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ESSENTIAL SPECTRA IN NON-ARCHIMEDEAN FIELDS

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In the paper we extend some aspects of the essential spectra theory of linear operators acting in non-Archimedean (or p-adic) Banach spaces. In particular, we establish sufficient conditions for the relations between the essential spectra of the sum of two bounded linear operators and the union of their essential spectra. Moreover, we give essential prerequisites by studying the duality between p-adic upper and p-adic lower semi-Fredholm operators. We close this paper by giving some properties of the essential spectra.

1. Introduction. Let $\mathbb{K}$ be a field with the unit element $1_{\mathbb{K}}$. A valuation on $\mathbb{K}$ is a map $|\cdot|\colon \mathbb{K} \longrightarrow \mathbb{R}_+$ satisfying: $|\alpha| = 0$ if, and only if, $\alpha = 0$; $|\alpha\beta| = |\alpha||\beta|$ for any α, $\beta \in \mathbb{K}$; and $|\alpha| \leq 1$ implies $|\alpha + 1_{\mathbb{K}}| \leq c$ for some constant $c \geq 1$ (independent of α). The valuation is non-Archimedean if $c = 1$, equivalently the ultrametric inequality ($|\alpha + \beta| \leq \max(|\alpha|, |\beta|)$ for any α, $\beta \in \mathbb{K}$) holds. The valuation $|\cdot|$ is trivial if $|\alpha| = 1$ for every non-zero α and $|0| = 0$. A field $\mathbb{K}$ is said to be non-Archimedean if it is endowed with a non-Archimedean valuation.

Throughout this paper $\mathbb{K}$ is a nontrivial non-Archimedean field which is complete (every Cauchy sequence of elements of $\mathbb{K}$ converges, where the valuation $|\cdot|$ induces a metric $d\colon \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{R}_+$ defined by $d(\alpha, \beta) = |\alpha - \beta|$, $\alpha, \beta \in \mathbb{K}$).

Let X be a vector space over $\mathbb{K}$. A non-Archimedean norm on X is a map $\|\cdot\|\colon X \longrightarrow \mathbb{R}_+$ such that: $\|x\| = 0$ if, and only if, $x = 0$; $\|\lambda x\| = |\lambda|\ \|x\|$ for any $x \in X$ and any $\lambda \in \mathbb{K}$; and $\|x + y\| \leq \max(\|x\|, \|y\|)$ for any x, $y \in X$. A vector space endowed with non-Archimedean norm is called non-Archimedean normed space. A non-Archimedean Banach space is a non-Archimedean normed vector space, which is complete (every Cauchy sequence of elements of X converges) with respect to the natural metric induced by the norm $d(x, y) = \|x - y\|$ for any $x, y \in X$.

Non-Archimedean valued fields are the fundamental of the non-Archimedean functional analysis, which has been studied by several authors, especially, due to A. F. Monna [9], W. H. Schikhof [13], and A. C. M. van Rooij [15]. One of the main objectives in this analysis consists in studying the spectral theory of bounded linear operators, which was constructed by M. M. Vishik [17] for a class of operators that he named analytic operators with compact spectrum (when the nontrivial valued field is algebraically closed).

In the classical Banach spaces (i.e. $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), one of modern approaches that is related to spectral analysis is the essential spectrum which has various uses in mathematics

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and physics. This approach was first inaugurated by H. Weyl [18] around 1909 when he defined the essential spectrum for a self-adjoint operator A on a Hilbert space as the set of all points of the spectrum of A that are not isolated eigenvalues of finite algebraic multiplicity. If A is not self-adjoint bounded operator (or just assumed to be closed and densely defined in an arbitrary Banach space), we can find several definitions of the essential spectrum which coincide with self adjoint operators on Hilbert spaces (see, for example, [6, 12]). Furthermore, for important characterizations concerning essential spectra in classical Banach spaces, we refer the reader to the book of A. Jeribi [7], which is a comprehensive survey of results concerning different types of essential spectra.

However, after the reading of literature about the essential spectrum, there are not many manuscripts in the non-Archimedean setting except in [4, 5], where it was defined only one type of the essential spectrum and proved that the latter is not affected by the addition of a completely continuous operator. This is our impetus to extend the other types of essential spectra in non-Archimedean fields, which we will examine in connection with various classes of linear operators defined through the kernels and ranges, the most important of these classes are p-adic Fredholm operators, p-adic upper semi-Fredholm operators, and p-adic lower semi-Fredholm operators.

In the second section, we give and establish some basic concepts of non-Archimedean functional analysis, which consist of some facts of the theory of linear operator in a non-Archimedean Banach space that is needed throughout this work.

In the third section, we present our main purpose by establishing sufficient conditions for the relations between essential spectra of the sum of two bounded linear operators defined on a non-Archimedean Banach space and the union of their essential spectra. Furthermore, we give some essential requirements by examining the duality between p-adic upper semi-Fredholm and p-adic lower semi-Fredholm and combining the obtained results to show that such relations transfers to the equality. Finally, we end up by giving some properties of the essential spectra.

2. Preliminaries and auxiliary results. This section aims to construct basic concepts of non-Archimedean functional analysis, including some results of the theory of linear operators in a non-Archimedean Banach space that is required in the sequel.

The symbol X, Y stand for a non-Archimedean Banach spaces over $\mathbb{K}$. For a linear operator A acting from X into Y, we write $D(A) \subseteq X$ to denote the domain of A. The operator A is called linear if $D(A)$ is a linear subspace of X and if $A(\alpha x+\beta y) = \alpha Ax + \beta Ay$ for any $\alpha, \beta \in \mathbb{K}$ and $x, y \in D(A)$.

Let $A\colon X \longrightarrow Y$ be a linear operator. The *kernel* $N(A) \subseteq D(A)$ and the *range* $R(A) \subseteq Y$, are respectively, defined by

$$N(A) = \{x \in D(A)\colon Ax = 0\}, \quad R(A) = \{Ax\colon x \in D(A)\}.$$

Let X, Y be two non-Archimedean Banach spaces.

(i) A linear operator $A\colon X \longrightarrow Y$ is said to be *bounded* if $D(A) = X$ and there exists $C > 0$ such that $\|Ax\| \leq C\|x\|$ for all $x \in X$.

(ii) The *norm* $\|A\|$ *of the bounded linear operator* A is defined by

$$\|A\| := \sup\left\{\frac{\|Ax\|}{\|x\|} : x \neq 0\right\}.$$

We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y.

Remark 1. The quotient space $\hat{X} = X/N(A)$ is a non-Archimedean Banach space endowed with the non-Archimedean norm

$$\|\tilde{x}\| = \inf\{\|x\|: x \in \tilde{x}\} = \inf: \{\|x - y\|: y \in N(A)\} := d(x, N(A)).$$

The bounded linear operator $\hat{A}$ defined by $\hat{A}\tilde{x} = Ax$ for each $x \in \tilde{x}$ has an inverse $\hat{A}^{-1}$. The reduced minimum modulus $\gamma(A)$ of A is defined by $\gamma(A) := \begin{cases} \|\hat{A}^{-1}\|^{-1}, & \text{if } A \neq 0, \\ \infty, & \text{if } A = 0. \end{cases}$

Let X, Y be two non-Archimedean Banach spaces. If $A \in \mathcal{L}(X, Y)$, then:
(i) A is said to be *injective* if $N(A) = \{0\}$.
(ii) A is said to be *surjective* if $R(A) = Y$.
(iii) A is said to be *invertible* if it is both injective and surjective.

Let X and Y be two non-Archimedean Banach spaces and let $A \in \mathcal{L}(X, Y)$.
(i) The *spectrum* $\sigma(A)$ of the linear operator A is defined by

$$\sigma(A) := \{\lambda \in \mathbb{K} : A - \lambda I \text{ has not a bounded inverse}\}.$$

(ii) The *resolvent set* $\rho(A)$ is the complement of the set $\sigma(A)$ in $\mathbb{K}$.

Let X be a non-Archimedean Banach space.
(i) $A \subset X$ is said to be of *countable type* if it is the closed linear hull of a countable set.
(ii) $A \subset X$ is called *compactoïd* if for every $\epsilon > 0$ there is a finite set $B \subset X$ such that

$$A \subset \mathrm{co}(B) + \{x \in X: \|x\| \leq \epsilon\},$$

where $\mathrm{co}(B)$ is the absolutely convex hull of B, i.e.,

$$\mathrm{co}(B) = \Big\{\sum_{x \in X} \lambda_x x: \lambda_x \in \mathbb{K}, |\lambda_x| \leq 1, X \text{ is finite subset of } B\Big\}.$$

Let X and Y be two non-Archimedean Banach spaces.
(i) $K \in \mathcal{L}(X, Y)$ is called a *compact operator* if $K(B_X)$ is compactoïd, where B_X denotes the closed unit ball of X. The collection of compact operators from X to Y will be denoted by $\mathcal{K}(X, Y)$.
(ii) $P \in \mathcal{L}(X)$ is called a *projection* if $P^2 = P$.

In the sequel, we denote by X' the topological dual of X (i.e., $X' = \mathcal{L}(X, \mathbb{K})$) and by X'' the bidual of X (i.e., $X'' = (X')'$).

Proposition 1. *Let X be a non-Archimedean Banach space. If X is a nonzero finite-dimensional space, then* $\dim X' \leq \dim X$.

Proof. We have two cases. First if $X' = \{0\}$, then $0 = \dim X' \leq \dim X$.

Second case, if $X' \neq \{0\}$, then $\dim X' = \dim X$. Indeed, suppose that $\dim X = n$ and let $\{x_1, ..., x_n\}$ be a basis of X. For each $i = 1, ..., n$, define a linear functional $f_i: X \longrightarrow \mathbb{K}$ by setting

$$(f_i, x_j) = \begin{cases} 1_{\mathbb{K}}, & \text{if } i = j \text{ else}, \\ 0_{\mathbb{K}}. \end{cases}$$

Note that $\{f_1, ..., f_n\}$ is linearly independent. In point of fact, suppose that $\alpha_1, ..., \alpha_n \in \mathbb{K}$ such that

$$\sum_{1 \leq i \leq n} \alpha_i f_i = 0 \tag{1}$$

Since $(\sum_{1 \leq i \leq n} \alpha_i f_i, x_j) = \sum_{1 \leq i \leq n} \alpha_i (f_i, x_j) = \alpha_i$ for all $i \in \{1, ..., n\}$, it follows from equality (1) that $\alpha_i = 0$ for all $i \in \{1, ..., n\}$. As a result, $\{f_i\}_{1 \leq i \leq n}$ is linearly independent.
Now, it remains to prove that $\{f_i\}_{1 \leq i \leq n}$ spans X', we show that for all $\varphi \in X'$

$$\varphi = \sum_{1 \leq i \leq n} (\varphi, x_i) f_i.$$

It suffices to show that both sides take the same value on every vector $x \in X$. So, it follows that
$$(\varphi, x) = \Big(\varphi, \sum_{1\le j\le n} x_j(f_j, x)\Big) = \sum_{1\le j\le n} (\varphi, x_j)(f_j, x).$$
On the other hand,
$$\Big(\sum_{1\le i\le n} (\varphi, x_i)f_i, x\Big) = \sum_{1\le i\le n} (\varphi, x_i)(f_i, x).$$
Hence, $\{f_1, ..., f_n\}$ spans X', and therefore forms a basis of X'. Consequently, $\dim X' = n$. That is, $\dim X' \le \dim X$. □

Let X, Y be two non-Archimedean Banach spaces and let $A \in \mathcal{L}(X, Y)$. The *adjoint* of A, denoted by A^*, is defined by
$$(A^* f, x) = (f, Ax) \text{ for all } f \in Y' \text{ and } x \in X.$$

Remark 2. In contrast with the classical operator theory, there exist bounded linear operators, which do not have adjoint (see [3]).

In what follow $\breve{\mathcal{L}}(X, Y)$ denotes the set of all bounded linear operators from X to Y whose their adjoint operators exist.

Let X be a non-Archimedean Banach space.

(i) Let M and N be two subspaces of X and X' respectively, we define their *orthogonal* by
$$M^\perp = \{f \in X' : (f, x) = 0 \ \text{ for all } \ x \in M\}, \quad {}^\perp N = \{x \in X : (f, x) = 0 \ \text{ for all } \ f \in N\}.$$
(ii) Let $J_X : X \longrightarrow X''$ be the natural map defined by
$$J_X(x)(f) = f(x) \ \text{ for all } \ x \in X \ \text{ and } \ f \in X''.$$
X is called *dual-separating* (respectively, *reflexive*) if J_X is injective (respevtively, bijective isometry).

(iii) A continuous seminorm q is said to be *polar* if $q = \sup\{|f| : f \in X', |f| \le q\}$.

(iv) X is said to be *polar* if its topology is defined by a polar norm.

(v) X is said to be *strongly polar* if every continuous seminorm q on X for which $q(X) \subset \overline{|\mathbb{K}|}$ is polar, where $\overline{|\mathbb{K}|}$ is the closure of $\{|\lambda| : \lambda \in |\mathbb{K}|\}$ in $[0, \infty)$.

(vi) A subspace D of X has the *weak extension property* (*wep* in short) in X if every $g \in D'$ has an extension $\overline{g} \in X'$.

Remark 3. (i) If X is reflexive, then X is polar. (ii) If X is polar, then X is dual-separating.

In the sequel of this paper, the nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y. The index of A is defined by $i(A) = \alpha(A) - \beta(A)$. The classes of p-adic upper semi-Fredholm and p-adic lower semi-Fredholm operators from X to Y are defined, respectively, by
$$\Phi_+(X, Y) = \big\{A \in \mathcal{L}(X, Y) : \alpha(A) < \infty, \ R(A) \ \text{ is closed in } \ Y\big\},$$
$$\Phi_-(X, Y) = \big\{A \in \mathcal{L}(X, Y) : \beta(A) < \infty, \ R(A) \ \text{ is closed in } \ Y\big\}.$$
The operators in $\Phi(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y)$ are called p-adic Fredholm operators from X to Y, while $\Phi_\pm(X, Y) = \Phi_+(X, Y) \cup \Phi_-(X, Y)$ denotes the set of p-adic semi-Fredholm operators from X to Y. If $X = Y$, then $\mathcal{L}(X, Y)$, $\Phi(X, Y)$, $\Phi_\pm(X, Y)$, $\Phi_+(X, Y)$, and $\Phi_-(X, Y)$ are replaced by $\mathcal{L}(X)$, $\Phi(X)$, $\Phi_\pm(X)$, $\Phi_+(X)$, and $\Phi_-(X)$, respectively.

In this paper, we are concerned with the following sets

$$\sigma_{e1}(A)=\big\{\lambda\in\mathbb{K}\colon A-\lambda I\notin\Phi_+(X)\big\},\quad \sigma_{e2}(A)=\big\{\lambda\in\mathbb{K}\colon A-\lambda I\notin\Phi_-(X)\big\},$$
$$\sigma_{e3}(A)=\big\{\lambda\in\mathbb{K}\colon A-\lambda I\notin\Phi_\pm(X)\big\},\quad \sigma_{e4}(A)=\big\{\lambda\in\mathbb{K}\colon A-\lambda I\notin\Phi(X)\big\},$$
$$\sigma_{e5}(A)=\mathbb{K}\setminus\rho_{e5}(A),\quad \sigma_{eap}(A)=\mathbb{K}\setminus\rho_{eap}(A),\quad \sigma_{e\delta}(A)=\mathbb{K}\setminus\rho_{e\delta}(A),$$

where

$$\rho_{e5}(A)=\big\{\lambda\in\mathbb{K}\colon A-\lambda I\in\Phi(X)\ \text{ and }\ i(A-\lambda I)=0\big\},$$
$$\rho_{eap}(A)=\big\{\lambda\in\mathbb{K}\colon A-\lambda I\in\Phi_+(X)\ \text{ and }\ i(A-\lambda I)\leq 0\big\},$$
$$\rho_{e\delta}(A)=\big\{\lambda\in\mathbb{K}\colon A-\lambda I\in\Phi_-(X)\ \text{ and }\ i(A-\lambda I)\geq 0\big\}.$$

In general, we have

$$\sigma_{e3}(A)=\sigma_{e1}(A)\cap\sigma_{e2}(A)\subseteq\sigma_{e4}(A)\subseteq\sigma_{e5}(A)=\sigma_{eap}(A)\cup\sigma_{e\delta}(A). \tag{2}$$

Let X be a non-Archimedean Banach space and let $F\in\mathcal{L}(X)$.
(i) The operator F is called a *p-adic Fredholm perturbation* if $U+F\in\Phi(X)$, whenever $U\in\Phi(X)$. The set of p-adic Fredholm perturbations is denoted by $\mathcal{F}(X)$.

(ii) The operator F is called a *p-adic upper* (respectively, *lower*) *semi-Fredholm perturbation* if $U+F\in\Phi_+(X)$ (respectively, $U+F\in\Phi_-(X)$) whenever $U\in\Phi_+(X)$ (respectively, $U\in\Phi_-(X)$). The set of p-adic upper (respectively, lower) semi-Fredholm perturbations is denoted by $\mathcal{F}_+(X)$ (respectively, $\mathcal{F}_-(X)$).

Proposition 2. *Let X be a non-Archimedean Banach space.*

(i) *If $A\in\Phi(X)$ and $F\in\mathcal{F}(X)$, then $A+F\in\Phi(X)$ and $i(A+F)=i(A)$.*

(ii) *If $A\in\Phi_+(X)$ and $F\in\mathcal{F}_+(X)$, then $A+F\in\Phi_+(X)$ and $i(A+F)=i(A)$.*

(iii) *If $A\in\Phi_-(X)$ and $F\in\mathcal{F}_-(X)$, then $A+F\in\Phi_-(X)$ and $i(A+F)=i(A)$.*

Proof. The proof may be checked in the same way as in the proof of [8, Lemma 2.1] for real or complex Banach spaces. □

Lemma 1. [14, Corollary 3.2] *Let X be a non-Archimedean Banach space.*

(i) *If $A\in\Phi_+(X)$ and $B\in\Phi_+(X)$, then $AB\in\Phi_+(X)$.*

(ii) *If $AB\in\Phi_+(X)$, then $B\in\Phi_+(X)$.*

Lemma 2. *Let X be a non-Archimedean Banach space. Let $A\in\mathcal{L}(X)$ and $B\in\mathcal{L}(X)$ such that $X/R(A)$ is dual separating. If $AB\in\Phi_-(X)$, then we have $A\in\Phi_-(X)$.*

Proof. On the one hand, as in the classical case, there exists a natural isomorphism between $Y^\perp$ and $(X/Y)'$ via the operator

$$T\colon Y^\perp\to(X/Y)',\quad f\to Tf(x+Y)=f(x),$$

where Y is a subspace of X. In fact, we prove that Tf is well defined. So, suppose that $x+Y=\acute{x}+Y$, then $x-\acute{x}\in Y$, thus $0=f(x-\acute{x})=f(x)-f(\acute{x})$, that is $f(x)=f(\acute{x})$, and hence $Tf(x+Y)=Tf(\acute{x}+Y)$.

Next, we claim that T is invertible. To prove that T is injective, assume that $f\in N(T)$. Then, $Tf=0$. Hence, $0=Tf(x+Y)=f(x)$, for all $x\in X$. So, $N(T)=0$, which implies that T is injective.

To show that T is surjective, let $g\in(X/Y)'$ and define an element $f\in X'$ by

$$f(x)=g(x+Y)\ \text{ for all }\ x\in X.$$

So, we claim that $f\in Y^\perp$. That is, if $x\in Y$, then $g(x+Y)=g(Y)=0$. Thus, $f(x)=0$. So, $f\in Y^\perp$. It follows that $Tf=g$. Therefore, T is surjective, and thus T is invertible.

On the other hand, it follows from Proposition 1 that $\beta(AB) < \infty$ implies that
$$\dim(X/R(AB))' < \infty.$$
Since there exists an isomorphism between $R(AB)^\perp$ and $(X/R(AB))'$, we conclude that $\dim R(AB)^\perp < \infty$. From $R(AB) \subseteq R(A)$ it follows $R(A)^\perp \subseteq R(AB)^\perp$, which implies that
$$\dim R(A)^\perp < \infty.$$
Using the fact that $R(A)^\perp$ and $(X/R(A))'$ are isomorphic, Proposition 1 assures that
$$\dim(X/R(A))'' < \infty.$$
Since $X/R(A)$ is dual separating, the rank theorem allows us to deduce that $\dim(X/R(A)) < \infty$. The use of [16, Proposition 3.2] leads to $A \in \Phi_-(X)$. □

Remark 4. In contrast with classical setting, the fact that $AB \in \Phi_-(X, Z)$ does not imply that $A \in \Phi_-(Y, Z)$, where $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$. In order to illustrate this situation, let us consider the canonical injection $A\colon c_0 \to l^\infty$ and the projection $B\colon l^\infty \to c_0$, where

$$l^\infty = \big\{x = (x_n)_{n\in\mathbb{N}}, x_n \in \mathbb{K}\colon \|x\| = \sup_{n\in\mathbb{N}}|x_n| < \infty\big\}, c_0 = \big\{x = (x_n)_{n\in\mathbb{N}}, x_n \in \mathbb{K}\colon \lim_n|x_n| = 0\big\},$$

and $\mathbb{K}$ is not spherically complete (there exists a decreasing sequence of balls in $\mathbb{K}$, which has an empty intersection). We have that $AB = I$, where I is the identity operator on l^∞. Hence, $AB \in \Phi_-(l^\infty)$. But $\beta(A) = \dim(l^\infty/c_0)$ which is not finite. So, we get $A \notin \Phi_-(c_0, l^\infty)$.

Using the same method of S. Vega [16], we can set forward the next lemma.

Lemma 3. *Let X be a non-Archimedean Banach space. Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(X)$ such that $A \in \Phi_-(X)$. Let $t \in (0.1)$ and $P\colon X \to R(A)$ be a continuous linear projection such that $\|P\| \le t^{-1}$. If for every $\lambda \in \mathbb{K}$*

$$\|B\| < \frac{t}{\|A\|}\gamma\big(\lambda PA\big), \tag{3}$$

then we have $(B - \lambda I) \in \Phi_-(X)$ for every $\lambda \in \mathbb{K}$.

Proof. Let $\lambda \in \mathbb{K}$. We have $P(\lambda A)$ which is a surjective map. Since the inequality (3) holds, it follows that $\|BA\| < t\gamma\big(\lambda PA\big)$, for every $\lambda \in \mathbb{K}$. Therefore, $\|PBA\| < \gamma\big(\lambda PA\big)$ for every $\lambda \in \mathbb{K}$. Using [16, Corollary 4.3], we obtain $P(BA - \lambda A)$ which is a surjective map. Applying [16, Lemma 5.1] to the operator $BA - \lambda A\colon X \to X$ and the subspace $W = R(A)$, we get $(B - \lambda I)A \in \Phi_-(X)$. Based on Lemma 2, we conclude that $(B - \lambda I) \in \Phi_-(X)$. □

Lemma 4 ([11], Lemma 4.2)**.** *If two of the three operators A, B, and AB have indices, then the third one has also an index and $i(AB) = i(A) + i(B)$.*

3. Main results. In this section, we establish sufficient conditions for the relations between essential spectra of the sum of two bounded linear operators defined on non-Archimedean Banach space and that of each operator. Moreover, we study some characterizations and properties of these essential spectra.

Theorem 1. *Let A and B be two bounded linear operators on a non-Archimedean Banach space X.*
(i) If $AB \in \mathcal{F}_+(X)$, then $\sigma_{ei}(A + B) \setminus \{0\} \subseteq [\sigma_{ei}(A) \cup \sigma_{ei}(B)] \setminus \{0\}$, $i \in \{1, ap\}$.
Further, if $BA \in \mathcal{F}_+(X)$, then $\sigma_{e1}(A + B) \setminus \{0\} = [\sigma_{e1}(A) \cup \sigma_{e1}(B)] \setminus \{0\}$.

(ii) If $X/R(A)$ is dual separating and one of the following conditions is satisfied:

(a) $AB \in \mathcal{F}_-(X)$ and $BA \in \mathcal{F}_-(X)$,

(b) $AB \in \Phi_-(X) \cap \mathcal{F}_-(X)$, $t \in (0,1)$, and P be a continuous linear projection from X onto $R(A)$ such that $\|P\| \leq t^{-1}$ and $\|B\| < \dfrac{t}{\|A\|}\gamma\big(\mu PA\big)$, for every $\mu \in \mathbb{K}$,

then

$$[\sigma_{e2}(A) \cup \sigma_{e2}(B)] \setminus \{0\} \subseteq \sigma_{e2}(A+B) \setminus \{0\}. \tag{4}$$

(iii) If $X/R(A)$ is dual separating and one of the following conditions is satisfied:

(c) $AB \in \mathcal{F}(X)$ and $BA \in \mathcal{F}(X)$,

(d) $AB \in \Phi_-(X) \cap \mathcal{F}(X)$, $BA \in \mathcal{F}_+(X)$, $t \in (0,1)$, and P be a continuous linear projection from X onto $R(A)$ such that $\|P\| \leq t^{-1}$ and $\|B\| < \dfrac{t}{\|A\|}\gamma\big(\mu PA\big)$, for every $\mu \in \mathbb{K}$,

then

$$[\sigma_{ei}(A) \cup \sigma_{ei}(B)] \setminus \{0\} \subseteq \sigma_{ei}(A+B) \setminus \{0\},\ i \in \{3,4,5\}. \tag{5}$$

Proof. Let $\lambda \in \mathbb{K}$. We have

$$(A-\lambda I)(B-\lambda I) = AB - \lambda(A+B-\lambda I) \tag{6}$$

and

$$(B-\lambda I)(A-\lambda I) = BA - \lambda(A+B-\lambda I). \tag{7}$$

(i) Let $\lambda \notin \sigma_{e1}(A) \cup \sigma_{e1}(B) \cup \{0\}$. Then, $(A-\lambda I) \in \Phi_+(X)$ and $(B-\lambda I) \in \Phi_+(X)$. By virtue of Lemma 1 (i), we obtain $(A-\lambda I)(B-\lambda I) \in \Phi_+(X)$. Since $AB \in \mathcal{F}_+(X)$, combining Proposition 2 (ii) with equality (6), we get $(A+B-\lambda I) \in \Phi_+(X)$. Then, $\lambda \notin \sigma_{e1}(A+B) \cup \{0\}$. So, $\sigma_{e1}(A+B) \setminus \{0\} \subseteq [\sigma_{e1}(A) \cup \sigma_{e1}(B)] \setminus \{0\}$.

Let us assume that $\lambda \notin \sigma_{eap}(A) \cup \sigma_{eap}(B) \cup \{0\}$. Then, $(A-\lambda I) \in \Phi_+(X)$ such that $i(A-\lambda I) \leq 0$ and $(B-\lambda I) \in \Phi_+(X)$ such that $i(B-\lambda I) \leq 0$. We have

$$i((A-\lambda I)(B-\lambda I)) = i(A-\lambda I) + i(B-\lambda I) \leq 0,$$

and the use of Lemma 1 (i) allows us to deduce that $(A-\lambda I)(B-\lambda I) \in \Phi_+(X)$. The fact that $AB \in \mathcal{F}_+(X)$, combining Proposition 2 (ii) with equality (6), we get $(A+B-\lambda I) \in \Phi_+(X)$ such that $i(A+B-\lambda I) \leq 0$. Then, $\lambda \notin \sigma_{eap}(A+B) \cup \{0\}$. As a result,

$$\sigma_{eap}(A+B) \setminus \{0\} \subseteq [\sigma_{eap}(A) \cup \sigma_{eap}(B)] \setminus \{0\}.$$

Now, let $BA \in \mathcal{F}_+(X)$. Suppose that $\lambda \notin \sigma_{e1}(A+B) \cup \{0\}$, then $(A+B-\lambda I) \in \Phi_+(X)$. Since $AB \in \mathcal{F}_+(X)$ and $BA \in \mathcal{F}_+(X)$, we can apply equalities (6), (7) and Proposition 2 (ii), which ensure that

$$(A-\lambda I)(B-\lambda I) \in \Phi_+(X) \ \text{ and } \ (B-\lambda I)(A-\lambda I) \in \Phi_+(X). \tag{8}$$

Relations (8) together with Lemma 1 (ii) allows us to deduce that $(A-\lambda I) \in \Phi_+(X)$ and $(B-\lambda I) \in \Phi_+(X)$. Then, $\lambda \notin [\sigma_{e1}(A) \cup \sigma_{e1}(B)] \setminus \{0\}$, so we obtain

$$[\sigma_{e1}(A) \cup \sigma_{e1}(B)] \setminus \{0\} \subseteq \sigma_{e1}(A+B) \setminus \{0\}.$$

Hence, we get $\sigma_{e1}(A+B) \setminus \{0\} = [\sigma_{e1}(A) \cup \sigma_{e1}(B)] \setminus \{0\}$.

(ii) Suppose that $\lambda \notin \sigma_{e2}(A+B) \cup \{0\}$, then $(A+B-\lambda I) \in \Phi_-(X)$.

(a) Since $AB \in \mathcal{F}_-(X)$ and $BA \in \mathcal{F}_-(X)$, we can apply equalities (6), (7) and Proposition 2 (iii), which imply that $(A-\lambda I)(B-\lambda I) \in \Phi_-(X)$ and $(B-\lambda I)(A-\lambda I) \in \Phi_-(X)$. Using Lemma 2, we conclude that $(A-\lambda I) \in \Phi_-(X)$ and $(B-\lambda I) \in \Phi_-(X)$. Then, $\lambda \notin [\sigma_{e2}(A) \cup \sigma_{e2}(B)] \setminus \{0\}$, so we get

$$[\sigma_{e2}(A) \cup \sigma_{e2}(B)] \setminus \{0\} \subseteq \sigma_{e2}(A+B) \setminus \{0\}.$$

(b) The fact that $AB \in \mathcal{F}_-(X)$, we can apply equality (6) and Proposition 2 (iii) which lead to

$$(A - \lambda I)(B - \lambda I) \in \Phi_-(X). \tag{9}$$

So, using Lemma 2 together with Lemma 3 (the operator A in Lemma 3 can be taken as $A - \lambda I$), we get $(A - \lambda I) \in \Phi_-(X)$ and $(B - \lambda I) \in \Phi_-(X)$. Hence, $\lambda \notin \sigma_{e2}(A) \cup \sigma_{e2}(B) \cup \{0\}$, thus

$$[\sigma_{e2}(A) \cup \sigma_{e2}(B)] \setminus \{0\} \subseteq \sigma_{e2}(A+B) \setminus \{0\}.$$

(iii) To prove the inclusion (5) for $i = 3$, we use the obtained results of (i) and (ii), for which we have $[\sigma_{e1}(A) \cup \sigma_{e1}(B)] \setminus \{0\} \subseteq \sigma_{e1}(A+B) \setminus \{0\}$, and $[\sigma_{e2}(A) \cup \sigma_{e2}(B)] \setminus \{0\} \subseteq \sigma_{e2}(A+B) \setminus \{0\}$.
Hence, $\sigma_{e1}(A) \setminus \{0\} \subseteq \sigma_{e1}(A+B) \setminus \{0\}$ and $\sigma_{e2}(A) \setminus \{0\} \subseteq \sigma_{e2}(A+B) \setminus \{0\}$. Based on relations (2), we conclude that $\sigma_{e3}(A) \setminus \{0\} \subseteq \sigma_{e3}(A+B) \setminus \{0\}$. In the same way, we have $\sigma_{e3}(B) \setminus \{0\} \subseteq \sigma_{e3}(A+B) \setminus \{0\}$. As a consequence, we deduce that

$$[\sigma_{e3}(A) \cup \sigma_{e3}(B)] \setminus \{0\} \subseteq \sigma_{e3}(A+B) \setminus \{0\}.$$

Using Proposition 2 (i), notice that the inclusion (5) for $i = 4$ can be checked in the same way as the proof of (i) and (ii). Now, it remains to prove that

$$[\sigma_{e5}(A) \cup \sigma_{e5}(B)] \setminus \{0\} \subseteq \sigma_{e5}(A+B) \setminus \{0\}.$$

So, let $\lambda \notin \sigma_{e5}(A+B) \cup \{0\}$. Then, $(A + B - \lambda I) \in \Phi(X)$ and $i(A + B - \lambda I) = 0$. Applying Proposition 2 (i) and reasoning in the same way, we get $(A - \lambda I) \in \Phi(X)$ and $(B - \lambda I) \in \Phi(X)$. Applying Lemma 4, we obtain

$$i((A - \lambda I)(B - \lambda I)) = i(B - \lambda I) + i(A - \lambda I) = i(A + B - \lambda I) = 0.$$

This prove that the inclusion (5) is valid for $i = 5$. As a result, we get

$$[\sigma_{ei}(A) \cup \sigma_{ei}(B)] \setminus \{0\} \subseteq \sigma_{ei}(A+B) \setminus \{0\},\ i \in \{3, 4, 5\}.$$

□

The reader could ask if the inverse inclusions of (4) and (5) hold. This question has a positive answer, if we impose assumptions on the operators in $\breve{\mathcal{L}}(X)$ and on the non-Archimedean Banach space, X. Indeed, note that in contrast to the classical case, the duality between p-adic upper and p-adic lower semi-Fredholm operators is failed (see [16, Example]). This fact allows us to apply the results from [10] to derive the following important properties.

Theorem 2. *Let X, Y be two non-Archimedean Banach spaces and suppose that $A \in \breve{\mathcal{L}}(X, Y)$.*
(i) If $A \in \Phi_+(X, Y)$ such that $R(A)$ has the wep in Y, then $A^* \in \Phi_-(Y', X')$.
(ii) If $A^* \in \Phi_-(Y', X')$ such that $R(A)$ is of countable type and has the wep in Y, then $A \in \Phi_+(X, Y)$.
(iii) If $A^* \in \Phi_+(Y', X')$ such that X is reflexive and $R(A)$ is weakly closed in Y, then $A \in \Phi_-(X, Y)$.

Proof. (i) Let $A \in \Phi_+(X, Y)$ such that $R(A)$ has the wep in Y. Then, $R(A)$ is closed, and thus by applying [10, Theorem 2.1], we get $R(A^*) = N(A)^\perp$. Hence,

$$^\perp R(A^*) = ^\perp (N(A)^\perp) = \overline{N(A)} = N(A).$$

It follows that $\beta(A^*) = \alpha(A) < \infty$. As a result, we conclude that $A^* \in \Phi_-(Y', X')$.

(ii) The fact that $A^* \in \Phi_-(Y', X')$ implies that $R(A^*)$ is closed in X'. As $R(A)$ is of countable type and has the wep in Y, and from [10, Theorem 3.1], we infer that $R(A)$ is closed in Y. So, we can apply [10, Theorem 2.1] to deduce that $\alpha(A) = \beta(A^*) < \infty$. As a consequence, we obtain $A \in \Phi_+(X, Y)$.

(iii) Let $R(A)$ be weakly closed in Y. Applying [10, Lemma 1.1], we get $R(A) =^\perp N(A^*)$, which implies that $R(A)^\perp = (^\perp N(A^*))^\perp$. Using the fact that $A^* \in \Phi_+(Y', X')$ together with the reflexivity of X, we obtain $R(A)^\perp = N(A^*)$, and thus $\beta(A) = \alpha(A^*) < \infty$. As a consequence, we achieve the desired result. □

Theorem 3. *Let X and Y be two non-Archimedean Banach spaces. Let $A \in \breve{\mathcal{L}}(X, Y)$. If one of the following conditions is satisfied:*
(i) Y is strongly polar,
(ii) Y is dual-separating and $R(A)$ is of countable type and has the wep in Y,
(iii) Y has a base and $R(A)$ is a subspace of of countable type of Y,
then, we have $A \in \Phi_{+,-}(X, Y)$ if and only if $A^* \in \Phi_{-,+}(Y', X')$.

Proof. It follows from [10, Theorem 4.1], the equivalence between the closedness of the range of a bounded linear operator, the closedness of the range of its adjoint, and the coincidence of this range with the orthogonal of an appropriate kernel. This gives the desired result. □

Corollary 1. *Let X be a non-Archimedean Banach space, $A \in \breve{\mathcal{L}}(X)$ such that $X/R(A)$ is dual separating and let $B \in \breve{\mathcal{L}}(X)$ such that either $AB \in \mathcal{F}(X)$ and $BA \in \mathcal{F}(X)$, or $AB \in \Phi_-(X) \cap \mathcal{F}(X)$, $BA \in \mathcal{F}_+(X)$, $t \in (0, 1)$, and P be a continuous linear projection from X onto $R(A)$ such that $\|P\| \leq t^{-1}$ and $\|B\| < \dfrac{t}{\|A\|}\gamma\big(\mu PA\big)$, for every $\mu \in \mathbb{K}$. If one of the following conditions is satisfied:*
(i) X is strongly polar,
(ii) both of $R(A)$ and $R(B)$ are of countable types and have the wep in X,
(iii) X has a base and $R(A)$ and $R(B)$ are subspaces of countable types of X,
then, $[\sigma_{ei}(A) \cup \sigma_{ei}(B)] \setminus \{0\} = \sigma_{ei}(A + B) \setminus \{0\}$, $i \in \{1, 2, 3, 4, 5, ap, \delta\}$.

Remark 5. The obtained results in Theorem 1 and Corollary 1 are extensions of some of those in [1, Theorem 2.4] to non-Archimedean Banach spaces.

Lemma 5. *Let X be a non-Archimedean Banach space and let $A \in \mathcal{L}(X)$ such that $X/R(A)$ is dual separating and let $B \in \mathcal{L}(X)$. Assume that $\mu \in \rho(A)$. Then, we have for $\lambda \neq \mu$*

$$\lambda \in \sigma_{ei}(A) \text{ if and only if } (\lambda - \mu)^{-1} \in \sigma_{ei}((A - \mu I)^{-1}),\ i \in \{1, 2, 3, 4, 5, ap, \delta\}.$$

Proof. We have the following identities:

$$\begin{aligned} A - \lambda I &= (\mu - \lambda)[(A - \mu I)^{-1} - (\lambda - \mu)^{-1} I](A - \mu I) && (10)\\ &= (\mu - \lambda)(A - \mu I)[(A - \mu I)^{-1} - (\lambda - \mu)^{-1} I]. && (11) \end{aligned}$$

Further, the fact that $\mu \in \rho(A)$ implies that

$$(A - \mu I) \in \Phi(X) \text{ and } i(A - \mu I) = 0. \tag{12}$$

Let $(\lambda - \mu)^{-1} \notin \sigma_{e1}((A - \mu I)^{-1})$ (respectively, $(\lambda - \mu)^{-1} \notin \sigma_{eap}((A - \mu I)^{-1})$). Hence, $((A - \mu I)^{-1} - (\lambda - \mu)^{-1} I) \in \Phi_+(X)$ (respectively, $((A - \mu I)^{-1} - (\lambda - \mu)^{-1} I) \in \Phi_+(X)$ and $i((A - \mu I)^{-1} - (\lambda - \mu)^{-1} I) \leq 0$). So, both operators on the right-hand side of equality (10) are p-adic upper semi-Fredholm. Consequently, by Lemma 1 (i) (respectively, by

Lemma 1 (i) and Lemma 4), we get $(A-\lambda I) \in \Phi_+(X)$ (respectively, $(A-\lambda I) \in \Phi_+(X)$ and $i(A-\lambda I) \leq 0$), then $\lambda \notin \sigma_{e1}(A)$ (respectively $\lambda \notin \sigma_{eap}(A)$).

Conversely, suppose that $\lambda \notin \sigma_{e1}(A)$ (respectively, $\lambda \notin \sigma_{eap}(A)$). We have $(A-\lambda I) \in \Phi_+(X)$ (respectively, $(A-\lambda I) \in \Phi_+(X)$ and $i(A-\lambda I) \leq 0$), so we apply in equality (11) Lemma 1 (ii) (respectively, by Lemma 1 (ii) and Lemma 4), we obtain $((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \in \Phi_+(X)$ (respectively, $((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \in \Phi_+(X)$ and $i((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \leq 0$). Then,

$$(\lambda-\mu)^{-1} \notin \sigma_{e1}((A-\mu I)^{-1}) \quad \text{(respectively } (\lambda-\mu)^{-1} \notin \sigma_{eap}((A-\mu I)^{-1})).$$

As a result,

$$\lambda \in \sigma_{ei}(A) \text{ if and only if } (\lambda-\mu)^{-1} \in \sigma_{ei}((A-\mu I)^{-1}), \ \ i \in \{1, ap\}. \tag{13}$$

If $\lambda \notin \sigma_{e2}(A)$ (respectively, $\lambda \notin \sigma_{e\delta}(A)$), then we have $A-\lambda I \in \Phi_-(X)$ (respectively, $A-\lambda I \in \Phi_-(X)$ and $i(A-\lambda I) \geq 0$). By applying Lemma 2 in equality (10) (respectively, Lemma 2 in equality (10) and Lemma 4), we get $((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \in \Phi_-(X)$ (respectively $((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \in \Phi_-(X)$ and $i((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \geq 0$), in other words, $(\lambda-\mu)^{-1} \notin \sigma_{ei}((A-\mu I)^{-1})$ $i \in \{2, \delta\}$.

Based upon equality (10) and the fact that $\mu \in \rho(A)$, it follows that

$$(A-\lambda I)(A-\mu I)^{-1} = (\mu-\lambda)[(A-\mu I)^{-1} - (\lambda-\mu)^{-1}I]. \tag{14}$$

So, if $(\lambda-\mu)^{-1} \notin \sigma_{e2}((A-\mu I)^{-1})$ (respectively, $(\lambda-\mu)^{-1} \notin \sigma_{e\delta}((A-\mu I)^{-1})$), then $((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \in \Phi_-(X)$ (respectively, $((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \in \Phi_-(X)$ and $i((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \geq 0$). The use of Lemma 2 in equality (14) (respectively, Lemma 2 in equality (14) and Lemma 4), leads to $(A-\lambda I) \in \Phi_-(X)$ (respectively, $(A-\lambda I) \in \Phi_-(X)$ and $i(A-\lambda I) \geq 0$), that is $\lambda \notin \sigma_{e2}(A)$ (respectively, $\lambda \notin \sigma_{e\delta}(A)$). Consequently,

$$\lambda \in \sigma_{ei}(A) \text{ if and only if } (\lambda-\mu)^{-1} \in \sigma_{ei}((A-\mu I)^{-1}) \ \ i \in \{2, \delta\}. \tag{15}$$

Using the fact that $\sigma_{e3}(A) = \sigma_{e1}(A) \cap \sigma_{e2}(A)$ and $\sigma_{e4}(A) = \sigma_{e1}(A) \cup \sigma_{e2}(A)$, it follows from relations (13) and (15) that

$$\lambda \in \sigma_{e3}(A) \text{ if and only if } (\lambda-\mu)^{-1} \in \sigma_{e3}((A-\mu I)^{-1}),$$
$$\lambda \in \sigma_{e4}(A) \text{ if and only if } (\lambda-\mu)^{-1} \in \sigma_{e4}((A-\mu I)^{-1}).$$

Now, suppose that $\lambda \notin \sigma_{e5}(A)$, then $(A-\lambda I) \in \Phi(X)$ and $i(A-\lambda I) = 0$.

Applying Lemma 2 in equality (10) and Lemma 1 (ii) in equality (11), we get

$$((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \in \Phi(X).$$

Relying on the relation (12) together with Lemma 4 leads to

$$i(A-\lambda I) = i(A-\mu I) + i((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) = i((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I).$$

Therefore, $i((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) = 0$. As a consequence, we deduce that

$$(\lambda-\mu)^{-1} \notin \sigma_{e5}((A-\mu I)^{-1}).$$

For the converse, let us assume that $(\lambda-\mu)^{-1} \notin \sigma_{e5}((A-\mu I)^{-1})$. We infer that

$$((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) \in \Phi(X) \text{ and } i((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) = 0.$$

Again, according relation (12) and using Lemma 1 (i) in equality (10) together with Lemma 2 in equality (14), allow us to deduce that $(A-\lambda I) \in \Phi(X)$. Moreover,

$$i(A-\lambda I) = i((A-\mu I)^{-1} - (\lambda-\mu)^{-1}I) = 0.$$

Thus, $\lambda \in \sigma_{e5}(A)$ if and only if $(\lambda-\mu)^{-1} \in \sigma_{e5}((A-\mu I)^{-1})$. $\square$

Theorem 4. *Let X be a non-Archimedean Banach space and let $A \in \mathcal{L}(X)$ such that $X/R(A)$ is dual separating and let $B \in \mathcal{L}(X)$. If, for some $\mu \in \rho(A) \cap \rho(B)$, the difference $((A-\mu I)^{-1} - (B-\mu I)^{-1}) \in \mathcal{K}(X)$, then $\sigma_{ei}(A) = \sigma_{ei}(B)$ $i \in \{1,2,3,4,5,ap,\delta\}$. and*

$$i(A-\lambda I) = i(B-\lambda I), \text{ for all } \lambda \notin \sigma_{ei}(A) \; i \in \{1,2,3,4,5,ap,\delta\}.$$

Proof. Since $\mu \in \rho(A) \cap \rho(B)$, we have $(A-\mu I)$ and $(B-\mu I)$ are p-adic Fredholm operators with index 0. Let $\lambda \in \mathbb{K}$ such that $\lambda \neq \mu$. By virtue of Lemma 5, we have

$$\lambda \in \sigma_{ei}(A) \text{ if, and only if, } (\lambda-\mu)^{-1} \in \sigma_{ei}((A-\mu I)^{-1}) \; i \in \{1,2,3,4,5,ap,\delta\}. \tag{16}$$

The fact that $((A-\mu I)^{-1} - (B-\mu I)^{-1}) \in \mathcal{K}(X)$ and the use of [2, Theorem 6.1] imply that

$$\begin{aligned} \sigma_{ei}((A-\mu I)^{-1}) &= \sigma_{ei}((A-\mu I)^{-1} - (B-\mu I)^{-1} + (B-\mu I)^{-1}) \\ &= \sigma_{ei}((B-\mu I)^{-1}) \;\; i \in \{1,2,3,4,5,ap,\delta\}. \end{aligned}$$

So, from relation (16) we obtain $\sigma_{ei}(A) = \sigma_{ei}(B)$ $i \in \{1,2,3,4,5,ap,\delta\}$. Now, note that the following identity hold $A-\lambda I = (\lambda-\mu)(A-\mu I)((\lambda-\mu)^{-1} - (A-\mu I)^{-1})$, so we obtain from Lemma 4 and the previous identity that

$$\begin{gathered} i(A-\lambda I) = i((\lambda-\mu)(A-\mu I)) + i((\lambda-\mu)^{-1} - (A-\mu I)^{-1}) = \\ = i((\lambda-\mu)^{-1} - (A-\mu I)^{-1}) = i((\lambda-\mu)^{-1} - (A-\mu I)^{-1} + (B-\mu I)^{-1} - (B-\mu I)^{-1}) = \\ = i((\lambda-\mu)^{-1} - (B-\mu I)^{-1}) = i(B-\lambda I). \end{gathered}$$

□

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