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ENTIRE BIVARIATE FUNCTIONS OF EXPONENTIAL TYPE II

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Let $f(z_1, z_2)$ be a bivariate entire function and C be a positive constant. If $f(z_1, z_2)$ satisfies the following inequality for non-negative integer M, for all non-negative integers k, l such that $k + l \in \{0, 1, 2, \ldots, M\}$, for some integer $p \geq 1$ and for all $(z_1, z_2) = (r_1 e^{\mathbf{i}\theta_1}, r_2 e^{\mathbf{i}\theta_2})$ with r_1 and r_2 sufficiently large:

$$\sum_{i+j=0}^{M} \frac{\left(\int_0^{2\pi}\int_0^{2\pi} |f^{(i+k,j+l)}(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{i!j!} \geq$$

$$\geq \sum_{i+j=M+1}^{\infty} \frac{\left(\int_0^{2\pi}\int_0^{2\pi} |f^{(i+k,j+l)}(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{i!j!},$$

then $f(z_1, z_2)$ is of exponential type not exceeding

$$2 + 2\log\left(1 + \frac{1}{C}\right) + \log[(2M)!/M!].$$

If this condition is replaced by related conditions, then also f is of exponential type.

1. Introduction. Let $f(z_1, z_2)$ be a bivariate entire function. Then the function at any point $(a, b) \in \mathbb{C}^2$ has a bivariate Taylor expansion

$$f(z_1, z_2) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{kl}(z_1 - a)^k (z_2 - b)^l,$$

where

$$c_{kl} = \frac{1}{k!l!}\left[\frac{\partial^{k+l} f(z_1, z_2)}{\partial z_1^k \partial z_2^l}\right]_{z_1=a;z_2=b} = \frac{1}{k!l!} f^{(k,l)}(a, b).$$

Recently, theory of functions having bounded index obtained the second breath. A great contribution was done by Lviv school of complex analysis. In particular, A. Bandura and O. Skaskiv extended the notion of bounded index to various classes of analytic function in multidimensional complex space. They used two approaches in their multidimensional generalizations. The first approach is based on a directional derivative in a definition of function having bounded index and leads to the notion of function of bounded L-index

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in direction where L is some positive continuous function defined in a unit ball or in $\mathbb{C}^n$. They implemented this approach for entire functions of several complex variables [7, 17] and for functions analytic in the unit ball (see [2, 6]). Another approach used all possible partial derivatives in a definition of a function having bounded index and led to the notion of functions of bounded $\mathbf{L}$-index in joint variables, where $\mathbf{L}$ is some vector-valued positive continuous function. This approach was proposed for entire functions of several complex variables [3, 8, 9], analytic functions in the unit ball [1], for analytic functions in the polydisc [4], for functions analytic in the Cartesian product of the complex plane and the unit ball [5]. These functions have applications in analytic theory of ordinary differential equations [11, 28–30], partial differential equations [6] and their system [9, 20]. In [7] last cited papers there were presented conditions providing boundedness of index for every analytic solution.

Similar to Gross [14] and Lepson [18] we presented in [21, 22] the following notion of bounded index of a bivariate entire function.

A bivariate entire function $f(z_1, z_2)$ is said to be of *bounded index* provided that there exist integers M and N independent of z_1 and z_2 such that

$$\max\left\{\frac{\left|f^{(k,l)}(z_1,z_2)\right|}{k!l!} : 0 \le k+l \le M+N\right\} \ge \frac{\left|f^{(i,j)}(z_1,z_2)\right|}{i!j!} \tag{1}$$

for all i and j and for all $(z_1, z_2) \in \mathbb{C}^2$.

Actually, the multi-dimensional generalization was firstly appeared in the papers of Salmassi, Krishna and Shah [19, 25].

We shall say the bivariate function f is of bounded index $M+N$, if $M+N$ is the smallest integers such that the above inequality holds. The least such integer $N+M$ is called the index of the function f and is denoted by $N(f)$. A bivariate entire function which is not of bounded index is said to be of unbounded index (see more details on unbounded index in [10]). One should observe that if f is a bivariate entire function of bounded index then there exist integer $M \ge 0$ and some $C > 0$,

$$\sum_{k+l=0}^{M} \frac{\left|f^{(k,l)}(z_1,z_2)\right|}{k!l!} \ge C\frac{\left|f^{(i,j)}(z_1,z_2)\right|}{i!j!}, \tag{2}$$

where $i+j \in \{M+1, M+2, \ldots\}$. In addition, if the inequality (1) in the Definition 1 holds then

$$\max\left\{\frac{\left|f^{(k,l)}(z_1,z_2)\right|}{k!l!} : 0 \le k \le M, 0 \le l \le N\right\} \ge \frac{1}{(M+1)(N+1)}\frac{\left|f^{(i,j)}(z_1,z_2)\right|}{i!j!},$$

where $i, j \in \mathbb{Z}_+$.

Let $f(z_1, z_2)$ be a bivariate entire function. Let us define

$$\tau = \varlimsup_{r_1, r_2 \to \infty} \frac{\log M_f(r_1, r_2)}{r_1 + r_2},$$

where

$$M_f(r_1, r_2) = \max\{|f(z_1, z_2)| \colon |z_1| = r_1, |z_2| = r_2\}.$$

The functions for which $\tau \ne 0$ is finite are said to be functions of exponential type τ. The definition slightly differs from a classic definition (see, for example, in [24, p.64–65]) where there is considered a ball exhaustion of $\mathbb{C}^2$. Here we consider a bidisc exhaustion of two-dimensional complex space in our definition (other interesting properties generated by the ball and the polydisc exhaustion in theory of functions having bounded index are obtained in [1]). In a recent paper Nuray and Patterson [21] considered interesting variations of condition (2) and proved the following theorem with slightly another definition of the type

$$\tau_1 = \overline{\lim_{r_1,r_2\to\infty}} \frac{\log M_f(r_1,r_2)}{r_1 r_2}.$$

Theorem 1. *Let $f(z_1,z_2)$ be a bivariate entire function and C be a positive constant. If f satisfies one of the following for $k\in\{0,1,2,3,\ldots,M\}$, $l\in\{0,1,2,3,\ldots,N\}$ and for all (z_1,z_2) with $|z_1|$ and $|z_2|$ sufficiently large:*

$$\sum_{i=0}^{M}\sum_{j=0}^{N}\frac{|f^{(i,j)}(z_1,z_2)|}{i!lj!} > C\sum_{i=M+1}^{\infty}\sum_{j=N+1}^{\infty}\frac{|f^{(i,j)}(z_1,z_2)|}{i!j!},$$

$$\sum_{i=0}^{M}\sum_{j=0}^{N}\frac{\left(\int_0^{2\pi}\int_0^{2\pi}|f^{(i+k,j+l)}(r_1e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{i!j!}\geq$$

$$\geq C\sum_{i=M+1}^{\infty}\sum_{j=N+1}^{\infty}\frac{\left(\int_0^{2\pi}\int_0^{2\pi}|f^{(i+k,j+l)}(r_1e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{i!j!} \quad \text{for some integer } p,$$

$$\sum_{i=0}^{M}\sum_{j=0}^{N}\frac{M(r_1,r_2,f^{(i,j)})}{i!lj!} > C\sum_{i=M+1}^{\infty}\sum_{j=N+1}^{\infty}\frac{M(r_1,r_2,f^{(i,j)})}{i!lj!},$$

then f is of exponential type $\tau_1<+\infty$.

Shah [27] and Hayman [16] is known that every entire functions of bounded index is of exponential type. Therefore, the functions of bounded index has properties of functions of exponential type (for example, see [12, 13, 23, 26, 31]).

2. Main result. The following results for functions of a sing variable having bounded index was firstly obtained in [26]. Here we deduce a two-dimensional analog.

Theorem 2. *Let $f(z_1,z_2)$ be a bivariate entire function, C be a positive constant. If $f(z_1,z_2)$ satisfies the following inequality for non-negative integer M, for all non-negative integers k, l such that $k+l\in\{0,1,2,\ldots,M\}$, for some integer $p\geq 1$ and for all $(z_1,z_2)=(r_1e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})$ with $r_1>r_1'>1$ and $r_2>r_2'>1$ sufficiently large:*

$$\sum_{i+j=0}^{M}\frac{\left(\int_0^{2\pi}\int_0^{2\pi}|f^{(i+k,j+l)}(r_1e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{i!j!}\geq$$

$$\geq C\sum_{i+j=M+1}^{\infty}\frac{\left(\int_0^{2\pi}\int_0^{2\pi}|f^{(i+k,j+l)}(r_1e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{i!j!}, \tag{3}$$

then $f(z_1,z_2)$ is of exponential type

$$\tau\leq 2+2\log(1+\frac{1}{C})+2\log\frac{(2M)!}{M!}.$$

Proof. One should observe that $\frac{(k+i)!}{i!} = (i+1)\cdot(i+2)\cdot\ldots\cdot(i+k) \le \frac{(2M)!}{M!}$ for $k,i \in \{0,1,2,\ldots,M\}$. We have for $r_1 > r_1'$ and $r_2 > r_2'$

$$\sum_{i+j=0}^{M} \frac{\left(\int_0^{2\pi}\int_0^{2\pi} |f^{(i+k,j+l)}(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{i!j!} =$$

$$= \sum_{i+j=0}^{M} \frac{\left(\int_0^{2\pi}\int_0^{2\pi} |f^{(i+k,j+l)}(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{(k+i)!(l+j)!}\frac{(k+i)!(l+j)!}{i!j!} \le$$

$$\le ((2M)!/M!)^2 \sum_{i+j=0}^{2M} \frac{\left(\int_0^{2\pi}\int_0^{2\pi} |f^{(i,j)}(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{i!j!} =$$

$$= ((2M)!/M!)^2 \left(\sum_{i+j=0}^{M} + \sum_{i+j=M+1}^{2M}\right) \frac{\left(\int_0^{2\pi}\int_0^{2\pi} |f^{(i,j)}(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{i!j!} \le$$

$$\le ((2M)!/M!)^2 \left(1+\frac{1}{C}\right) \sum_{i+j=0}^{M} \frac{\left(\int_0^{2\pi}\int_0^{2\pi} |f^{(i,j)}(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}}{i!j!}. \tag{4}$$

Here we have used the hypothesis (3) with $k = 0$ and $l = 0$ to obtain the last inequality. In the Taylor expansion of the function $f^{(k,l)}$ in the variable z_1

$$f^{(k,l)}(a+h_1, b) = \sum_{i=0}^{\infty} \frac{f^{(k+i,l)}(a,b)}{i!} h_1^i$$

we put $a = (r_1-1)e^{\mathbf{i}\theta_1}$, $h_1 = e^{\mathbf{i}\theta_1}$, and obtain

$$|f^{(k,l)}(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p = \left|\sum_{i=0}^{\infty} \frac{f^{(i+k,l)}((r_1-1)e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})e^{\mathbf{i}\theta_1 i}}{i!}\right|^p, \tag{5}$$

and so

$$\left(\int_0^{2\pi}\int_0^{2\pi} |f^{(k,l)}(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}} \le$$

$$\le \left(\int_0^{2\pi}\int_0^{2\pi} \left\{\sum_{i=0}^{\infty} \frac{\left|f^{(i+k,l)}((r_1-1)e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})\right|}{i!}\right\}^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}. \tag{6}$$

Applying Minkowski's inequality [15, p. 148] to right-hand side of inequality (6), we have for $r_1 > r_1' + 1$ and $r_2 > r_2'$

$$\left(\int_0^{2\pi}\int_0^{2\pi} |f^{(k,l)}(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}} \le$$

$$\le \sum_{i=0}^{\infty} \left(\int_0^{2\pi}\int_0^{2\pi} \left\{\frac{\left|f^{(i+k,l)}((r_1-1)e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})\right|}{i!}\right\}^p d\theta_1 d\theta_2\right)^{\frac{1}{p}} =$$

$$= \left(\sum_{i=0}^{M} + \sum_{i=M+1}^{\infty} \right) \frac{\left(\int_0^{2\pi} \int_0^{2\pi} \left| f^{(i+k,l)}((r_1-1)e^{\mathbf{i}\theta_1}, r_2 e^{\mathbf{i}\theta_2}) \right|^p d\theta_1 d\theta_2 \right)^{\frac{1}{p}}}{i!}. \tag{7}$$

Applying inequality (3) with $j = 0$ in the right-hand side to the sum $\sum_{i=M+1}^{\infty}$ in (7) we deduce

$$\left(\int_0^{2\pi} \int_0^{2\pi} |f^{(k,l)}(r_1 e^{\mathbf{i}\theta_1}, r_2 e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2 \right)^{\frac{1}{p}} \leq$$

$$\leq \left(1 + \frac{1}{C}\right) \sum_{i+j=0}^{M} \frac{\left(\int_0^{2\pi} \int_0^{2\pi} |f^{(i+k,j+l)}((r_1-1)e^{\mathbf{i}\theta_1}, r_2 e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2 \right)^{1/p}}{i!j!}. \tag{8}$$

Applying inequality (4) to (8), we obtain

$$\left(\int_0^{2\pi} \int_0^{2\pi} |f^{(k,l)}(r_1 e^{\mathbf{i}\theta_1}, r_2 e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2 \right)^{\frac{1}{p}} \leq$$

$$\leq \left(1 + \frac{1}{C}\right)^2 ((2M)!/M!)^2 \sum_{i+j=0}^{M} \frac{\left(\int_0^{2\pi} \int_0^{2\pi} |f^{(i,j)}((r_1-1)e^{\mathbf{i}\theta_1}, r_2 e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2 \right)^{1/p}}{i!j!}.$$

Hence,

$$\sum_{k+l=0}^{M} \frac{\left(\int_0^{2\pi} \int_0^{2\pi} |f^{(k,l)}(r_1 e^{\mathbf{i}\theta_1}, r_2 e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2 \right)^{\frac{1}{p}}}{k!l!} \leq$$

$$\leq \left(1 + \frac{1}{C}\right)^2 ((2M)!/M!)^2 \sum_{k+l=0}^{M} \frac{1}{k!l!} \cdot \sum_{i+j=0}^{M} \frac{\left(\int_0^{2\pi} \int_0^{2\pi} |f^{(i,j)}((r_1-1)e^{\mathbf{i}\theta_1}, r_2 e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2 \right)^{1/p}}{i!j!} \leq$$

$$\leq e^2 \left(1 + \frac{1}{C}\right)^2 ((2M)!/M!)^2 \sum_{i+j=0}^{M} \frac{\left(\int_0^{2\pi} \int_0^{2\pi} |f^{(i,j)}((r_1-1)e^{\mathbf{i}\theta_1}, r_2 e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2 \right)^{1/p}}{i!j!}. \tag{9}$$

Denote

$$\lambda = \left(e\left(1 + \frac{1}{C}\right)(2M)!/M! \right)^2 > 1$$

and

$$\xi(r_1, r_2) = \sum_{k+l=0}^{M} \frac{\left(\int_0^{2\pi} \int_0^{2\pi} |f^{(k,l)}(r_1 e^{\mathbf{i}\theta_1}, r_2 e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2 \right)^{1/p}}{k!l!}.$$

Then inequality (9) can be rewritten as $\xi(r_1, r_2) \leq \lambda \xi(r_1 - 1, r_2)$. Applying this inequality $[r_1 - r_1']$ times, we obtain

$$\xi(r_1, r_2) \leq \lambda^{[r_1 - r_1']} \xi(r_1 - [r_1 - r_1'], r_2) \leq \lambda^{r_1} \xi(r_1' + \{r_1 - r_1'\}, r_2), \tag{10}$$

where $[x]$ is the entire part of a real x, $\{x\}$ is the fractional part of a real x.

Replacing the Taylor expansion in the variable z_1 by the Taylor expansion in the variable z_2 in equation (5) and repeating other considerations from (6) up to (9) it can be proved $\xi(r_1, r_2) \le \lambda\xi(r_1, r_2 - 1)$. Again Applying the last inequality $[r_2 - r_2']$ times in variable r_2 to (10) again we deduce

$$\xi(r_1, r_2) \le \lambda^{r_1+[r_2-r_2']}\xi(r_1' + \{r_1 - r_1'\}, r_2 - [r_2 - r_2']) \le \lambda^{r_1+r_2}\xi(r_1' + \{r_1 - r_1'\}, r_2' + \{r_2 - r_2'\}).$$

Therefore, for $r_1 > r_1'$ and $r_2 > r_2'$ we get

$$\xi(r_1, r_2) \le C_1\lambda^{r_1+r_2}, \tag{11}$$

where

$$C_1 = C_1(M, p, r_1', r_2', f) := \max_{\substack{s_1\in[r_1';r_1'+1],\\ s_2\in[r_2';r_2'+1]}} \sum_{k+l=0}^{M} \frac{\left(\int_0^{2\pi}\int_0^{2\pi} |f^{(k,l)}(s_1e^{\mathbf{i}\theta_1}, s_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{1/p}}{k!l!}$$

is a constant.

Now write

$$f(z_1, z_2) = \sum_{m+n=0}^{\infty} a_{mn} z_1^m z_2^n, \quad \mu(r_1, r_2) = \max\{|a_{mn}| r_1^m r_2^n \colon m, n \in \mathbb{Z}_+\}.$$

Then from Cauchy's and Hölder's inequalities one has

$$|a_{mn}| r_1^m r_2^m \le \frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} |f(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})| d\theta_1 d\theta_2 \le$$
$$\le \frac{1}{4\pi^2}\left(\int_0^{2\pi}\int_0^{2\pi} |f(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}}\left(\int_0^{2\pi}\int_0^{2\pi} d\theta_1 d\theta_2\right)^{\frac{1}{q}} =$$
$$= (4\pi^2)^{-1/p}\left(\int_0^{2\pi}\int_0^{2\pi} |f(r_1e^{\mathbf{i}\theta_1}, r_2e^{\mathbf{i}\theta_2})|^p d\theta_1 d\theta_2\right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence, estimate (11) gives

$$\mu(r_1, r_2) \le (4\pi^2)^{-\frac{1}{p}} C_1\lambda^{r_1+r_2}. \tag{12}$$

It is known that for $\alpha > 1$ one has

$$\mu(r_1, r_2) \le M_f(r_1, r_2) \le \sum_{k+l=0}^{\infty} |c_{k,l}| r_1^k r_2^l = \sum_{k+l=0}^{\infty} |c_{k,l}| (\alpha r_1)^k (\alpha r_2)^l \cdot \frac{1}{\alpha^{k+l}} \le$$
$$\le \mu(\alpha r_1, \alpha r_2) \sum_{k+l=0}^{\infty} \frac{1}{\alpha^{k+l}} = \left(\frac{\alpha}{\alpha - 1}\right)^2 \mu(\alpha r_1, \alpha r_2).$$

Then

$$\varlimsup_{r_1,r_2\to\infty} \frac{\log\log\mu(r_1, r_2)}{\log(r_1 + r_2)} = \varlimsup_{r_1,r_2\to\infty} \frac{\log\log M_f(r_1, r_2)}{\log(r_1 + r_2)} = \rho.$$

In view of (12) the order ρ is finite and $\rho \le 1$. Therefore, we yield that $f(z_1, z_2)$ is of exponential type and

$$\tau = \overline{\lim_{r_1,r_2\to\infty}} \frac{\log M_f(r_1,r_2)}{r_1+r_2} \le \overline{\lim_{r_1,r_2\to\infty}} \frac{\log((\frac{\alpha}{\alpha-1})^2\mu(\alpha r_1,\alpha r_2))}{r_1+r_2} = \overline{\lim_{r_1,r_2\to\infty}} \frac{\log(\mu(\alpha r_1,\alpha r_2))}{r_1+r_2} \le$$
$$\le \overline{\lim_{r_1,r_2\to\infty}} \frac{\log((4\pi^2)^{-\frac{1}{p}}C_1\lambda^{\alpha(r_1+r_2)})}{r_1+r_2} = \alpha\log\lambda.$$

In view of arbitrariness $\alpha > 1$, we tends α to 1 and conclude that $\tau \le \log\lambda$. The proof is completed. □

Theorem 3. *Let $f(z_1, z_2)$ be a bivariate entire function and C be a positive constant. If f satisfies the following inequality for non-negative integer M and for non-negative integers k, l such that $k+l \in \{0,1,2,\dots,M\}$ and for all (z_1, z_2) with $|z_1| > r_1'$ and $|z_2| > r_2'$ sufficiently large:*

$$\sum_{i+j=0}^{M} \frac{|f^{(k+i,l+j)}(z_1,z_2)|}{i!j!} \ge C \sum_{i+j=M+1}^{\infty} \frac{|f^{(k+i,l+j)}(z_1,z_2)|}{i!j!}, \tag{13}$$

then $f(z_1, z_2)$ is of exponential type and

$$\tau \le 2 + 2\log\Big(1+\frac{1}{C}\Big)+2\log\left(\frac{(2M)!}{M!}\right).$$

Proof. For any entire function $F : \mathbb{C}^2 \to \mathbb{C}$ and $(z_1', z_2') \in \mathbb{C}^2$ one has

$$F(z_1,z_2) = \sum_{i+j=0}^{\infty} \frac{F^{(i,j)}(z_1',z_2')}{i!j!}(z_1-z_1')^i(z_2-z_2')^j. \tag{14}$$

Let n_1 be any integer, a_1, $\xi_1 \in \mathbb{C}$ with $|\xi_1| = 1$, $|a_1| \le r_1'$. Choosing $z_1' = (n_1-1)\xi_1 + a_1$, $z_1 = n_1\xi_1 + a_1$ and $F = f^{(i,j)}$ we obtain in (14)

$$|f^{(i,j)}(a_1+n_1\xi_1,z_2)| \le \sum_{k=0}^{\infty} \frac{|f^{(i+k,j)}(a_1+(n_1-1)\xi_1,z_2)|}{k!}. \tag{15}$$

for $i \in \mathbb{Z}_+$. Applying (13) to (15) one has

$$|f^{(i,j)}(a_1+n_1\xi_1,z_2)| \le \sum_{k=0}^{\infty} \frac{|f^{(i+k,j)}(a_1+(n_1-1)\xi_1,z_2)|}{k!} \le$$
$$\le \sum_{k+l=0}^{\infty} \frac{|f^{(i+k,j+l)}(a_1+(n_1-1)\xi_1,z_2)|}{k!l!} = \left(\sum_{k+l=0}^{M} + \sum_{k+l=M+1}^{\infty}\right) \frac{|f^{(i+k,j+l)}(a_1+(n_1-1)\xi_1,z_2)|}{k!l!} \le$$
$$\le (1+\frac{1}{C}) \sum_{k+l=0}^{M} \frac{|f^{(i+k,j+l)}(a_1+(n_1-1)\xi_1,z_2)|}{k!l!} \tag{16}$$

for $i+j \in \{0,1,2,\dots,M\}$. We remark that for $i+j \le M$

$$\sum_{k+l=0}^{M} \frac{|f^{(i+k,j+l)}(z_1,z_2)|}{k!l!} = \sum_{k+l=0}^{M} \frac{|f^{(i+k,j+l)}(z_1,z_2)|}{(i+k)!(j+l)!} \cdot \frac{(i+k)!(j+l)!}{k!l!} \le$$

$$\leq \sum_{s+p=0}^{2M} \frac{|f^{(s,p)}(z_1,z_2)|}{s!p!} \cdot \left(\frac{(2M)!}{M!}\right)^2 = \left(\sum_{s+p=0}^{M} + \sum_{s+p=M+1}^{2M}\right) \frac{|f^{(s,p)}(z_1,z_2)|}{s!p!} \cdot \left(\frac{(2M)!}{M!}\right)^2.$$

We apply estimate (13) to the last inequality, and we obtain

$$\sum_{k+l=0}^{M} \frac{|f^{(i+k,j+l)}(z_1,z_2)|}{k!l!} \leq \left(1+\frac{1}{C}\right)\left(\frac{(2M)!}{M!}\right)^2 \sum_{s+p=0}^{M} \frac{|f^{(s,p)}(z_1,z_2)|}{s!p!}. \tag{17}$$

Equations (16) and (17) with $z_1 = a_1 + n_1\xi_1$ together give

$$\sum_{i+j=0}^{M} \frac{|f^{(i,j)}(a_1+n_1\xi_1,z_2)|}{i!j!} \leq \left(1+\frac{1}{C}\right) \sum_{i+j=0}^{M} \frac{1}{i!j!} \sum_{k+l=0}^{M} \frac{|f^{(i+k,j+l)}(a_1+(n_1-1)\xi_1,z_2)|}{k!l!} \leq$$
$$\leq \left(1+\frac{1}{C}\right)^2 \left(\frac{(2M)!}{M!}\right)^2 \sum_{i+j=0}^{M} \frac{1}{i!j!} \sum_{s+p=0}^{M} \frac{|f^{(s,p)}(a_1+(n_1-1)\xi_1,z_2)|}{s!p!} \leq$$
$$\leq e^2\left(1+\frac{1}{C}\right)^2 \left(\frac{(2M)!}{M!}\right)^2 \sum_{s+p=0}^{M} \frac{|f^{(s,p)}(a_1+(n_1-1)\xi_1,z_2)|}{s!p!}. \tag{18}$$

Denoting $\lambda = e^2(1+\frac{1}{C})^2\left(\frac{(2M)!}{M!}\right)^2 > 1$ and using (18) recursively we have

$$\sum_{i+j=0}^{M} \frac{|f^{(i,j)}(a_1+n_1\xi_1,z_2)|}{i!j!} \leq \lambda^{n_1} \sum_{s+p=0}^{M} \frac{|f^{(s,p)}(a_1,z_2)|}{s!p!}.$$

Let n_2 be any integer, a_2, $\xi_2 \in \mathbb{C}$ with $|\xi_2| = 1$, $|a_2| \leq r_2'$. As above, choosing $z_2' = (n_2-1)\xi_2 + a_2$, $z_2 = n_2\xi_2 + a_2$ and repeating all considerations from (15) to (18) in variable z_2 we can recursively prove that

$$\sum_{i+j=0}^{M} \frac{|f^{(i,j)}(a_1+n_1\xi_1,a_2+n_2\xi_2)|}{i!j!} \leq \lambda^{n_1+n_2} \sum_{s+p=0}^{M} \frac{|f^{(s,p)}(a_1,a_2)|}{s!p!}.$$

Hence, for $|a_1| < 1$, $|a_2| \leq 1$ we get

$$\sum_{i+j=0}^{M} \frac{|f^{(i,j)}(a_1+n_1\xi_1,a_2+n_2\xi_2)|}{i!j!} \leq C\lambda^{n_1+n_2},$$

where $C = C(M,f) = \max\{\sum_{s+p=0}^{M} \frac{|f^{(s,p)}(a_1,a_2)|}{s!p!} : |a_1| \leq r_1', |a_2| \leq r_2'\}$ is a constant.

Letting $z_1 = a_1 + n_1\xi_1$, $z_1 = a_2 + n_2\xi_2$, $r_1 = |z_1|$, $r_2 = |z_2|$, we get $n_1 = \frac{z_1-a_1}{\xi_1} = |\frac{z_1-a_1}{\xi_1}| \leq |z_1| + |a_1| \leq r_1 + 1$, $n_2 = \frac{z_2-a_2}{\xi_2} = |\frac{z_2-a_2}{\xi_2}| \leq |z_2| + |a_2| \leq r_2 + 1$. Therefore, we deduce

$$\sum_{i+j=0}^{M} \frac{|f^{(i,j)}(z_1,z_2)|}{i!j!} \leq C\lambda^{n_1+n_2} \leq C\lambda^{r_1+r_2+2}.$$

Then $|f(z_1,z_2)| \leq C\lambda^{r_1+r_2+2}$. Hence, f must be of exponential type and its type τ do not exceed $\log\lambda$. $\square$

Theorem 4. *Let $f(z_1,z_2)$ be a bivariate entire function and C be a positive constant. If f satisfies the following inequality for non-negative integer M, for all non-negative integers k, l such that $k+l\in\{0,1,2,\dots,M\}$ and for all $(z_1,z_2)=(r_1e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})$ with $r_1>r_1'>1$ and $r_2>r_2'>1$ sufficiently large:*

$$\sum_{i+j=0}^{M}\frac{|M(r_1,r_2,f^{(i+k,j+l)})|}{i!j!}>C\sum_{i+j=M+1}^{\infty}\frac{|M(r_1,r_2,f^{(i+k,j+l)})|}{i!j!},\tag{19}$$

then $f(z_1,z_2)$ is of exponential type and

$$\tau\le 2+2\log\left(1+\frac{1}{C}\right)+2\log\left(\frac{(2M)!}{M!}\right).$$

Proof. The proof is similar to that of Theorem 2. We have for $r_1>r_1'$ and $r_2>r_2'$

$$\begin{gathered}\sum_{i+j=0}^{M}\frac{M(r_1,r_2,f^{(i+k,j+l)})}{i!j!}=\sum_{i+j=0}^{M}\frac{M(r_1,r_2,f^{(i+k,j+l)})}{(k+i)!(l+j)!}\frac{(k+i)!(l+j)!}{i!j!}\le\\ \le((2M)!/M!)^2\sum_{i+j=0}^{2M}\frac{M(r_1,r_2,f^{(i,j)})}{i!j!}=((2M)!/M!)^2\left(\sum_{i+j=0}^{M}+\sum_{i+j=M+1}^{2M}\right)\frac{M(r_1,r_2,f^{(i,j)})}{i!j!}\le\\ \le((2M)!/M!)^2\left(1+\frac{1}{C}\right)\sum_{i+j=0}^{M}\frac{M(r_1,r_2,f^{(i,j)})}{i!j!}.\end{gathered}\tag{20}$$

Here we have used the hypothesis (19) with $k=0$ and $l=0$ to obtain the last inequality. In the Taylor expansion of the function $f^{(k,l)}$ in the variable z_1

$$f^{(k,l)}(a+h_1,b)=\sum_{i=0}^{\infty}\frac{f^{(k+i,l)}(a,b)}{i!}h_1^i$$

we put $a=(r_1-1)e^{\mathbf{i}\theta_1}$, $h_1=e^{\mathbf{i}\theta_1}$, and obtain

$$|f^{(k,l)}(r_1e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})|=\left|\sum_{i=0}^{\infty}\frac{f^{(i+k,l)}((r_1-1)e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})e^{\mathbf{i}\theta_1 i}}{i!}\right|\tag{21}$$

and so

$$\begin{gathered}M(r_1,r_2,f^{(k,l)})=\max_{(\theta_1,\theta_2)\in[0,2\pi]^2}|f^{(k,l)}(r_1e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})|\le\\ \le\max_{(\theta_1,\theta_2)\in[0,2\pi]^2}\left|\sum_{i=0}^{\infty}\frac{f^{(i+k,l)}((r_1-1)e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})e^{\mathbf{i}\theta_1 i}}{i!}\right|\le\\ \le\sum_{i=0}^{\infty}\frac{\max_{(\theta_1,\theta_2)\in[0,2\pi]^2}|f^{(i+k,l)}((r_1-1)e^{\mathbf{i}\theta_1},r_2e^{\mathbf{i}\theta_2})|}{i!}=\sum_{i=0}^{\infty}\frac{M(r_1-1,r_2,f^{(i+k,l)})}{i!}=\\ =\left(\sum_{i=0}^{M}+\sum_{i=M+1}^{\infty}\right)\frac{M(r_1-1,r_2,f^{(i+k,l)})}{i!}.\end{gathered}\tag{22}$$

Applying inequality (19) with $j = 0$ in the right-hand side to the sum $\sum_{i=M+1}^{\infty}$ in (22) we deduce

$$M(r_1, r_2, f^{(k,l)}) \le \Big(1 + \frac{1}{C}\Big) \sum_{i+j=0}^{M} \frac{M(r_1 - 1, r_2, f^{(i+k,j+l)})}{i!j!}. \tag{23}$$

Applying inequality (20) to (23), we obtain

$$M(r_1, r_2, f^{(k,l)}) \le \Big(1 + \frac{1}{C}\Big)^2 ((2M)!/M!)^2 \le \sum_{i+j=0}^{M} \frac{M(r_1 - 1, r_2, f^{(i,j)})}{i!j!}.$$

Hence,

$$\sum_{k+l=0}^{M} \frac{M(r_1, r_2, f^{(k,l)})}{k!l!} \le \Big(1 + \frac{1}{C}\Big)^2 ((2M)!/M!)^2 \sum_{k+l=0}^{M} \frac{1}{k!l!} \cdot \sum_{i+j=0}^{M} \frac{M(r_1 - 1, r_2, f^{(i,j)})}{i!j!} \le$$
$$\le e^2 \Big(1 + \frac{1}{C}\Big)^2 ((2M)!/M!)^2 \sum_{i+j=0}^{M} \frac{M(r_1 - 1, r_2, f^{(i,j)})}{i!j!}. \tag{24}$$

Denote

$$\lambda = \Big(e\Big(1 + \frac{1}{C}\Big)(2M)!/M!\Big)^2 > 1$$

and

$$\xi(r_1, r_2) = \sum_{k+l=0}^{M} \frac{M(r_1, r_2, f^{(k,l)})}{k!l!}.$$

Then inequality (24) can be rewritten as $\xi(r_1, r_2) \le \lambda \xi(r_1 - 1, r_2)$. Applying this inequality $[r_1 - r_1']$ times, we obtain

$$\xi(r_1, r_2) \le \lambda^{[r_1 - r_1']} \xi(r_1 - [r_1 - r_1'], r_2) \le \lambda^{r_1} \xi(r_1' + \{r_1 - r_1'\}, r_2), \tag{25}$$

where $[x]$ is the entire part of a real x, $\{x\}$ is the fractional part of the real x.

Replacing the Taylor expansion in the variable z_1 by the Taylor expansion in the variable z_2 in equation (21) and repeating other considerations from (22) up to (24) it can be proved $\xi(r_1, r_2) \le \lambda \xi(r_1, r_2 - 1)$. Again applying the last inequality $[r_2 - r_2']$ times in variable r_2 to (10) we deduce

$$\xi(r_1, r_2) \le \lambda^{r_1 + [r_2 - r_2']} \xi(r_1' + \{r_1 - r_1'\}, r_2 - [r_2 - r_2']) \le \lambda^{r_1 + r_2} \xi(r_1' + \{r_1 - r_1'\}, r_2' + \{r_2 - r_2'\}).$$

Therefore, for $r_1 > r_1'$ and $r_2 > r_2'$ we get

$$\xi(r_1, r_2) \le C_1 \lambda^{r_1 + r_2} \tag{26}$$

where $C_1 = C_1(M, p, r_1', r_2', f) := \max\Big\{ \sum_{k+l=0}^{M} \frac{M(s_1, s_2, f^{(k,l)})}{k!l!} \colon s_1 \in [r_1'; r_1'+1], s_2 \in [r_2'; r_2'+1] \Big\}$ is a constant. Inequality (26) yields

$$M(r_1, r_2, f) \le C_1 \lambda^{r_1 + r_2}. \tag{27}$$

In view of (27) the order ρ is finite and $\rho \le 1$. Then $f(z_1, z_2)$ is of exponential type and $\tau \le \log \lambda$. □

Remark 1. Proof of Theorem 2 is similar to proof of corresponding theorem for entire functions in [26]. Other results from [26] use Wiman-Valiron's theory for entire function of one variable: $\frac{M(r_n,f^{(q)})}{M(r_n,f^{(j)})} \sim (\frac{\nu(r_n,f)}{r})^{q-j}$, where $r_n \to +\infty$ as $n \to \infty$, $\nu(r,f)$ is the central index. At the present moment, the authors do not know applicable multidimensional analog of the theory in this case. Therefore, Theorems 3 and 4 are obtained by methods from [13] with similar harder conditions.

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