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VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS WITH RELATIVE (k,n) VALIRON DEFECT ON ANNULI

A. Rathod. *Value distribution of meromorphic functions with relative (k,n) Valiron defect on annuli*, Mat. Stud. **57** (2022), 172–185.

In the paper, we study and compare relative (k,n) Valiron defect with the relative Nevanlinna defect for meromorphic function where k and n are both non negative integers on annuli. The results we proved are as follows
1. Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ and $\sum_{a\neq\infty}\delta_0(a,f)+\delta_0(\infty,f)=2$. Then
$$\lim_{R\to\infty}\frac{T_0(R,f^{(k)})}{T_0(R,f)}=(1+k)-k\delta_0(\infty,f).$$
2. Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ such that $m_0(r,f)=S(r,f)$. If a, b and c are three distinct complex numbers, then for any two positive integer k and n
$$3_R\delta^{(0)}_{0(n)}(a,f)+2_R\delta^{(0)}_{0(n)}(b,f)+3_R\delta^{(0)}_{0(n)}(c,f)+5_R\Delta^{(k)}_{0(n)}(\infty,f)\leq 5_R\Delta^{(0)}_{0(n)}(\infty,f)+5_R\Delta^{(k)}_{0(n)}(0,f).$$
3. Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ such that $m_0(r,f)=S(r,f)$. If a, b and c are three distinct complex numbers, then for any two positive integer k and n
$${}_R\delta^{(0)}_{0(n)}(0,f)+_R\Delta^{(k)}_{0(n)}(\infty,f)+_R\delta^{(0)}_{0(n)}(c,f)\leq_R\Delta^{(0)}_{0(n)}(\infty,f)+2_R\Delta^{(k)}_{0(n)}(0,f).$$
4. Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ such that $m_0(r,f)=S(r,f)$. If a and d are two distinct complex numbers, then for any two positive integer k and p with $0\leq k\leq p$
$${}_R\delta^{(0)}_{0(n)}(d,f)+_R\Delta^{(p)}_{0(n)}(\infty,f)+_R\delta^{(k)}_{0(n)}(a,f)\leq_R\Delta^{(k)}_{0(n)}(\infty,f)+_R\Delta^{(p)}_{0(n)}(0,f)+_R\Delta^{(k)}_{0(n)}(0,f),$$
where n is any positive integer.
5.Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$. Then for any two positive integers k and n,
$${}_R\Delta^{(0)}_{0(n)}(\infty,f)+_R\Delta^{(k)}_{0(n)}(0,f)\geq_R\delta^{(0)}_{0(n)}(0,f)+_R\delta^{(0)}_{0(n)}(a,f)+_R\Delta^{(k)}_{0(n)}(\infty,f),$$
where a is any non zero complex number.

1. Introduction and basic notations in the Nevanlinna theory on annuli. The uniqueness theory of meromorphic functions is an interesting problem in the value distribution theory. In 2005, A. Ya. Khrystiyanyn and A. A. Kondratyuk have proposed Nevanlinna Theory for meromorphic functions on annuli (see [5,6]). In 2009, Cao and Yi [1] investigated the uniqueness of meromorphic functions sharing some values on annuli. On the characteristic function of derivative of f(z) with maximum deficiency sum has been studied by S. K. Singh, Kulkarni and A. Weitsman [18,19] and others have done lots of work in this area([2–4], [7–17] and [20–33]). After this work, it is natural to ask whether we study and compare relative

2010 *Mathematics Subject Classification:* 30D35.
Keywords: Nevanlinna theory; Valiron defect; meromorphic function; the annuli.
doi:10.30970/ms.57.2.172-185

(k,n) Valiron defect with the relative Nevanlinna defect for meromorphic function on annuli where k and n are both non negative integers.

Let $f(z)$ be a meromorphic function on the annulus $\mathbb{A} = \left\{z\colon \frac{1}{R_0} < |z| < R_0\right\}$. We recall classical notations of Nevanlinna theory as follows

$$N(R,f) = \int_0^R \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)\log R,$$
$$m(R,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(Re^{i\theta})|d\theta, \qquad T(R,f) = N(R,f) + m(R,f),$$

where $\log^+ x = \max\{\log x, 0\}$, and $n(t,f)$ is the counting function of poles of the function f in $\{z\colon |z| \le t\}$. Here we show the notations of the Nevanlinna theory on annuli. Let

$$N_1(R,f) = \int_{\frac{1}{R}}^1 \frac{n_1(t,f)}{t}dt, \qquad N_2(R,f) = \int_1^R \frac{n_2(t,f)}{t}dt,$$
$$m_0(R,f) = m(R,f) + m\left(\frac{1}{R}, f\right) - 2m(1,f), \qquad N_0(R,f) = N_1(R,f) + N_2(R,f),$$

where $n_1(t,f)$ and $n_2(t,f)$ are the counting functions of the poles of the function f in $\{z\colon t < |z| \le 1\}$ and $\{z\colon 1 < |z| \le t\}$, respectively. The Nevanlinna charecteristic of f on the annulus $\mathbb{A}$ is defined [7] by

$$T_0(R,f) = m_0(r,f) + N_0(R,f).$$

Let $f(z)$ be a non-constant meromorphic function on the annulus $\mathbb{A}(R_0) = \{z\colon 1/R_0 < |z| < R_0\}$, where $1 < R_0 < +\infty$. The function f is called [4] a *transcedental* or *admissible meromorphic function* on the annulus $\mathbb{A}(R_0)$ provided that

$$\overline{\lim_{R\to+\infty}} \frac{T_0(R,f)}{\log R} = \infty, \quad 1 < R < R_0 = +\infty$$

or

$$\overline{\lim_{R\to R_0}} \frac{T_0(R,f)}{-\log(R_0 - R)} = \infty, \quad 1 < R < R_0 < +\infty,$$

respectively.

Let $f(z)$ be a non-constant meromorphic function on the annulus $\mathbb{A}(R_0) = \{z\colon 1/R_0 < |z| < R_0\}$, where $1 < R_0 < +\infty$. Then, the *order* of $f(z)$ is defined by

$$\sigma(f) = \overline{\lim_{r\to R_0}} \frac{\log T_0(r,f)}{\log r}.$$

Let $f(z)$ be a non-constant meromorphic function on the annulus $\mathbb{A}(R_0) = \{z\colon 1/R_0 < |z| < R_0\}$, where $1 < R_0 < +\infty$. Then, the value

$$\delta_0(a,f) = \underline{\lim_{r\to R_0}} \frac{m_0(r, \frac{1}{f-a})}{T_0(r,f)}$$

is called the *deficiency of the function* $f(z)$ for the value a. For $a = \infty$, we set

$$\delta_0(\infty,f) = \underline{\lim_{r\to R_0}} \frac{m_0(r,f)}{T_0(r,f)} = 1 - \overline{\lim_{r\to R_0}} \frac{N_0(r,f)}{T_0(r,f)}.$$

If $\delta_0(a,f)>0,\ a\in\mathbb{C}_\infty$, we call a is a deficient value of $f(z)$.

Let $f(z)$ be a non-constant meromorphic function on the annulus $\mathbb{A}(R_0)=\{z\colon 1/R_0<|z|<R_0\}$, where $1<R_0<+\infty$. Then, the value

$$\Theta_0(a,f)=1-\varlimsup_{r\to R_0}\frac{\overline{N}_0(r,\frac{1}{f-a})}{T_0(r,f)}\quad\text{and}\quad \theta_0(a,f)=\varliminf_{r\to R_0}\frac{N_0(r,\frac{1}{f-a})-\overline{N}_0(r,\frac{1}{f-a})}{T_0(r,f)}$$

is called the *reduced deficiency of the function* $f(z)$ for the value a.

The *order* ρ_f *of meromorphic function* on the annulus $\mathbb{A}(R_0)=\{z\colon 1/R_0<|z|<R_0\}$, where $1<R_0<+\infty$ is defined as follows

$$\rho_f=\varlimsup_{r\to R_0}\frac{\log T(r,f)}{\log r}.$$

If $\rho_f<\infty$, then f is of finite order.

The *Nevanlinna defect* $\delta(a,f)$ and *Valiron defect* $\Delta(a,f)$ of a for meromorphic function are respectively defined on the annulus $\mathbb{A}(R_0)=\{z\colon 1/R_0<|z|<R_0\}$, where $1<R_0<+\infty$ as follows

$$\delta_0(a,f)=\varliminf_{r\to R_0}\frac{m_0(r,a,f)}{T_0(r,f)}=\varlimsup_{r\to R_0}\frac{N_0(r,a,f)}{T_0(r,f)}$$

and

$$\Delta_0(a,f)=\varliminf_{r\to R_0}\frac{m_0(r,a,f)}{T_0(r,f)}=\varlimsup_{r\to R_0}\frac{N_0(r,a,f)}{T_0(r,f)}.$$

The *relative Nevanlinna defect* of α for meromorphic function on the annulus $\mathbb{A}(R_0)=\{z\colon 1/R_0<|z|<R_0\}$, where $1<R_0<+\infty$, *with respect* to $f^{(k)}$ is defined as follows

$${}_R\delta_0^{(k)}(a,f)=\varliminf_{r\to R_0}\frac{m_0(r,a,f^{(k)})}{T_0(r,f)}=\varlimsup_{r\to R_0}\frac{N_0(r,a,f^{(k)})}{T_0(r,f)},$$

for $k=1,2,3,...$

The *relative* (k,n) *Nevanlinna defect* of α for meromorphic function on the annulus $\mathbb{A}(R_0)=\{z\colon 1/R_0<|z|<R_0\}$, where $1<R_0<+\infty$, *with respect* to $f^{(k)}$ for $k=1,2,3,...$ and $n=0,1,2,3,...$ is defined as follows

$${}_R\delta_{0(n)}^{(k)}(\alpha,f)=\varliminf_{r\to R_0}\frac{m_0(r,\alpha,f^{(k)})}{T_0(r,f^{(n)})}=\varlimsup_{r\to R_0}\frac{N_0(r,\alpha,f^{(k)})}{T_0(r,f^{(n)})}$$

and the *relative* (k,n) *Valiron defect* of α for meromorphic function on the annulus $\mathbb{A}(R_0)=\{z\colon 1/R_0<|z|<R_0\}$, where $1<R_0<+\infty$, *with respect* to $f^{(k)}$ for $k=1,2,3,...$ and $n=0,1,2,3,...$ is defined as follows

$${}_R\Delta_{0(n)}^{(k)}(\alpha,f)=\varlimsup_{r\to R_0}\frac{m_0(r,\alpha,f^{(k)})}{T_0(r,f^{(n)})}=\varliminf_{r\to R_0}\frac{N_0(r,\alpha,f^{(k)})}{T_0(r,f^{(n)})}.$$

Next, we have

$$\overline{N_0}\left(r,\frac{1}{f-a}\right)=\overline{N_1}\left(r,\frac{1}{f-a}\right)+\overline{N_2}\left(r,\frac{1}{f-a}\right)=\int_{\frac{1}{R}}^{1}\frac{\overline{n_1}\left(t,\frac{1}{f-a}\right)}{t}dt+\int_1^R\frac{\overline{n_2}\left(t,\frac{1}{f-a}\right)}{t}dt$$

in which each zero of the function $f-a$ is counted only once.

Theorem A ([6], The First Fundamental Theorem)**.** *Let $f(z)$ be a non-constant meromorphic function in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$. Then*

$$T_0\left(r, \frac{1}{f-a}\right) = T_0(r, f) + O(1) \tag{1}$$

for any fixed $a \in \mathbb{C}$.

Theorem B ([7], Lemma on the Logarithmic Derivative)**.** *Let $f(z)$ be a non-constant meromorphic function in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ and $\alpha \geq 0$. Then*
1. In the case, $R_0 = +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log(RT_0(R, f))\right) \tag{2}$$

for $R \in (1, +\infty)$ except for the set $\triangle_R$ such that $\int_{\triangle_R} R^{\alpha-1}dR < +\infty$;
2. In the case, $R_0 < +\infty$,

$$m_0\left(r, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right) \tag{3}$$

for $R \in (1, R_0)$ except for the set $\triangle'_R$ such that $\int_{\triangle'_R} \frac{dR}{(R_0 - R^{\alpha-1})} < +\infty$.

For any non-constant meromorphic function $f(z)$ in the punctured plane, Khrystiyanyn and Kondrutyuk [6] proved that there are at most countably many deficient values of $f(z)$, and

$$\sum_{a\in\mathbb{C}} \delta_0(a, f) + \delta_0(\infty, f) \leq 2.$$

If equality holds in the above inequality , then we say that $f(z)$ has maximal deficiency sum. Following Lemmas are required to prove our main results

Lemma 1. *Let $f(z)$ be a transcendental meromorphic function in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ and k is a positive integer. Then*

$$(k-1)\overline{N}_0(r, f) \leq (1+\varepsilon)N_0\left(r, \frac{1}{f^{(k)}}\right) + (1+\varepsilon)(N_0(r, f) - \overline{N}_0(r, f)) + S(r, f)$$

where ε is any fixed positive number.

Proof. Proof of Lemma 1.1 follows on similar lines as in Lemma (p.30, [16]). □

Lemma 2. *Let $f(z)$ be a transcendental meromorphic function in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$. Then for each positive number ε and each positive integer k, we have*

$$k\overline{N}_0(r, f) \leq N_0\left(r, \frac{1}{f^{(k)}}\right) + N_0(r, f) + 2\varepsilon T_0(r, f^{(k)}) + S(r, f)$$

Proof. By Lemma 1.1, we have

$$(k-1)\overline{N}_0(r, f) \leq (1+\varepsilon)N_0\left(r, \frac{1}{f^{(k)}}\right) + (1+\varepsilon)(N_0(r, f) - \overline{N}_0(r, f)) + S\left(r, f\right). \tag{4}$$

Noting that

$$N_0\left(r,\frac{1}{f^{(k)}}\right)\leq T_0(r,f^{(k)})+O(1) \tag{5}$$

and

$$N_0(r,f)\leq T_0(r,f^{(k)}). \tag{6}$$

Now equation (4) can be written as follows

$$(k-1)\overline{N}_0(r,f)\leq N_0\left(r,\frac{1}{f^{(k)}}\right)+N_0(r,f)-\overline{N}_0(r,f)\leq$$
$$\leq\varepsilon\left[N_0\left(r,\frac{1}{f^{(k)}}\right)+N_0(r,f)-\overline{N}_0(r,f)\right]+S(r,f).$$

Therefore

$$k\overline{N}_0(r,f)\leq N_0\left(r,\frac{1}{f^{(k)}}\right)+N_0(r,f)\leq\varepsilon\left[N_0\left(r,\frac{1}{f^{(k)}}\right)+N_0(r,f)\right]+S(r,f). \tag{7}$$

From $(5),(6)$ and (7), we get

$$k\overline{N}_0(r,f)\;\;\leq\;\;N_0\left(r,\frac{1}{f^{(k)}}\right)+N_0(r,f)+2\varepsilon T_0\left(r,f^{(k)}\right)+S(r,f).$$

□

2. Main results. In this paper, we will prove following theorems

Theorem 1. *Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1<R_0\leq+\infty$ and*

$$\sum_{a\neq\infty}\delta_0(a,f)+\delta_0(\infty,f)=2.$$

Then

$$\lim_{r\to R_0}\frac{T_0(r,f^{(k)})}{T_0(r,f)}=(1+k)-k\delta_0(\infty,f).$$

Proof. From [17], we have

$$T_0(r,f^{(k)})=T_0\left(r,f\frac{f^{(k)}}{f}\right)\leq T_0(r,f)+T_0\left(r,\frac{f^{(k)}}{f}\right)+O(1)=$$
$$=T_0(r,f)+m_0\left(r,\frac{f^{(k)}}{f}\right)+N_0\left(r,\frac{f^{(k)}}{f}\right)-2m\left(1,\frac{f^{(k)}}{f}\right)+O(1)\leq$$
$$\leq T_0(r,f)+k\overline{N}_0(r,f)+S(r,f). \tag{8}$$

That implies

$$\frac{T_0(r,f^{(k)})}{T_0(r,f)} \leq 1+k-k+k\frac{\overline{N}_0(r,f)}{T_0(r,f)}+\frac{S(r,f)}{T_0(r,f)} \leq (1+k)-k\left(1-\frac{\overline{N}_0(r,f)}{T_0(r,f)}\right)+\frac{S(r,f)}{T_0(r,f)}.$$

Therefore

$$\overline{\lim_{R\to r_0}}\frac{T_0(r,f^{(k)})}{T_0(r,f)} \leq (1+k)-k\left(1-\underline{\lim_{R\to R_0}}\frac{\overline{N}_0(r,f)}{T_0(r,f)}\right)+\overline{\lim_{R\to R_0}}\frac{S(r,f)}{T_0(r,f)} \leq$$
$$\leq 1+k-k\delta_0(\infty,f). \tag{9}$$

$f(z)$ has at most countably infinitely many deficient values and we denote them by a_i. For any positive p, Wu and Chen [15] prove the following inequality

$$\sum_{i=1}^{p} m_0\left(r,\frac{1}{f(z)-a_i}\right) \leq T_0(r,f^{(k)})-N_0\left(r,\frac{1}{f^{(k)}}\right)+S(r,f). \tag{10}$$

From (10) and Lemma 2, we get

$$\sum_{i=1}^{p} m_0\left(r,\frac{1}{f(z)-a_i}\right) \leq T_0(r,f^{(k)})+N_0(r,f)-k\overline{N}_0\,(r,f)+$$
$$+2\varepsilon T_0(r,f^{(k)})+S(r,f). \tag{11}$$

By the first fundamental theorem on annuli and $\varepsilon\to 0$ in (11), we obtain

$$\begin{aligned} pT_0(r,f) \;\; &\leq \;\; T_0(r,f^{(k)})+\sum_{i=1}^{p} N_0\left(r,\frac{1}{f(z)-a_i}\right)+N_0(r,f) \\ &\quad -k\overline{N}_0\,(r,f)+S(r,f). \end{aligned} \tag{12}$$

and hence

$$\frac{T_0(r,f^{(k)})}{T_0(r,f)} \geq \sum_{i=1}^{p}\delta_0(a_i,f)+(k-1)(1-\delta_0(\infty,f)).$$

As p is arbitrary, we have

$$\underline{\lim_{r\to R_0}}\frac{T_0(r,f^{(k)})}{T_0(r,f)} \geq (1+k)-k\delta_0(\infty,f). \tag{13}$$

Therefore, using equations (9) and (13), we have

$$\lim_{r\to R_0}\frac{T_0(r,f^{(k)})}{T_0(r,f)} = (1+k)-k\delta_0(\infty,f). \tag{14}$$

Hence (i) follows. □

Theorem 2. *Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ and $\sum_a \delta_0(a,f) = 1$ and $\delta_0(\infty,f) = 1$, then for a non-negative integer k,*

$$\lim_{r\to R_0}\frac{T_0(r,f^{(k)})}{T_0(r,f)} = 1.$$

Proof. We omit the proof of Theorem 2.2 because it can be carried out in the similar lines of Theorem 1. □

Theorem 3. *Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ and $\sum_a \delta_0(a, f) = 1$ and $\delta_0(\infty, f) = 1$, then for a non-negative integer k,*

$$\lim_{r\to R_0} \frac{T_0(r, f^{(k)})}{T_0(r, f^{(n)})} = 1.$$

Proof. We omit the proof of Theorem 3 because it can be carried out in the similar lines of Theorem 2. □

Theorem 4. *Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ and $\sum_a \delta_0(a, f) = 1$ and $\delta_0(\infty, f) = 1$, then for any α,*

$${}_R\delta^{(k)}_{0(n)} = \underline{\lim}_{r\to R_0} \frac{m_0(r, \alpha, f^{(k)})}{T_0(r, f^{(n)})}.$$

Proof. We have

$$\begin{aligned} {}_R\delta^{(k)}_{0(n)} &= 1 - \overline{\lim}_{r\to R_0} \frac{N_0(r, \alpha, f^{(k)})}{T_0(r, f^{(n)})} = 1 - \overline{\lim}_{r\to R_0} \frac{N_0(r, \alpha, f^{(k)})}{T_0(r, f^{(k)})} \lim_{r\to +\infty} \frac{T_0(r, f^{(k)})}{T_0(r, f^{(n)})} = \\ &= 1 - \overline{\lim}_{r\to R_0} \frac{N_0(r, \alpha, f^{(k)})}{T_0(r, f^{(k)})} = \underline{\lim}_{r\to R_0} \frac{m_0(r, \alpha, f^{(k)})}{T_0(r, f^{(k)})} = \\ &= \underline{\lim}_{r\to R_0} \frac{m_0(r, \alpha, f^{(k)})}{T_0(r, f^{(n)})} \lim_{r\to +\infty} \frac{T_0(r, f^{(n)})}{T_0(r, f^{(k)})} = \underline{\lim}_{r\to R_0} \frac{m_0(r, \alpha, f^{(k)})}{T_0(r, f^{(n)})}. \end{aligned}$$

□

Theorem 5. *Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ and $\sum_a \delta_0(a, f) = 2$ and $\delta_0(\infty, f) = 1$, then for any α,*

$${}_R\Delta^{(k)}_{0(n)} = \overline{\lim}_{r\to R_0} \frac{m_0(r, \alpha, f^{(k)})}{T_0(r, f^{(n)})}.$$

Theorem 6. *Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ such that $m_0(r, f) = S(r, f)$. If a, b and c are three distinct complex numbers, then for any two positive integer k and n*

$$3{}_R\delta^{(0)}_{0(n)}(a, f) + 2{}_R\delta^{(0)}_{0(n)}(b, f) + 3{}_R\delta^{(0)}_{0(n)}(c, f) + 5{}_R\Delta^{(k)}_{0(n)}(\infty, f) \leq 5{}_R\Delta^{(0)}_{0(n)}(\infty, f) + 5{}_R\Delta^{(k)}_{0(n)}(0, f).$$

Proof. For any positive integer k, let us consider the following identity

$$\frac{b-a}{f-a} = \left[\frac{f^{(k)}}{f-a}\left(\frac{f-a}{f^{(k)}} - \frac{f-b}{f^{(k)}}\right) - \frac{f-c}{f^{(k)}}.\frac{f^{(k)}}{f}.\frac{f^{(k)}}{f-a}\left(\frac{f-a}{f^{(k)}} - \frac{f-b}{f^{(k)}}\right)\right]\frac{f}{c}$$

Since

$$m_0\left(r, \frac{1}{f-a}\right) \leq m_0\left(r, \frac{b-a}{f-a}\right) + O(1)$$

and

$$m_0\left(r,\frac{f}{c}\right)\le m_0(r,f)+O(1).$$

From above identity, we get

$$m_0\left(r,\frac{b-a}{f-a}\right)\le m_0\left(r,\frac{f-a}{f^{(k)}}\right)+m_0\left(r,\frac{f-b}{f^{(k)}}\right)+m_0\left(r,\frac{f-c}{f^{(k)}}\right)+$$
$$+m_0\left(r,\frac{f-a}{f^{(k)}}\right)+m_0\left(r,\frac{f-b}{f^{(k)}}\right)+m_0\left(r,\frac{f}{c}\right)+S(r,f),$$

implies

$$m_0\left(r,\frac{1}{f-a}\right)\le 2m_0\left(r,\frac{f-a}{f^{(k)}}\right)+2m_0\left(r,\frac{f-b}{f^{(k)}}\right)+m_0\left(r,\frac{f-c}{f^{(k)}}\right)+m_0(r,f)+S(r,f).$$

From above equation, we have

$$m_0\left(r,\frac{1}{f-a}\right)\le 2T_0\left(r,\frac{f-a}{f^{(k)}}\right)-2N_0\left(r,\frac{f-a}{f^{(k)}}\right)+2T_0\left(r,\frac{f-b}{f^{(k)}}\right)-2N_0\left(r,\frac{f-b}{f^{(k)}}\right)+$$
$$+T_0\left(r,\frac{f-c}{f^{(k)}}\right)-N_0\left(r,\frac{f-c}{f^{(k)}}\right)+m_0(r,f)+S(r,f). \tag{15}$$

By first fundamental theorem of Nevanlinna for meromorphic function annuli and in view of Theorem 2, it follows from (15) that

$$m_0\left(r,\frac{1}{f-a}\right)\le 2T_0\left(r,\frac{f^{(k)}}{f-a}\right)-2N_0\left(r,\frac{f-a}{f^{(k)}}\right)+2T_0\left(r,\frac{f^{(k)}}{f-b}\right)-2N_0\left(r,\frac{f-b}{f^{(k)}}\right)+$$
$$+T_0\left(r,\frac{f^{(k)}}{f-c}\right)-N_0\left(r,\frac{f-c}{f^{(k)}}\right)+m_0(r,f)+S(r,f).$$

Therefore,

$$m_0\left(r,\frac{1}{f-a}\right)\le$$
$$\le 2\left(N_0\left(r,\frac{f^{(k)}}{f-a}\right)-N_0\left(r,\frac{f-a}{f^{(k)}}\right)\right)+2\left(N_0\left(r,\frac{f^{(k)}}{f-b}\right)-N_0\left(r,\frac{f-b}{f^{(k)}}\right)\right)+$$
$$+N_0\left(r,\frac{f^{(k)}}{f-c}\right)-N_0\left(r,\frac{f-c}{f^{(k)}}\right)+m_0(r,f)+S(r,f). \tag{16}$$

By Jensen formula for meromorphic function on annuli, we have

$$m_0\left(r,\frac{1}{f-a}\right)\le 2N\left(r,f^{(k)}\right)+2N_0\left(r,\frac{1}{f-a}\right)-2N_0\left(r,f-a\right)-2N_0\left(r,\frac{1}{f^{(k)}}\right)+$$
$$+2N_0\left(r,f^{(k)}\right)+2N_0\left(r,\frac{1}{f-b}\right)-2N_0\left(r,f-b\right)-2N_0\left(r,\frac{1}{f^{(k)}}\right)+$$
$$+N_0\left(r,f^{(k)}\right)+N_0\left(r,\frac{1}{f-c}\right)-N_0\left(r,f-c\right)-N_0\left(r,\frac{1}{f^{(k)}}\right)+m_0(r,f)+S(r,f).$$

Therefore

$$m_0\left(r,\frac{1}{f-a}\right)\le 5N_0\left(r,f^{(k)}\right)-5N_0\left(r,f\right)-5N_0\left(r,\frac{1}{f^{(k)}}\right)+2N_0\left(r,\frac{1}{f-a}\right)+$$
$$+2N_0\left(r,\frac{1}{f-b}\right)+N_0\left(r,\frac{1}{f-c}\right)+m_0(r,f)+S(r,f). \tag{17}$$

Applying the condition $m_0(r,f)=S(r,f)$, then from (17) we have

$$m_0\left(r,\frac{1}{f-a}\right)\le 5N_0\left(r,f^{(k)}\right)-5N_0\left(r,f\right)-5N_0\left(r,\frac{1}{f^{(k)}}\right)+2N_0\left(r,\frac{1}{f-a}\right)+$$
$$+2N_0\left(r,\frac{1}{f-b}\right)+N_0\left(r,\frac{1}{f-c}\right)+S(r,f).$$

Therefore

$$\varliminf_{r\to R_0}\frac{m_0\left(r,\frac{1}{f-a}\right)}{T_0(r,f^{(n)})}\le 5\varliminf_{r\to R_0}\frac{N_0\left(r,f^{(k)}\right)}{T_0(r,f^{(n)}}-5\varliminf_{r\to R_0}\frac{N_0\left(r,f\right)}{T_0(r,f^{(n)})}-$$
$$-5\varliminf_{r\to R_0}\frac{N_0\left(r,\frac{1}{f^{(k)}}\right)}{T_0(r,f^{(n)})}+2\varlimsup_{r\to R_0}\frac{N_0\left(r,\frac{1}{f-a}\right)}{T_0(r,f^{(n)})}+2\varlimsup_{r\to R_0}\frac{N_0\left(r,\frac{1}{f-b}\right)}{T_0(r,f^{(n)})}+\varlimsup_{r\to R_0}\frac{N_0\left(r,\frac{1}{f-c}\right)}{T_0(r,f^{(n)})}$$

implies

$$_R\delta^{(0)}_{0(n)}(a,f)\le 5[1-_R\Delta^{(k)}_{0(n)}(\infty,f)]-5[1-_R\Delta^{(0)}_{0(n)}(\infty,f)]-5[1-_R\Delta^{(k)}_{0(n)}(0,f)]$$
$$+2[1-_R\delta^{(0)}_{0(n)}(a,f)]+2[1-_R\delta^{(0)}_{0(n)}(b,f)]+[1-_R\delta^{(0)}_{0(n)}(c,f)].$$

Hence

$$3_R\delta^{(0)}_{0(n)}(a,f)+2_R\delta^{(0)}_{0(n)}(b,f)+3_R\delta^{(0)}_{0(n)}(c,f)+5_R\Delta^{(k)}_{0(n)}(\infty,f)\le 5_R\Delta^{(0)}_{0(n)}(\infty,f)+5_R\Delta^{(k)}_{0(n)}(0,f).$$

□

Theorem 7. *Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1<R_0\le+\infty$ such that $m_0(r,f)=S(r,f)$. If a, b and c are three distinct complex numbers, then for any two positive integer k and n*

$$_R\delta^{(0)}_{0(n)}(0,f)+_R\Delta^{(k)}_{0(n)}(\infty,f)+_R\delta^{(0)}_{0(n)}(c,f)\le_R\Delta^{(0)}_{0(n)}(\infty,f)+2_R\Delta^{(k)}_{0(n)}(0,f).$$

Proof. Let us consider the following identity

$$\frac{c}{f}=\left[\left(1-\frac{f-c}{f^{(k)}}\frac{f^{(k)}}{f}\right)\left(\frac{f^{(k)}}{f-a}\frac{1}{f^{(k)}}\right)\right](f-a).$$

Since

$$m_0\left(r,\frac{1}{f}\right)\le m_0\left(r,\frac{c}{f}\right)+O(1)$$

and

$$m_0(r,f-a)\le m_0(r,f)+O(1),$$

then from the above identity we have

$$m_0\left(r,\frac{c}{f}\right)\le m_0\left(r,\frac{f-c}{f^{(k)}}\right)+m_0\left(r,\frac{c}{f^{(k)}}\right)+m_0(r,f)+S(r,f).$$

Therefore

$$m_0\left(r,\frac{1}{f}\right)\le T_0\left(r,\frac{f-c}{f^{(k)}}\right)-N_0\left(r,\frac{f-c}{f^{(k)}}\right)+T_0\left(r,\frac{c}{f^{(k)}}\right)-$$
$$-N_0\left(r,\frac{c}{f^{(k)}}\right)+m_0(r,f)+S(r,f). \tag{18}$$

By Nevanlinna first fundamental theorem for meromorphic function on annuli, we have

$$m_0\left(r,\frac{1}{f}\right)\le T_0\left(r,\frac{f^{(k)}}{f-c}\right)-N_0\left(r,\frac{f-c}{f^{(k)}}\right)+T_0\left(r,f^{(k)}\right)-$$
$$-N_0\left(r,\frac{c}{f^{(k)}}\right)+m_0(r,f)+S(r,f).$$

Therefore

$$m_0\left(r,\frac{1}{f}\right)\le N_0\left(r,\frac{f^{(k)}}{f-c}\right)-N_0\left(r,\frac{f-c}{f^{(k)}}\right)-$$
$$-N_0\left(r,\frac{1}{f^{(k)}}\right)+T_0\left(r,f^{(k)}\right)+m_0(r,f)+S(r,f). \tag{19}$$

By Jensen formula for meromorphic function on annuli and from (19) it follows that

$$m_0\left(r,\frac{1}{f}\right)\le N_0\left(r,f^{(k)}\right)+N_0\left(r,\frac{1}{f-c}\right)-N_0\left(r,f-c\right)-N_0\left(r,\frac{1}{f^{(k)}}\right)-$$
$$-N_0\left(r,\frac{1}{f^{(k)}}\right)+T_0\left(r,f^{(k)}\right)+m_0(r,f)+S(r,f). \tag{20}$$

Applying the condition $m_0(r,f)=S(r,f)$ and from (21) it follows that

$$m_0\left(r,\frac{1}{f}\right)\le N_0\left(r,f^{(k)}\right)+N_0\left(r,\frac{1}{f-c}\right)-N_0\left(r,f-c\right)-2N_0\left(r,\frac{1}{f^{(k)}}\right)$$
$$+T_0\left(r,f^{(k)}\right)+S(r,f).$$

Implies

$$\varliminf_{r\to R_0}\frac{m_0\left(r,\frac{1}{f}\right)}{T_0(r,f^{(n)})}\le\varliminf_{r\to R_0}\frac{N_0\left(r,f^{(k)}\right)}{T_0(r,f^{(n)})}+\overline{\lim_{r\to R_0}}\frac{N_0\left(r,\frac{1}{f-c}\right)}{T_0(r,f^{(n)}}-$$
$$-\varliminf_{r\to R_0}\frac{N_0\left(r,f\right)}{T_0(r,f^{(n)}}-2\varliminf_{r\to R_0}\frac{N_0\left(r,\frac{1}{f^{(k)}}\right)}{T_0(r,f^{(n)}}+\lim_{r\to R_0}\frac{T_0\left(r,f^{(k)}\right)}{T_0(r,f^{(n)}}.$$

Therefore

$${}_R\delta^{(0)}_{0(n)}(0,f)\le[1-{}_R\Delta^{(k)}_{0(n)}(\infty,f)]+[1-{}_R\delta^{(0)}_{0(n)}(c,f)]-[1-{}_R\Delta^{(0)}_{0(n)}(\infty,f)]-2[1-{}_R\Delta^{(k)}_{0(n)}(0,f)].$$

Hence

$${}_R\delta^{(0)}_{0(n)}(0,f)+{}_R\Delta^{(k)}_{0(n)}(\infty,f)+{}_R\delta^{(0)}_{0(n)}(c,f)\le{}_R\Delta^{(0)}_{0(n)}(\infty,f)+2{}_R\Delta^{(k)}_{0(n)}(0,f).$$

□

Theorem 8. *Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ such that $m_0(r,f) = S(r,f)$. If a and d are two distinct complex numbers, then for any two positive integer k and p with $0 \leq k \leq p$*

$$ {}_R\delta^{(0)}_{0(n)}(d,f) +_R \Delta^{(p)}_{0(n)}(\infty,f) +_R \delta^{(k)}_{0(n)}(a,f) \leq_R \Delta^{(k)}_{0(n)}(\infty,f) +_R \Delta^{(p)}_{0(n)}(0,f) +_R \Delta^{(k)}_{0(n)}(0,f), $$

where n is any positive integer.

Proof. Let us consider the following identity

$$ \frac{1}{f-d} = \left[\frac{1}{a}\left(\frac{f^{(k)}}{f-a} - \frac{f^{(k)}-a}{f^{(p)}}.\frac{f^{(p)}}{f-a}\right)\left(\frac{f^{(k)}}{f-d}\frac{1}{f^{(k)}}\right)\right](f-a). $$

Since $m(r,f-a) \leq m(r,f) + O(1)$ and from the above identity, we have

$$ m_0\left(r,\frac{1}{f-d}\right) \leq m_0\left(r,\frac{f^{(k)}-a}{f^{(p)}}\right) + m_0\left(r,\frac{1}{f^{(k)}}\right) + m_0(r,f) + S(r,f). \tag{21} $$

By Nevanlinna first fundamental theorem for meromorphic function on annuli and from (21), we have

$$ m_0\left(r,\frac{1}{f-d}\right) \leq T_0\left(r,\frac{f^{(k)}-a}{f^{(p)}}\right) - N_0\left(r,\frac{f^{(k)}-a}{f^{(p)}}\right) + $$
$$ +T_0\left(r,\frac{1}{f^{(k)}}\right) - N_0\left(r,\frac{1}{f^{(k)}}\right) + m_0(r,f) + S(r,f). $$

Therefore

$$ m_0\left(r,\frac{1}{f-d}\right) \leq T_0\left(r,\frac{f^{(p)}}{f^{(k)}-a}\right) - N_0\left(r,\frac{f^{(k)}-a}{f^{(p)}}\right) + $$
$$ +T_0\left(r,f^{(k)}\right) - N_0\left(r,\frac{1}{f^{(k)}}\right) + m_0(r,f) + S(r,f). $$

By Lemma of Logarithmic derivative for meromorphic function on annuli, we have

$$ m_0\left(r,\frac{1}{f-d}\right) \leq N_0\left(r,\frac{f^{(p)}}{f^{(k)}-a}\right) - N_0\left(r,\frac{f^{(k)}-a}{f^{(p)}}\right) + $$
$$ +T_0\left(r,f^{(k)}\right) - N_0\left(r,\frac{1}{f^{(k)}}\right) + m_0(r,f) + S(r,f). $$

By Jenson formula for meromorphic function on annuli, we have

$$ m_0\left(r,\frac{1}{f-d}\right) \leq N_0\left(r,f^{(p)}\right) + N_0\left(r,\frac{1}{f^{(k)}-a}\right) - N_0\left(r,f^{(k)}-a\right) - N_0\left(r,\frac{1}{f^{(p)}}\right) + $$
$$ +T_0\left(r,f^{(k)}\right) - N_0\left(r,\frac{1}{f^{(k)}}\right) + m_0(r,f) + S(r,f). \tag{22} $$

Applying the condition $m_0(r,f) = S(r,f)$ and from (), we have

$$ m_0\left(r,\frac{1}{f-d}\right) \leq N_0\left(r,f^{(p)}\right) + N_0\left(r,\frac{1}{f^{(k)}-a}\right) - N_0\left(r,f^{(k)}-a\right) - N_0\left(r,\frac{1}{f^{(p)}}\right) + $$

$$+T_0\left(r, f^{(k)}\right) - N_0\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).$$

Therefore

$$\underline{\lim}_{r\to R_0} \frac{m_0\left(r, \frac{1}{f-d}\right)}{T_0(r, f^{(n)})} \le \underline{\lim}_{r\to R_0} \frac{N_0\left(r, f^{(p)}\right)}{T_0(r, f^{(n)})} + \overline{\lim}_{r\to R_0} \frac{N_0\left(r, \frac{1}{f^{(k)}-a}\right)}{T_0(r, f^{(n)})} \le$$

$$\le - \underline{\lim}_{r\to R_0} \frac{N_0\left(r, f^{(k)}\right)}{T_0(r, f^{(n)})} - \underline{\lim}_{r\to R_0} \frac{N_0\left(r, \frac{1}{f^{(p)}}\right)}{T_0(r, f^{(n)})} \le - \underline{\lim}_{r\to R_0} \frac{N_0\left(r, \frac{1}{f^{(k)}}\right)}{T_0(r, f^{(n)})} + \underline{\lim}_{r\to R_0} \frac{T_0\left(r, f^{(k)}\right)}{T_0(r, f^{(n)}}.$$

Implies

$$_R\delta^{(0)}_{0(n)}(d, f) \le [1 -_R \Delta^{(p)}_{0(n)}(\infty, f)] - [1 -_R \Delta^{(k)}_{0(n)}(\infty, f)] - [1 -_R \Delta^{(p)}_{0(n)}(0, f)]-$$
$$-[1 -_R \Delta^{(k)}_{0(n)}(0, f)] + [1 -_R \delta^{(k)}_{0(n)}(a, f)] + 1.$$

Hence

$$_R\delta^{(0)}_{0(n)}(d, f) +_R \Delta^{(p)}_{0(n)}(\infty, f) +_R \delta^{(k)}_{0(n)}(a, f) \le_R \Delta^{(k)}_{0(n)}(\infty, f) +_R \Delta^{(p)}_{0(n)}(0, f) +_R \Delta^{(k)}_{0(n)}(0, f).$$

□

Theorem 9. *Let $f(z)$ be a transcendental or admissible meromorphic function of finite order in $\mathbb{A}(R_0)$, where $1 < R_0 \le +\infty$. Then for any two positive integers k and n,*

$$_R\Delta^{(0)}_{0(n)}(\infty, f) +_R \Delta^{(k)}_{0(n)}(0, f) \ge_R \delta^{(0)}_{0(n)}(0, f) +_R \delta^{(0)}_{0(n)}(a, f) +_R \Delta^{(k)}_{0(n)}(\infty, f),$$

where a is any non zero complex number.

Proof. Let us consider the following inequality

$$\frac{a}{f} = 1 - \frac{f-a}{f^{(k)}}.\frac{f^{(k)}}{f}.$$

Since

$$m_0\left(r, \frac{1}{f}\right) \le m_0\left(r, \frac{a}{f}\right) + O(1).$$

From the above identity, we have

$$m_0\left(r, \frac{1}{f}\right) \le m_0\left(r, \frac{f-a}{f^{(k)}}\right) + S(r, f) \tag{23}$$

By Nevanlinna first fundamental theorem for meromorphic function on annuli and from (24), we have

$$m_0\left(r, \frac{1}{f}\right) \le T_0\left(r, \frac{f-a}{f^{(k)}}\right) - N_0\left(r, \frac{f-a}{f^{(k)}}\right) + S(r, f),$$

implies

$$m_0\left(r, \frac{1}{f}\right) \le T_0\left(r, \frac{f^{(k)}}{f-a}\right) - N_0\left(r, \frac{f-a}{f^{(k)}}\right) + S(r, f).$$

Therefore

$$m_0\left(r,\frac{1}{f}\right)\le N_0\left(r,\frac{f^{(k)}}{f-a}\right)-N_0\left(r,\frac{f-a}{f^{(k)}}\right)+S(r,f). \tag{24}$$

By Jensen formula for meromorphic function on annuli and from (), we have

$$m_0\left(r,\frac{1}{f}\right)\le N_0\left(r,f^{(k)}\right)+N_0\left(r,\frac{1}{f-a}\right)-N_0\left(r,f-a\right)-N_0\left(r,\frac{1}{f^{(k)}}\right)+S(r,f).$$

Therefore

$$\underline{\lim}_{r\to R_0}\frac{m_0\left(r,\frac{1}{f}\right)}{T_0(r,f^{(n)})}\le$$

$$\le\underline{\lim}_{r\to R_0}\frac{N_0\left(r,f^{(k)}\right)}{T_0(r,f^{(n)})}-\underline{\lim}_{r\to R_0}\frac{N_0\left(r,f\right)}{T_0(r,f^{(n)})}-\underline{\lim}_{r\to R_0}\frac{N_0\left(r,\frac{1}{f^{(k)}}\right)}{T_0(r,f^{(n)})}+\overline{\lim}_{r\to R_0}\frac{N_0\left(r,\frac{1}{f-a}\right)}{T_0(r,f^{(n)})}.$$

Therefore

$${}_R\delta^{(0)}_{0(n)}(0,f)\le[1-{}_R\,\Delta^{(k)}_{0(n)}(\infty,f)]-[1-{}_R\,\Delta^{(0)}_{0(n)}(\infty,f)]-1-{}_R\,\Delta^{(k)}_{0(n)}(0,f)+1-{}_R\,\delta^{(0)}_{0(n)}(a,f).$$

Hence ${}_R\Delta^{(0)}_{0(n)}(\infty,f)+{}_R\,\Delta^{(k)}_{0(n)}(0,f)\ge{}_R\,\delta^{(0)}_{0(n)}(0,f)+{}_R\,\delta^{(0)}_{0(n)}(a,f)+{}_R\,\Delta^{(k)}_{0(n)}(\infty,f)$. □

Remark. The sign $\ge$ in Theorem 9 cannot be replaced by $>$. This we can see by following example.

Example. Let $f(z)=\exp z$. Then ${}_R\Delta^{(0)}_{0(n)}(\infty,f)={}_R\,\Delta^{(k)}_{0(n)}(\infty,f)={}_R\,\Delta^{(k)}_{0(n)}(0,f)=1$ and ${}_R\delta^{(0)}_{0(n)}(\infty,f)={}_R\,\delta^{(k)}_{0(n)}(\infty,f)=1$. So ${}_R\delta^{(0)}_{0(n)}(a,f)=0$. Then

$${}_R\Delta^{(0)}_{0(n)}(\infty,f)+{}_R\,\Delta^{(k)}_{0(n)}(0,f)=2={}_R\,\delta^{(0)}_{0(n)}(0,f)+{}_R\,\delta^{(0)}_{0(n)}(a,f)+{}_R\,\Delta^{(k)}_{0(n)}(\infty,f).$$

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