^1)$, $k_1(u^1)$ is obtained from the decomposition

$$\frac{d^2\vec{\varphi}(u^1)}{(du^1)^2} = k_1(u^1)\frac{d\vec{\varphi}(u^1)}{du^1} + k_2(u^1)\vec{\varphi}(u^1),$$

which is a consequence of regularity immersion condition

$$|\vec{\varphi}(u^1), \frac{d\vec{\varphi}(u^1)}{du^1}, \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n| \neq 0 \ \ \textit{for all } u^1. \tag{22}$$

Proof. We take tangent and transversal frame as in Lemmas 1 and 2 (**A**). From equations (26) we get $\frac{\partial}{\partial u^i}h_{11}^1 = 0$ and $\Gamma_{ij}^1 = 0, \ i, j = \overline{2,n}$. Thus, $h_{11}^1 = h(u^1)$ and $\Gamma_{ij}^k = 0, \ i, j, k = \overline{1,n}$.

Since we have transversal distribution of type (**A**), the only nonzero transversal form can be $\tau_1^2(e_1)$. The h^2-Codazzi equations (5) for the tangent frame vectors ($j, k \neq 1$) are

$$\frac{\partial}{\partial u^i}h_{jk}^2 = \frac{\partial}{\partial u^j}h_{ik}^2 \ \text{ for } \ i, j, k = \overline{1,n}.$$

Therefore there exists a function $g(u^1,\dots,u^n)$ such that components of the affine fundamental form are

$$h^2_{ij}=\frac{\partial^2 g}{\partial u^i\partial u^j}.$$

Let us remind that ξ_1 depends on u^1 only, ξ_2 is a constant vector field. We have the Gauss decomposition

$$\vec{r}_{11}=h(u^1)\xi_1(u^1)+\frac{\partial^2 g}{\partial u^1\partial u^1}\xi_2.$$

Denote $h(u^1)\xi_1(u^1)=\frac{d}{du^1}\vec{\varphi}(u^1)$, $\xi_2=\vec{a}_1$. By integrating the Gauss decompositions

$$\vec{r}_{11}=\frac{d\vec{\varphi}(u^1)}{du^1}+\frac{\partial^2 g}{\partial u^1\partial u^1}\vec{a}_1,\quad \vec{r}_{1j}=\frac{\partial^2 g}{\partial u^1\partial u^j}\vec{a}_1,\ j=\overline{2,n},\quad \vec{r}_{ij}=\frac{\partial^2 g}{\partial u^i\partial u^j}\vec{a}_1,\quad i,j=\overline{2,n},$$

we get

$$\vec{r}_1=\vec{\varphi}(u^1)+\frac{\partial g}{\partial u^1}\vec{a}_1,\ \vec{r}_i=\frac{\partial g}{\partial u^i}\vec{a}_1+\vec{a}_i,\ i=\overline{2,n}.$$

Thus, a position-vector of immersion in case **A** takes the form (21)

$$\vec{r}=g(u^1,\dots,u^n)\vec{a}_1+\int\vec{\varphi}(u^1)du^1+\sum_{i=2}^{n}u^i\vec{a}_i,$$

where the regularity condition holds (22):

$$\left|\vec{\varphi}(u^1),\frac{d\vec{\varphi}(u^1)}{du^1},\vec{a}_1,\vec{a}_2,\dots,\vec{a}_n\right|\neq 0\ \forall u^1.$$

Since the affine immersion is equiaffine immersion with one-dimensional Weingarten mapping, we have the Weingarten decomposition (34).

In our case $\xi_1=\frac{1}{h(u^1)}\frac{d}{du^1}\vec{\varphi}(u^1)$, then

$$\frac{d\xi_1}{du^1}=\frac{d}{du^1}\left(\frac{1}{h(u^1)}\frac{d\vec{\varphi}(u^1)}{du^1}\right)=-\frac{1}{h^2(u^1)}\frac{dh(u^1)}{du^1}\frac{d\vec{\varphi}(u^1)}{du^1}+\frac{1}{h(u^1)}\frac{d^2\vec{\varphi}(u^1)}{(du^1)^2}.$$

Due to the form of immersion position-vector and (22), we can choose the coordinate system as follows (here 1 is at $(i+2)$-th coordinate place for $\vec{a}_i$):

$$\vec{\eta}(u^1)=\{\varphi^1(u^1),\varphi^2(u^1),0,\dots,0\},\ \vec{a}_i=\{0,\dots,0,1,0,\dots,0\},i=\overline{1,n}.$$

Since the vectors $\vec{\varphi}(u^1)$ and $\frac{d}{du^1}\vec{\varphi}(u^1)$ are linearly independent, we have

$$\frac{d^2\vec{\varphi}}{(du^1)^2}=k_1(u^1)\frac{d\vec{\varphi}}{du^1}+k_2(u^1)\vec{\varphi}.$$

To satisfy (34), it is necessary

$$\frac{1}{h(u^1)}\frac{dh(u^1)}{du^1}=k_1(u^1),\ \text{i.e. } h(u^1)=h^1_{11}=\exp\Big(\int k_1(u^1)du^1\Big).$$

Thus,

$$\frac{d\xi_1}{du^1}=\frac{k_2(u^1)}{h(u^1)}\Big(\vec{r}_1-\frac{\partial g}{\partial u^1}\xi_2\Big).$$

Therefore, the coordinate functions for the Weingarten mapping image and $\tau_1^2(e_1)$ are

$$s^1(e_1) = -\frac{k_2(u^1)}{h(u^1)}, \quad s^i(e_1) = 0, \quad \tau_1^2(e_1) = -\frac{\partial g}{\partial u^1}\frac{k_2(u^1)}{h(u^1)}, \tag{23}$$

which completely agree with the Codazzi equations (6) for S_1, namely

$$\nabla_{e_i}(S_1 e_1) = 0, \text{ i.e. } \frac{\partial s^k(e_1)}{\partial u^i} = 0, \ k = \overline{1,n}, \quad i = \overline{2,n}.$$

Thus, we get the parametrization of affine immersion with given properties. □

4. Parametrization in the case of type B.

Theorem 2. *Let $f\colon (M^n, \nabla) \to (\mathbb{R}^{n+2}, D)$ be an affine immersion with pointwise codimension 2, equiaffine structure, flat connection ∇, one-dimensional Weingarten mapping, and none of the transversal frame vectors is constant. Then there exists a parametrization*

$$\vec{r} = g(u^2,\dots,u^n)\int \lambda_1(u^1)\vec{\eta}(u^1)du^1 + \int \left(v(u^1) - u^1\lambda_1(u^1)\right)\vec{\eta}(u^1)du^1 + \\ + \sum_{i=1}^{n} u^i \int \lambda_i(u^1)\vec{\eta}(u^1)du^1. \tag{24}$$

The transversal frame is $\bar{\xi}_1 = \frac{1}{c(u^1)}\frac{d}{du^1}\vec{\eta}(u^1)$, $\bar{\xi}_2 = \vec{\eta}(u^1)$, $c(u^1) = \exp(\int k_1(u^1)du^1)$, where $k_1(u^1)$ is obtained from the decomposition

$$\frac{d^2\vec{\eta}(u^1)}{(du^1)^2} = k_1(u^1)\frac{d\vec{\eta}(u^1)}{du^1} + k_2(u^1)\vec{\eta}(u^1) + \sum_{i=1}^{n} k_{i+2}(u^1)\int \lambda_i(u^1)\vec{\eta}(u^1)du^1,$$

which is a consequence of the regularity immersion condition

$$|\vec{\eta}(u^1), \frac{d\vec{\eta}(u^1)}{du^1}, \int \lambda_1(u^1)\vec{\eta}(u^1)du^1, \dots, \int \lambda_n(u^1)\vec{\eta}(u^1)du^1| \neq 0 \ \forall u^1. \tag{25}$$

Proof. We take tangent and transversal frames as in Lemmas 1 and 2 (**B**). Then we have the properties of main affine characteristics as in Lemma 1, as well as (19) and the transversal frame (20).

The h^1-Codazzi equations (5) for the tangent frame vectors ($j, k \neq 1$) are

$$\frac{\partial}{\partial u^j}h^1_{11} = \tau^1_2(e_1)h^2_{1j}, \quad \tau^1_2(e_1)h^2_{jk} = -\Gamma^1_{jk}h^1_{11}, \tag{26}$$

The h^2-Codazzi equations (5) for the tangent frame vectors ($i, j, k \neq 1$) are

$$\frac{\partial}{\partial u^1}h^2_{j1} = \frac{\partial}{\partial u^j}h^2_{11}, \quad \frac{\partial}{\partial u^i}h^2_{j1} = \frac{\partial}{\partial u^j}h^2_{i1}, \\ \frac{\partial}{\partial u^1}h^2_{jk} = \frac{\partial}{\partial u^j}h^2_{1k} - \Gamma^1_{jk}h^2_{11}, \\ \frac{\partial}{\partial u^i}h^2_{jk} - \Gamma^1_{ik}h^2_{j1} = \frac{\partial}{\partial u^j}h^2_{ik} - \Gamma^1_{jk}h^2_{i1}. \tag{27}$$

Flatness of ∇ implies $\frac{\partial}{\partial u^1}\Gamma^1_{ij} = 0$, $\frac{\partial}{\partial u^i}\Gamma^1_{jk} = \frac{\partial}{\partial u^j}\Gamma^1_{ik}$, $i,j,k \neq 1$. Therefore, there exists a function $g(u^2,\dots,u^n)$ such that

$$\Gamma^1_{ij} = \frac{\partial^2 g}{\partial u^i \partial u^j}, \quad i,j = \overline{2,n}. \tag{28}$$

Using the Codazzi equations (26) and (28) for h^1 we conclude that the components of h^2 now are

$$\bar{h}^2_{1j} = \frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^j}, \quad \bar{h}^2_{jk} = -\frac{1}{c(u^1)}\frac{\partial^2 g}{\partial u^i \partial u^j}\bar{h}^1_{11}, \tag{29}$$

where $c(u^1) = \tau^1_2(e_1) \neq 0$. From the first group of system (27) and (29) we get

$$\frac{\partial}{\partial u^1}\left(\frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^j}\right) = \frac{\partial \bar{h}^2_{11}}{\partial u^j}, \quad -\frac{1}{c^2(u^1)}\frac{dc(u^1)}{du^1}\frac{\partial \bar{h}^1_{11}}{\partial u^j} + \frac{1}{c(u^1)}\frac{\partial^2 \bar{h}^1_{11}}{\partial u^j \partial u^1} = \frac{\partial \bar{h}^2_{11}}{\partial u^j},$$

$$\frac{\partial}{\partial u^j}\left(-\frac{1}{c^2(u^1)}\frac{dc(u^1)}{du^1}\bar{h}^1_{11} + \frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^1}\right) = \frac{\partial \bar{h}^2_{11}}{\partial u^j},$$

$$-\frac{1}{c^2(u^1)}\frac{dc(u^1)}{du^1}\bar{h}^1_{11} + \frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^1} = \bar{h}^2_{11} + \mu(u^1),$$

$$\bar{h}^2_{11} = -\frac{1}{c^2(u^1)}\frac{dc(u^1)}{du^1}\bar{h}^1_{11} + \frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^1} - \mu(u^1). \tag{30}$$

Using the third group of equations of system (27)–(30) we obtain

$$\frac{\partial}{\partial u^1}\left(-\frac{1}{c(u^1)}\frac{\partial^2 g}{\partial u^i \partial u^j}\bar{h}^1_{11}\right) =$$

$$= \frac{\partial}{\partial u^i}\left(\frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^j}\right) - \frac{\partial^2 g}{\partial u^i \partial u^j}\left(-\frac{1}{c^2(u^1)}\frac{dc(u^1)}{du^1}\bar{h}^1_{11} + \frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^1} - \mu(u^1)\right),$$

$$0 = \frac{\partial}{\partial u^i}\left(\frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^j}\right) + \frac{\partial^2 g}{\partial u^i \partial u^j}\mu(u^1), \tag{31}$$

$$\bar{h}^2_{1j} = \frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^j} = -\frac{\partial g}{\partial u^j}\mu(u^1) + \lambda_j(u^1), \tag{32}$$

$$\bar{h}^1_{11} = c(u^1)\Big(- g\mu(u^1) + \sum_{j=2}^{n} \lambda_j(u^1)u^j + v(u^1)\Big). \tag{33}$$

One can see that the second and forth groups of equations of system (27) are fulfilled. From (30), (20), and the Gauss decomposition

$$\vec{r}_{11} = \bar{h}^1_{11}\frac{1}{c(u^1)}\frac{d\vec{\eta}(u^1)}{du^1} + \Big(- \frac{1}{c^2(u^1)}\frac{dc(u^1)}{du^1}\bar{h}^1_{11} + \frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^1} - \mu(u^1)\Big)\vec{\eta}(u^1),$$

$$\vec{r}_{1j} = \frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^j}\vec{\eta}(u^1), \; j = \overline{2,n},$$

we obtain

$$\vec{r}_1 = \frac{1}{c(u^1)}\bar{h}^1_{11}\vec{\eta}(u^1) - \int \mu(u^1)\vec{\eta}(u^1)du^1.$$

Using this equality and (28), (29) we get the remaining equations of the Gauss decomposition

$$\vec{r}_{ij} = \frac{\partial^2 g}{\partial u^i \partial u^j}\left(\frac{1}{c(u^1)}\bar{h}^1_{11}\vec{\eta}(u^1) - \int \mu(u^1)\vec{\eta}(u^1)du^1\right) - \frac{1}{c(u^1)}\frac{\partial^2 g}{\partial u^i \partial u^j}\bar{h}^1_{11}\vec{\eta}(u^1) =$$
$$= -\frac{\partial^2 g}{\partial u^i \partial u^j}\int \mu(u^1)\vec{\eta}(u^1)du^1, \quad i,j = \overline{2,n}.$$

By using (32) and integrating the following equalities

$$\vec{r}_{i1} = \frac{1}{c(u^1)}\frac{\partial \bar{h}^1_{11}}{\partial u^i}\vec{\eta}(u^1) = \left(-\frac{\partial g}{\partial u^i}\mu(u^1) + \lambda_i(u^1)\right)\vec{\eta}(u^1), \quad i = \overline{2,n},$$
$$\vec{r}_{ij} = -\frac{\partial^2 g}{\partial u^i \partial u^j}\int \mu(u^1)\vec{\eta}(u^1)du^1, \quad i,j = \overline{2,n},$$

we get

$$\vec{r}_i = -\frac{\partial g}{\partial u^i}\int \mu(u^1)\vec{\eta}(u^1)du^1 + \int \lambda_i(u^1)\vec{\eta}(u^1)du^1, \quad i = \overline{2,n}.$$

By integrating

$$\vec{r}_1 = \Big(- g\mu(u^1) + \sum_{j=2}^{n} \lambda_j(u^1)u^j + v(u^1)\Big)\vec{\eta}(u^1) - \int \mu(u^1)\vec{\eta}(u^1)du^1,$$
$$\vec{r}_i = -\frac{\partial g}{\partial u^i}\int \mu(u^1)\vec{\eta}(u^1)du^1 + \int \lambda_i(u^1)\vec{\eta}(u^1)du^1, \quad i = \overline{2,n}$$

we get an immersion parametrization

$$\vec{r} = -(g(u^2,\dots,u^n) + u^1)\int \mu(u^1)\vec{\eta}(u^1)du^1 +$$
$$+\int v(u^1)\vec{\eta}(u^1)du^1 + \int u^1\mu(u^1)\vec{\eta}(u^1)du^1 + \sum_{i=2}^{n} u^i \int \lambda_i(u^1)\vec{\eta}(u^1)du^1.$$

Denote $\lambda_1(u^1) = -\mu(u^1)$ and get the parametrization (24). We need (25) for all u^1 to provide the immersion regularity, namely

$$|\vec{\eta}(u^1), \frac{d\vec{\eta}(u^1)}{du^1}, \int \lambda_1(u^1)\vec{\eta}(u^1)du^1, \dots, \int \lambda_n(u^1)\vec{\eta}(u^1)du^1| \neq 0.$$

Denote the image of Weingarten mapping as $S_1(e_1) = U = s^i(e_1)e_i$. The one-dimensional property of Weingarten mapping and equiaffine structure imply

$$\frac{d\bar{\xi}_1}{du^1} = -s^i(e_1)\vec{r}_i + \tau^2_1(e_1)\bar{\xi}_2. \tag{34}$$

Since in our case $\bar{\xi}_1 = \frac{1}{c(u^1)}\frac{d\vec{\eta}(u^1)}{du^1}$, $\bar{\xi}_2 = \vec{\eta}(u^1)$ then

$$\frac{d\bar{\xi}_1}{du^1} = \frac{d}{du^1}\left(\frac{1}{c(u^1)}\frac{d\vec{\eta}(u^1)}{du^1}\right) = -\frac{1}{c^2(u^1)}\frac{dc(u^1)}{du^1}\frac{d\vec{\eta}(u^1)}{du^1} + \frac{1}{c(u^1)}\frac{d^2\vec{\eta}(u^1)}{(du^1)^2}.$$

Because of (25) we have the following decompositions

$$\frac{d^2\vec{\eta}(u^1)}{(du^1)^2} = k_1(u^1)\frac{d\vec{\eta}(u^1)}{du^1} + k_2(u^1)\vec{\eta}(u^1) + \sum_{i=1}^{n} k_{i+2}(u^1)\int \lambda_i(u^1)\vec{\eta}(u^1)du^1.$$

The Weingarten decomposition (34) implies

$$\frac{1}{c(u^1)}\frac{dc(u^1)}{du^1} = k_1(u^1), \text{ i.e. } c(u^1) = \exp\Big(\int k_1(u^1)du^1\Big).$$

We have

$$\int \lambda_1(u^1)\vec{\eta}(u^1)du^1 = \vec{r}_1 - \Big(g\lambda_1(u^1) + \sum_{j=2}^{n}\lambda_j(u^1)u^j + v(u^1)\Big)\vec{\eta}(u^1),$$
$$\int \lambda_i(u^1)\vec{\eta}(u^1)du^1 = \vec{r}_i - \frac{\partial g}{\partial u^i}\int \lambda_1(u^1)\vec{\eta}(u^1)du^1, \quad i = \overline{2,n}.$$

Since $g\lambda_1(u^1) + \sum\limits_{j=2}^{n}\lambda_j(u^1)u^j + v(u^1) = \dfrac{\bar{h}^1_{11}}{c(u^1)}$, we see that

$$\int \lambda_1(u^1)\vec{\eta}(u^1)du^1 = \vec{r}_1 - \frac{\bar{h}^1_{11}}{c(u^1)}\vec{\eta}(u^1),$$
$$\int \lambda_i(u^1)\vec{\eta}(u^1)du^1 = \vec{r}_i - \frac{\partial g}{\partial u^i}\vec{r}_1 + \frac{\partial g}{\partial u^i}\frac{\bar{h}^1_{11}}{c(u^1)}\vec{\eta}(u^1), \quad i = \overline{2,n}.$$

Therefore,

$$\begin{aligned}\frac{d\bar{\xi}_1}{du^1} &= \frac{k_2(u^1)}{c(u^1)}\vec{\eta}(u^1) + \frac{k_3(u^1)}{c(u^1)}\Big(\vec{r}_1 - \frac{\bar{h}^1_{11}}{c(u^1)}\vec{\eta}(u^1)\Big) + \\ &+ \sum_{i=2}^{n}\frac{k_{i+2}(u^1)}{c(u^1)}\Big(\vec{r}_i - \frac{\partial g}{\partial u^i}\vec{r}_1 + \frac{\partial g}{\partial u^i}\frac{\bar{h}^1_{11}}{c(u^1)}\vec{\eta}(u^1)\Big) = \\ &= \Big(\frac{k_3(u^1)}{c(u^1)} - \sum_{i=2}^{n}\frac{k_{i+2}(u^1)}{c(u^1)}\frac{\partial g}{\partial u^i}\Big)\vec{r}_1 + \sum_{i=2}^{n}\frac{k_{i+2}(u^1)}{c(u^1)}\vec{r}_i + \\ &+ \Big(\frac{k_2(u^1)}{c(u^1)} - \frac{k_3(u^1)}{c^2(u^1)}\bar{h}^1_{11} + \sum_{i=2}^{n}\frac{k_{i+2}(u^1)}{c^2(u^1)}\frac{\partial g}{\partial u^i}\bar{h}^1_{11}\Big)\vec{\eta}(u^1).\end{aligned}$$

Thus, we have the coordinate functions of the image of Weingarten mapping and $\tau^2_1(e_1)$ as

$$\begin{aligned} s^1(e_1) &= -\frac{k_3(u^1)}{c(u^1)} + \sum_{i=2}^{n}\frac{k_{i+2}(u^1)}{c(u^1)}\frac{\partial g}{\partial u^i}, \\ s^i(e_1) &= -\frac{k_{i+2}(u^1)}{c(u^1)}, \quad i = \overline{2,n}, \\ \tau^2_1(e_1) &= \frac{k_2(u^1)}{c(u^1)} - \frac{k_3(u^1)}{c^2(u^1)}\bar{h}^1_{11} + \sum_{i=2}^{n}\frac{k_{i+2}(u^1)}{c^2(u^1)}\frac{\partial g}{\partial u^i}\bar{h}^1_{11}, \end{aligned} \qquad (35)$$

which satisfy the Codazzi equations (6) for S_1:

$$\nabla_{e_i}(S_1e_1) = 0, \text{ i.e. } \frac{\partial s^k(e_1)}{\partial u^i} + s^l(u^1)\Gamma^k_{il} = 0,\ k = \overline{1,n},\quad i = \overline{2,n}.$$

Thus, we get the parametrization of an affine immersion with given properties. □

Corollary 1. *Let $f\colon (M^n,\nabla) \to (\mathbb{R}^{n+2}, D)$ be an affine immersion with pointwise codimension 2, equiaffine structure, flat connection ∇, one-dimensional Weingarten mapping, and none of the transversal frame vectors is constant. Then it can be locally parameterized by (36) or (37):*

$$\vec r = (g(u^2,\dots,u^n) + u^1)\vec a + \int v(u^1)\vec\eta(u^1)du^1 + \sum_{i=2}^{n} u^i \int \lambda_i(u^1)\vec\eta(u^1)du^1, \tag{36}$$

where the regularity immersion condition

$$|\vec a, \vec\eta(u^1), \frac{d\vec\eta(u^1)}{du^1}, \int \lambda_2(u^1)\vec\eta(u^1)du^1, \dots, \int \lambda_n(u^1)\vec\eta(u^1)du^1| \neq 0\ \forall u^1$$

holds, the transversal frame is $\bar\xi_1 = \exp(-\int k_1(u^1)du^1)\frac{d}{du^1}\vec\eta(u^1)$, $\bar\xi_2 = \vec\eta(u^1)$, where $k_1(u^1)$ is obtained from the decomposition

$$\frac{d^2\vec\eta(u^1)}{(du^1)^2} = k_1(u^1)\frac{d\vec\eta(u^1)}{du^1} + k_2(u^1)\vec\eta(u^1) + k_3(u^1)\vec a + \sum_{i=2}^{n} k_{i+2}(u^1)\int \lambda_i(u^1)\vec\eta(u^1)du^1,$$

which is a consequence of the regularity immersion condition;

$$\vec r = (g(u^2,\dots,u^n) + u^1)\vec\rho(u^1) + \int (v(u^1) - u^1)\frac{d\vec\rho(u^1)}{du^1}du^1 + \sum_{i=2}^{n} u^i \int \lambda_i(u^1)\frac{d\vec\rho(u^1)}{du^1}du^1, \tag{37}$$

where the following regularity immersion condition holds for all u^1:

$$|\vec\rho(u^1), \vec\rho\,'(u^1), \vec\rho\,''(u^1), \int \lambda_2(u^1)\vec\rho\,'(u^1)du^1, \dots, \int \lambda_n(u^1)\vec\rho\,'(u^1)du^1| \neq 0,$$

the transversal frame is $\bar\xi_1 = \exp(-\int k_1(u^1)du^1)\vec\rho\,''(u^1)$, $\bar\xi_2 = \vec\rho\,'(u^1)$, where $k_1(u^1)$ is obtained from the decomposition

$$\vec\rho\,'''(u^1) = k_1(u^1)\vec\rho\,''(u^1) + k_2(u^1)\vec\rho\,'(u^1) + k_3(u^1)\vec\rho(u^1) + \sum_{i=2}^{n} k_{i+2}(u^1)\int \lambda_i(u^1)\vec\rho\,'(u^1)du^1,$$

which is a consequence of the regularity immersion condition.

Proof. The first parametrization is obtained from (24) when $\lambda_1(u^1) \equiv 0$, the second one when $\lambda_1(u^1) \neq 0$. □

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