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THE NUMBER OF STANDARD FORMS OF MATRICES OVER IMAGINARY EUCLIDEAN QUADRATIC RINGS WITH RESPECT TO THE (z,k)–EQUIVALENCE

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The (z,k)–equivalence of matrices over imaginary Euclidean quadratic rings is investigated. The classes of matrices over these rings are selected for which the standard form with respect to (z,k)–equivalence is uniquely defined and equal to the Smith normal form. It is established that the number of standard forms over imaginary Euclidean quadratic rings is finite. Bounds for a number of standard forms are established.

1. Introduction and preliminaries. The classical notion of equivalence of matrices are well known, starting with H. Smith (1861), who established a canonical form of matrices over a ring of integers $\mathbb{Z}$ called the Smith normal form, and distributed by many authors to other rings.

In many problems other types of equivalences of matrices over various rings: arise elementary equivalence of matrices over Euclidean rings [1], scalar [2] and semiscalar equivalences of polynomial matrices [3, 4] or the $(P,S)-$ equivalence [5]. The established special triangular form of polynomial matrices with respect to the semiscalar equivalence is used in the theory of matrix factorization [3, 6], for solving of matrix Sylvester-type equations [7], etc.

We investigate the equivalence of matrices over quadratic rings. Matrices over quadratic rings are used in the number theory and other sections of mathematics. The structure of such matrices was studied only over certain quadratic rings, in particular over Euclidean quadratic rings, the ring of Gaussian integers.

In [8] the problem of similarity of matrices over the ring of Gaussian integers $\mathbb{Z}[i]$ is considered. Canonical forms with respect to the transformation of similarity for 2×2 matrices over $\mathbb{Z}[i]$ with reducible characteristic polynomial, i.e. $\det(I\lambda - A) = d(\lambda)$ and $d(\lambda) = (\lambda-\alpha_1)(\lambda-\alpha_2)$, where $\alpha_1 \neq \alpha_2$ or $\alpha_1=\alpha_2=\alpha$ and $d(\lambda)=(\lambda-\alpha)^2$ are constructed.

In number theory, is well known, the concept of the Kloosterman sum, i.e. an exponential sum over a redused residue system modulo q:

$$K\,(u,v;q) = \Sigma'_{x \pmod q} e^{2\pi i \frac{ux+vx^{-1}}{q}},$$

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where q is a positive integer, u, v is fixed integers here and in seque x^{-1} denote the reciprocal to x modulo q, $xx^{-1} \equiv 1 \pmod{q}$. In [9], [10] generalized Kloosterman sum distributed over a ring of matrices over the ring of integers and the ring of Gaussian integers $\mathbb{Z}[i]$ were studied. Estimates of these amounts are given.

In [11], [12] the so-called cyclotomic matrices over quadratic rings are studied, in particular over Gaussian integers. An Hermitian matrix with integral characteristic polynomial whose zeros are contained in the interval $[-2; 2]$ is called cyclotomic matrix.

We investigate the equivalence of matrices over quadratic rings. In [13, 14], a special equivalence of matrices over these rings is investigated and simpler forms of matrices with respect to the this equivalence are established. In [15], the notion of (z, k)–equivalence of matrices was introduced and standard forms of matrices with respect to this equivalence was established.

Let $\mathbb{Z}$ be ring of integers. Then $\mathbb{K} = \mathbb{Z}[\sqrt{k}]$ is a quadratic ring, where $k \neq 0, 1$ is a square-free element of $\mathbb{Z}$. The quadratic ring $\mathbb{K} = \mathbb{Z}[\sqrt{k}]$ is called real if $k > 0$. If $k < 0$, $\mathbb{K} = \mathbb{Z}[\sqrt{k}]$ is called an imaginary quadratic ring. It is established that there are five Euclidean quadratic rings for $k = -1, -2, -3, -7, -11$. It is known that among the quadratic rings there are Euclidean quadratic rings, among them there are quadratic principal ideal rings which are non-Euclidean, for example, the rings $\mathbb{Z}\left[\sqrt{-19}\,\right]$ is a principal ideal ring, but it is non-Euclidean. There are some quadratic rings that are not the principal ideal rings, for example, the ring $\mathbb{Z}[\sqrt{-5}]$.

The elements $a \in \mathbb{Z}[\sqrt{k}]$ and their algebraic norm $N(a) \in \mathbb{Z}$ are determined in the following way:

if $k \equiv 2 \pmod 4$ or $k \equiv 3 \pmod 4$, then

$$\mathbb{Z}\left[\sqrt{k}\,\right] = \left\{a_1 + a_2\sqrt{k} \;\middle|\; a_1, a_2 \in \mathbb{Z}\right\}, \quad N(a_1 + a_2\sqrt{k}) = a_1^2 - ka_2^2,$$

if $k \equiv 1 \pmod 4$, then

$$\mathbb{Z}\left[\sqrt{k}\right] = \left\{\frac{a_1}{2} + \frac{a_2}{2}\sqrt{k} \;\middle|\; a_1, a_2 \in \mathbb{Z},\ (a_1 - a_2) \,\vdots\, 2\right\}, \quad N\left(\frac{a_1}{2} + \frac{a_2}{2}\sqrt{k}\right) = \frac{1}{4}(a_1^2 - ka_2^2).$$

Let $\mathbb{K}$ be a Euclidean quadratic ring. Then the Euclidean norm $\mathcal{E}(a) \in \mathbb{N}$ of an element $a \in \mathbb{K}$ can be defined as follow:

$$\begin{cases} \mathcal{E}(a) = N(a), & \text{if } \mathbb{K} \text{ is an imaginary;} \\ \mathcal{E}(a) = |N(a)|, & \text{if } \mathbb{K} \text{ is a real Euclidean quadratic ring,} \end{cases}$$

where $N(a)$ is algebraic norm of an element $a \in \mathbb{K}$.

It what follows $M(n, \mathbb{K})$ will denote the ring of $n \times n$ matrices over the quadratic ring $\mathbb{K}$.

Every an $n \times n$ matrix A over a Euclidean quadratic ring $\mathbb{K} = \mathbb{Z}[\sqrt{k}\,]$ is equivalent to the Smith normal form, i.e.

$$UAV = S^A = \operatorname{diag}(\varphi_1, \varphi_2, \ldots, \varphi_n), \quad \varphi_i | \varphi_{i+1},$$

$i = 1, \ldots, n-1$, where $U, V \in GL(n, \mathbb{K})$.

Definition 1. Matrices $A, B \in M(n, \mathbb{K})$ are called (z, k)-*equivalent* if there exist invertible matrices $S \in GL(n, \mathbb{Z})$ over ring of integers $\mathbb{Z}$ and a matrix $Q \in GL(n, \mathbb{K})$ over the quadratic ring $\mathbb{K}$, such that $A = SBQ$.

In [15] it is established that each matrix over Euclidean quadratic ring is reduced by (z,k)–equivalent transformations to such a special triangular form.

Theorem 1. *Let A be an $n\times n$ matrix over Euclidean quadratic ring $\mathbb{K}$, $\det A \neq 0$. Then the matrix A is (z,k)–equivalent to the triangular matrix T^A, i.e. there exist an invertible matrix $S\in GL(n,\mathbb{Z})$ and an invertible matrix $Q\in GL(n,\mathbb{K})$ such that*

$$T^A = SAQ = \begin{Vmatrix} \varphi_1 & 0 & \dots & 0 \\ t_{21}\varphi_1 & \varphi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ t_{n1}\varphi_1 & t_{n2}\varphi_2 & \dots & \varphi_n \end{Vmatrix} \qquad (1)$$

where

$$\begin{cases} t_{ij}=0, \quad \text{if } \varphi_i=1, \quad i,j=1,...,n, \quad j<i; \\ \mathcal{E}(t_{ij}) < \frac{\mathcal{E}(\varphi_i)}{\mathcal{E}(\varphi_j)}, \ \text{if } t_{ij}\neq 0, \ \ i,j=1,...,n, \ j<i. \end{cases}$$

In [14] a similar triangular form with invariant factors on the main diagonal for matrices over quadratic rings of the principal ideals is established.

A triangular form T^A of the form (1) is called the standard form of the matrix A with respect to the (z,k)–equivalence. In particular, standard forms of matrices are used for solving matrix linear bilateral equations over quadratic rings and for descriptions of the structure of solutions these equations [15, 16].

The standard form is defined ambiguously. Therefore, there is a problem to distinguish classes of matrices with uniquely defined standard form. In this paper it is established that the number of matrices over imaginary Euclidean quadratic rings is finite and given estimate this number.

2. Main results. Let $\mathbb{K}=\mathbb{Z}[\sqrt{k}]$ is an imaginary Euclidean quadratic ring, i.e. $k = -1,-2,-3,-7,-11$. Let $\Phi = \operatorname{diag}(\varphi_1,\varphi_2,...,\varphi_{\mathrm{n}})$, $\varphi_i|\varphi_{i+1}$, $i=1,\dots,n-1$, be the d-matrix [3]. We consider a set of such matrices

$$T_\Phi = \begin{Vmatrix} \varphi_1 & 0 & \dots & 0 & 0 \\ t_{21} & \varphi_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ t_{n-1,1} & t_{n-1,2} & \dots & \varphi_{n-1} & 0 \\ t_{n1} & t_{n2} & \dots & t_{n,n-1} & \varphi_n \end{Vmatrix}, \qquad (2)$$

where $t_{ij}=0$ and $\mathcal{E}(t_{ij})<\mathcal{E}(\varphi_i)$, if $t_{ij}\neq 0$, $j<i$, $i=2,...,n$. Let us denote this set by $\mathcal{M}_{T_\Phi}$. In the case if $\mathbb{K}$ is an imaginary Euclidean quadratic ring the set $\mathcal{M}_{T_\Phi}$ is finite, i.e. $|\mathcal{M}_{T_\Phi}|=m$. Standard forms T^A of a matrix $A\in M(n,\mathbb{K})$, with respect to the (z,k)–equivalence with the Smith normal form $S^A=\operatorname{diag}(\varphi_1,\varphi_2,...,\varphi_{\mathrm{n}})$, $\varphi_i|\varphi_{i+1}$, $i=1,\dots,n-1$ is contained in the set $\mathcal{M}_{T_\Phi}$. Therefore, the number r of standard forms of matrices over $\mathbb{K}$ is finite.

We give the estimate of the number of standard form of a matrix A.

Lemma 1. *The set $\mathcal{M}_{T_\Phi}$ contains $m=\prod_{i=2}^{n} m_i^{i-1}$ matrices of the form (2), where m_i is the number of elements of Euclidean norms for which are less than the Euclidean norms of the corresponding invariant factors φ_i, $i=2,...,n$.*

Proof. The (i,j) elements of matrix (2) are m_i with the Euclidean norm less than the Euclidean norm φ_i, $i=2,...,n$, $j=1,...,n-1$. Therefore, the number of matrices T_Φ is equal to $m=\prod_{i=2}^{n} m_i^{i-1}$. □

Theorem 2. *Let $A \in M(n,\mathbb{K})$, where $\mathbb{K}$ is an imaginary Euclidean quadratic ring,*

$$S^A = \Phi = \operatorname{diag}(\varphi_1, \varphi_2, ..., \varphi_n),\ \varphi_i \mid \varphi_{i+1},\ i=1,...,n-1 \tag{3}$$

is the Smith normal form of matrix A and the Euclidean norm its determinant $\det A$ are less the four, that is $\mathcal{E}(\mathrm{detA})<4$. Then standard form T^A of matrix A is a matrix with the set $\mathcal{M}_{T_\Phi}$, i.e. $T^A=T_\Phi$, $T_\Phi \in \mathcal{M}_{T_\Phi}$ and the number of standard forms $r=m=\prod_{i=2}^{n} m_i^{i-1}$ is maximum.

Proof. Proposition of the theorem, if the quadratic ring $\mathbb{K}$ is the ring of Gaussian integers $\mathbb{Z}[i]$ is proved in the article [17].

Let A be the invertible matrix, i.e. $\mathcal{E}(\mathrm{detA})=1$. Then the set $\mathcal{M}_{T_\Phi}$ consists of the identity matrix and its right associated matrices

$$\left\| \begin{matrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & \varphi \end{matrix} \right\|,$$

where φ is an invertible element of $\mathbb{K}$. For this case, the statement of the theorem for matrices over each imaginary Euclidean quadratic rings $\mathbb{K}$ is proved.

Consider the matrix A of $\mathcal{E}(\mathrm{detA})=2$ over $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-7}]$. Then, obviously, the Smith normal forms are $S^A=\operatorname{diag}(1,\dots,1,\varphi)$, where $\mathcal{E}(\varphi)=2$. The set of matrices M_{T_Φ} consists of matrices of the form

$$T_\Phi = \left\| \begin{matrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ t_1 & t_2 & \dots & t_{n-1} & \varphi \end{matrix} \right\|,$$

where $t_i = 0, 1, -1$, $i=1,...,n-1$.

We show that such matrices are (z,k)–equivalent to the diagonal matrix S^A.

If $t_1=0$, then consider t_2. Let $t_2=1$, then adding to the last row the second row multiplied by (-1) in place of t_2 we get zero. Similarly, using elementary operations on rows reduce these matrices to the Smith normal form. Therefore, each matrix of the set $\mathcal{M}_{T_\Phi}$ will be (z, k)–equivalent to the Smith normal form S^A of the matrix A.

Let the matrix $A \in M(n,\mathbb{Z}[\sqrt{-3}])$ such that $\mathcal{E}(\mathrm{detA})=3$. The Smith normal form of a matrix A is $S^A=\operatorname{diag}(1,\dots,1,\sqrt{-3})$. Then the set of matrices M_{T_Φ} consists of matrices of the form

$$T_\Phi = \left\| \begin{matrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ t_1 & t_2 & \dots & t_{n-1} & \sqrt{-3} \end{matrix} \right\|, \tag{4}$$

where

$$t_i = 0, 1, -1, \frac{1}{2} + \frac{1}{2}\sqrt{-3}, -\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \frac{1}{2} - \frac{1}{2}\sqrt{-3}, -\frac{1}{2} - \frac{1}{2}\sqrt{-3},\ i = 1, ..., n-1. \qquad (5)$$

If $t_i = 0, 1, -1,\ i = 1, 2, ..., n-1$, the proof is made as in the previous case. Let for some $i,\ i = 1, ..., n-1\quad t_i = \frac{1}{2} + \frac{1}{2}\sqrt{-3}$. Using elementary columns operations over $\mathbb{K}$, namely by adding to the i-th column n-th, multiplied by $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ in place of t_i in the new matrix we get -1. Then, as in the previous case, using elementary row operations over $\mathbb{Z}$ in place t_i we get zero. These transformations are (z, k)–equivalent transformations. Similarly, by means of such transformations, on the place any t_i with (5) in the matrix (4) we get zero. Hence this matrix (z, k)–equivalent to the diagonal $S^A = \mathrm{diag}(1, \ldots, 1, \sqrt{-3})$.

Let $A \in M(n, \mathbb{Z}[\sqrt{-2}])$ and $\mathcal{E}(\mathrm{detA}) = 3$. Let $S^A = \mathrm{diag}(1, \ldots, 1, 1 + \sqrt{-2})$. If $S^A = \mathrm{diag}(1, \ldots, 1, 1 - \sqrt{-2})$ the proof of theorem is similar.

The set of matrices M_{T_Φ} consists of matrices of the form

$$T_\Phi = \begin{Vmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \\ t_1 & t_2 & \ldots & t_{n-1} & 1 + \sqrt{-2} \end{Vmatrix},$$

where $\quad t_i = 0, 1, -1, \sqrt{-2}, -\sqrt{-2}$.

Let there be $i,\ i = 1, ..., n-1$ such that $t_i = \sqrt{-2}$. Using elementary row operations over $\mathbb{Z}$ and elementary columns operations over $\mathbb{K}$, namely by adding to the n-th row i-th row and adding n-th to the newly formed i-th column the column is multiplied by (-1), in place of t_i we get zero.

Similarly, we obtain zero in place of the other $t_i,\ i = 1, ..., n-2$. For this case the proof of theorem is proved.

Since in $\mathbb{Z}[\sqrt{-11}]$ exist only trivial invertible elements, then it is easy to prove that each matrix $A \in M(n, \mathbb{Z}[\sqrt{-11}])$, such that an Euclidean norm of its determinant $\mathcal{E}(\mathrm{detA})$ is equal to 3, is (z, k)–equivalent to its Smith norm form.

Therefore, each matrices from the set $\mathcal{M}_{T_\Phi}$, where $\mathcal{E}(\mathrm{detA}) < 4$ are pairwise (z, k)–equiva- lent and every matrix of this set is a standard form. The number of standard forms for all such matrices over imaginary Euclidean quadratic rings is the maximum and is equal to $m = \prod_{i=2}^{n} m_i^{i-1}$. The proof of the theorem is completed.

Theorem 2 implies the following corollaries.

Corollary 1. *The matrix A over an imaginary Euclidean quadratic ring $\mathbb{K}$, such that $\mathcal{E}(\mathrm{detA}) < 4$, is (z, k)–equivalent to the Smith normal form S^A of matrix A and S^A is the standard form of the matrix A which is unique.*

The following corollary indicates the relationship (z, k)–equivalence and equivalence of matrices over a quadratic ring $\mathbb{K}$.

Corollary 2. *Matrices A and B over an imaginary Euclidean quadratic ring $\mathbb{K}$, such that $\mathcal{E}(\mathrm{detA}) < 4$, $\mathcal{E}(\mathrm{detB}) < 4$ are (z, k)–equivalent if and only if these matrices are equivalent, i.e. $S^A = S^B$.*

The following theorem establish the boundaries for the number of standard forms of a matrix A.

Theorem 3. *Let A be a matrix over an imaginary Euclidean quadratic ring $\mathbb{K}$ with the Smith normal form* (3). *Then the matrix A has r standard forms T^A, where r satisfies condition*

$$1 \le r \le m = \prod_{i=2}^{n} m_i^{i-1}.$$

Proof. If the matrix A is an invertible matrix, then $\mathcal{E}(\mathrm{detA}) = 1$ and this matrix has one standard form $r = 1$. The proof is obviously, since, the Smith normal form of the matrix A is the identity matrix $S^A = \mathrm{diag}(1, ..., 1)$.

It follows from the theorem 2 that there are matrices with the maximum number of standard forms.

Now we show that there are matrices with the Smith normal form Φ, for which not every matrix from the set $\mathcal{M}_{T_\Phi}$ is the standard form of the matrix A. Consider such a matrix over the ring $\mathbb{Z}[\sqrt{-2}]$

$$A = \begin{Vmatrix} -8+6\sqrt{-2} & 6+3\sqrt{-2} \\ -16+11\sqrt{-2} & 11+6\sqrt{-2} \end{Vmatrix}.$$

The Euclidean norm of the determinant detA is equal to 4, i.e. $\mathcal{E}(\mathrm{detA}) = 4$. Since, the Smith normal form of matrix A is a matrix $S^A = \Phi = \mathrm{diag}(1, 2)$, then the set of matrices M_Φ consists of matrices of the form

$$T_\Phi = \begin{Vmatrix} 1 & 0 \\ t & 2 \end{Vmatrix},$$

where $t \in \{0, 1, -1, \sqrt{-2}, -\sqrt{-2}, 1+\sqrt{-2}, 1-\sqrt{-2}, -1+\sqrt{-2}, -1-\sqrt{-2}\}$. This set is finite and contains 9 elements. The standard form of the matrix A is a diagonal matrix $\mathrm{diag}(1, 2)$, which is also its Smith normal form and is contained in the set $\mathcal{M}_{T_\Phi}$.

We show that the matrix

$$\begin{Vmatrix} 1 & 0 \\ 1+\sqrt{-2} & 2 \end{Vmatrix}$$

of the set $\mathcal{M}_{T_\Phi}$ is not (z, k)–equivalent to matrix $\mathrm{diag}(1, 2)$ from the set M_Φ. Suppose that these matrices are (z, k)–equivalent, i.e. there exist the inverse matrix $S = \|s_{ij}\|_{i,j=1}^2$ over the ring of integers $\mathbb{Z}$ and inverse matrix $Q = \|q_{ij}\|_{i,j=1}^2$ over the quadratic ring $\mathbb{Z}[\sqrt{-2}]$, such that

$$S \begin{Vmatrix} 1 & 0 \\ 1+\sqrt{-2} & 2 \end{Vmatrix} Q = \begin{Vmatrix} 1 & 0 \\ 0 & 2 \end{Vmatrix}. \tag{6}$$

From this equation we obtain a system of equations over the ring $\mathbb{Z}[\sqrt{-2}]$:

$$\begin{cases} s_{11}q_{11} + s_{12}q_{11}(1+\sqrt{-2}) + 2s_{12}q_{21} = 1, \\ s_{21}q_{11} + s_{22}q_{11}(1+\sqrt{-2}) + 2s_{22}q_{21} = 0, \\ s_{11}q_{12} + s_{12}q_{12}(1+\sqrt{-2}) + 2s_{12}q_{22} = 0, \\ s_{21}q_{12} + s_{22}q_{12}(1+\sqrt{-2}) + 2s_{22}q_{22} = 2. \end{cases} \tag{7}$$

Solving this system, we obtain that the system is unsolvable. Our the assumption is incorrect and therefore these matrices do not (z, k)–equivalent.

So, the number of r standard forms of the matrix A is not the maximum. In a similar way, we prove that exist matrices over other imaginary Euclidean quadratic rings for which the number of standard forms is not the maximum. □

REFERENCES

1. Zabavskii B.V., Romaniv O.M. *Rings with elementary reduction of matrices*// Ukr. Math. J. – 2000. – V. 52, №12. – P. 1872–1881.
2. Mal'tsev A.I. Foundations of Linear Algebra. – Nauka, Moscow, 1970. – 400 p. (in Russian)
3. Kazimirs'kyi P.S. Decomposition of matrix polynomials into factors. – Lviv: Pidstryhach Inst. Appl. Probl. Mech. and Math. of NAS of Ukrainian, 2015 – 282 p. (in Ukrainian)
4. Petrychkovich V.M. *Semiscalar equivalence and the Smith normal form of polynomial matrices*// J. Sov. Math. – 1993. – V.66, №1. – P. 2030–2033.
5. Dias da Silva J.A., Laffey T.J. *On simultaneous similarity of matrices and related questions*// Linear Algebra Appl. – 1999. – V. 291. – P. 167–184.
6. Petrychkovych V.M. Generalized equivalence of matrices and their sets and the factorization of matrices over rings. — Lviv: Pidstryhach Inst. Appl. Probl. Mech. and Math. of NAS of Ukrainian, 2015 – 312 p. (in Ukrainian)
7. Dzhaliuk N.S., Petrychkovych V.M. *Solutions of the matrix linear bilateral polynomial equation and their structure*// Algebra Discrete Math. – 2019. – V. 27,№2. – P. 243–251.
8. Sidorov S.V. *On similarity of* 2×2 *matrices over the ring of Gaussian integers with reducible characteristic polynomial*// Vestn. Nizhni Novgorod Univ. – 2008. – V. 4. – P. 122–126. (in Russian)
9. Velichko I.N. *Generalized Kloosterman sum over the matrix ring* $M_n(\mathbb{Z}[i])$// Visn. Odes. Nats. Univ., Ser. Mat. and Mekh. – 2010. – V.1, №19. – P. 9–20. (in Russian)
10. Savastru O., Varbanets S. *Norm Kloosterman sums over* $\mathbb{Z}[i]$// Algebra Discrete Math. – 1995. – V.80. – P. 105–137.
11. Taylor G. *Cyclotomic matrices and graphs over the ring of integers of some imaginary quadratic fields*// J. Algebra. – 2011. – V.331. – P. 523–545.
12. Greaves G. *Cyclotomic matrices over the Eisenstein and Gaussian integers*// J. Algebra. – 2012. – V.372. – P. 560–583.
13. Ladzoryshyn N.B. *On equivalence of pairs of matrices, in which determinants are orders of primes, over quadratic Euclidean rings*// Carpathian Math. Publ. – 2013. – V.5, №1. – P. 63–69, https:// doi:10.15330/cmp.5.1.63-69. (in Ukrainian)
14. Ladzoryshyn N., Petrychkovych V. *Equivalence of pairs of matrices with relatively prime determinants over quadratic rings of principal ideals*// Bul. Acad. Ştiinţe Repub. Mold. Mat. – 2014. – №3. – P. 38–48.
15. Ladzoryshyn N.B., Petrychkovych V.M. *Standard form of matrices over quadratic rings with respect to the* (z,k)*–equivalence and the structure of solutions of bilateral matrix linear equations*// J. Math. Sci. – 2021. – V.253., №1. – P. 54–62. https://doi.org/10.1007/s10958-021-05212-w
16. Ladzoryshyn N.B., Petrychkovych V.M., Zelisko H.V. *Matrix Diophantine equations over quadratic rings and their solutions* // Carpathian Math. Publ. – 2020. – V.12, №2. – P. 368–375. https://doi.org/10.15330/cmp.12.2.368-375
17. Petrychkovych V.M., Zelisko H.V., Ladzoryshyn N.B. *The standard form of matrices over the ring of Gaussian integers with respect to* (z,k)*–equivalence* // Appl. Probl. Mech. and Math. – 2020. – V. 18. – P. 5–10. https://doi.org/10.15407/apmm2020.18.5-10 (in Ukrainian)

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