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A NEW APPROACH TO NEARLY PARACOMPACT SPACES

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The pre-open sets are a generalization of open sets of topological spaces. In this paper, we introduce and study a notion of po-paracompact spaces via pre-open sets on topological spaces. We see that po-paracompact spaces are equivalent to nearly paracompact spaces. However, we find new characterizations to nearly paracompact spaces when we study it in the sense of po-paracompact spaces. We see that a topological space is nearly paracompact if and only if each regularly open cover of the topological space has a locally finite pre-open refinement. We also show that four statements involving pre-open sets on an almost regular topological space are equivalent. A result on a subspace of a topological space is also obtained in term of pre-open sets.

1. Introduction. Let X be a nonempty set and $\mathscr{P}$ be a topology on X. Sometimes, we simply write X to denote the topological space $(X, \mathscr{P})$. For a subset A of the topological space X, $\mathrm{Int}(A)$ (resp. $\mathrm{Cl}(A)$) represents the interior (resp. closure) of A in X.

The notion of locally dense sets are introduced and studied by Corson and Michael [2] are one among several generalizations of open sets in topological spaces. A subset A of a topological space X is called locally dense [2] if there exists an open set U such that $A \subset U \subset \mathrm{Cl}(A)$. Thereafter, Mashhour et al. [3] introduced and studied pre-open sets in topological spaces. It is seen that pre-open sets are equivalent to locally sets. As locally dense sets are widely studied after the name pre-open sets, we agree to put into use the term pre-open sets to mean locally dense sets also. It is easy to see that a subset A of X is pre-open in X if and only if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$. The complement of a pre-open set in X is called a pre-closed set [3]. For any subset A of a topological space X, the intersection of all pre-closed sets containing A is called the pre-closure of A and it is denoted by $\mathrm{pCl}(A)$.

If A is a subset of X, then we write $(A, \mathscr{P}_A)$ to denote the subspace A of X. Also the collection of all pre-open sets in X is expressed by $PO(X)$. Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers.

2. Po-paracompactness. If $A = \mathrm{Int}(\mathrm{Cl}(A))$ for a subset A of X, then A is called regularly open in X. The complement of a regularly open set in X is called a regularly closed set. So a subset E of X is regularly closed if $E = \mathrm{Cl}(\mathrm{Int}(E))$. It follows that if A is open (resp. closed) in a topological space X, then $\mathrm{Int}(\mathrm{Cl}(A))$ (resp. $\mathrm{Cl}(\mathrm{Int}(A))$) is regularly open (resp. regularly closed) in X.

We agree to write 'open collection' and 'pre-open collection' to mean a collection consisting only open sets and pre-open sets of a topological space, respectively. A cover of a topological space X is a collection $\mathscr{A}$ of subsets of X such that $\bigcup_{A\in\mathscr{A}} A = X$. $\mathscr{A}$ is called an

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open cover (resp. pre-open cover) of X if $\mathscr{A}$ is an open collection (resp. pre-open collection) of X and covers X. The terms 'regularly open collection', 'regularly open cover' are apparent. A collection $\mathscr{A}$ of subsets of X is called a weak cover of X if $\mathrm{Cl}\left(\bigcup_{A\in\mathscr{A}} A\right) = X$.

Let $\mathscr{U}$ and $\mathscr{V}$ be two covers of X. The cover $\mathscr{V}$ is called a refinement [9, p. 144] of the cover $\mathscr{U}$ if for each $V \in \mathscr{V}$, there exists a $U \in \mathscr{U}$ such that $V \subset U$. If the covers $\mathscr{U}$ and $\mathscr{V}$ both are open covers of X, then $\mathscr{V}$ is called an open refinement of $\mathscr{U}$. A collection $\mathscr{U}$ of subsets of X is called locally finite if each $x \in X$ has a neighbourhood meeting only finitely many members of $\mathscr{U}$. A topological space X is called paracompact [9, p. 146] if each open cover of X has a locally finite open refinement.

Definition 1 (Singal and Arya [7]). A topological space X is called *nearly paracompact* if each regularly open cover of X has a locally finite open refinement.

Theorem 1 (Singal and Arya [7]). *A topological space X is nearly paracompact if and only if for each open cover $\mathscr{U}$ of X, there exists a locally finite collection $\mathscr{V}$ of open sets such that each $V \in \mathscr{V}$ is contained in some $U \in \mathscr{U}$ and $\{\mathrm{Int}(\mathrm{Cl}(V)) \mid V \in \mathscr{V}\}$ covers X.*

Definition 2 (Bagchi et al. [6]). Let $\mathscr{S}$ be a pre-open collection of X. We define $\mathscr{U} = \{U \mid A \in \mathscr{S}, A \subset U \subset \mathrm{Cl}(A)\}$. Then $\mathscr{U}$ is said to be an *open super-collection* of $\mathscr{S}$.

It follows that there always exists an open super-collection for a pre-open collection of a topological space X. We note that $\mathscr{U}$ may be a cover of X even if $\mathscr{S}$ is not a cover of X. If $\mathscr{U}$ is a cover of X then $\mathscr{U}$ is said to be an open super-cover of $\mathscr{S}$.

We now introduce the following.

Definition 3. A collection $\mathscr{V}$ of subsets of X is said to be a *weak refinement* of the cover $\mathscr{U}$ of X if for each $V \in \mathscr{V}$, there exists a $U \in \mathscr{U}$ such that $V \subset U$. $\mathscr{V}$ is said to be *open* (resp. *pre-open*) *weak refinement* if members of $\mathscr{V}$ are open (resp. pre-open) in X.

Note that in this case, $\mathscr{V}$ may not be a cover of X.

Definition 4. A topological space X is said to be *po-paracompact* if for each open cover $\mathscr{U}$ of X, there exists a locally finite pre-open weak refinement $\mathscr{S}$ such that $\mathscr{S}$ has an open super-cover.

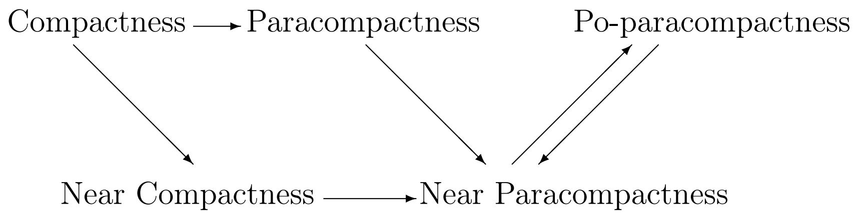
We see that paracompact spaces are po-paracompact spaces. For it, let $\mathscr{U}$ be an open cover of a paracompact space X. By paracompactness of X, there exists a locally finite open refinement $\mathscr{V}$ of $\mathscr{U}$. Since open sets in X are pre-open sets in X, $\mathscr{V}$ is a locally finite pre-open weak refinement of $\mathscr{U}$. As $\mathscr{V}$ is an open cover of X, $\mathscr{V}$ itself is an open super-cover of $\mathscr{V}$. So X is po-paracompact.

Theorem 2. *A topological space X is po-paracompact if and only if X is nearly paracompact.*

Proof. Firstly, let X be nearly paracompact and $\mathscr{U}$ be an open cover of X. As X is near paracompact, $\mathscr{U}$ has a locally finite open weak refinement $\mathscr{V}$ such that $\{\mathrm{Int}(\mathrm{Cl}(V)) \mid V \in \mathscr{V}\}$ covers X. Since open sets are pre-open sets and for each $V \in \mathscr{V}$, we have $V \subset \mathrm{Int}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(V)$, $\mathscr{V}$ is a locally finite pre-open weak refinement of $\mathscr{U}$ and $\{\mathrm{Int}(\mathrm{Cl}(V)) \mid V \in \mathscr{V}\}$ is an open super-cover of $\mathscr{V}$. So X is po-paracompact.

Conversely, let $\mathscr{U}$ be an open cover of a po-paracompact space X. By po-paracompactness of X, we obtain a locally finite pre-open weak refinement $\mathscr{S}$ of $\mathscr{U}$ such that $\mathscr{S}$ has an open super-cover $\mathscr{V}$. For each $A \in \mathscr{S}$, there exist a $U_A \in \mathscr{U}$ and a $V_A \in \mathscr{V}$ such that $A \subset U_A$ and $A \subset V_A \subset \mathrm{Cl}(A)$. We write $W_A = U_A \cap V_A$. Then $W_A \subset U_A$ and $A \subset W_A \subset V_A \subset \mathrm{Cl}(A)$. It means that $W_A \subset V_A \subset \mathrm{Int}(\mathrm{Cl}(W_A))$. Putting $\mathscr{W} = \{W_A \mid W_A = U_A \cap V_A, A \in \mathscr{S}\}$, we see that $\{\mathrm{Int}(\mathrm{Cl}(W)) \mid W \in \mathscr{W}\}$ is a cover of X as $\mathscr{V}$ covers X. Since $\mathscr{S}$ is locally finite, the collection $\{\mathrm{Cl}(A) \mid A \in \mathscr{S}\}$ and hence the collection $\mathscr{W}$ is also locally finite. So $\mathscr{W}$ is a locally finite open weak refinement of $\mathscr{U}$ such that $\{\mathrm{Int}(\mathrm{Cl}(W)) \mid W \in \mathscr{W}\}$ covers X. So X is nearly paracompact. □

The following diagram depicts the position of po-paracompactness with some other covering properties of a topological space.



Note: An arrow between two notions in above diagram stands to mean 'implies that'.

As po-paracompactness and near paracompactness are equivalent by Theorem 2, the following two results are obvious:

(i) A topological space X is po-paracompact if and only if each regularly open cover of X has a locally finite open refinement.

(ii) A topological space X is nearly paracompact if and only if for each open cover $\mathscr{U}$ of X, there exists a locally finite pre-open weak refinement $\mathscr{S}$ such that $\mathscr{S}$ has an open super-cover.

Since the notion near paracompactness is well established and po-paracompactness is equivalent to near paracompactness, we study near paracompactness in the sense of po-paracompactness to obtain some new characterizations on near paracompactness.

Lemma 1 (Al-Zoubi and Al-Ghour [1]). *If A is pre-open, then* $\mathrm{Int}(\mathrm{Cl}(A))$ *is regularly open.*

Corollary 1. *Corresponding to an open cover $\mathscr{U}$ of a nearly paracompact space X, there exists a locally finite regularly open cover generated from a locally finite pre-open weak refinement.*

Proof. By near paracompactness of X, we obtain a locally finite pre-open weak refinement $\mathscr{S}$ of $\mathscr{U}$. For each $A \in \mathscr{S}$, $\mathrm{Int}(\mathrm{Cl}(A))$ is regularly open by Lemma 1. Proceeding like in the second part of the proof of Theorem 2, we find that $\{\mathrm{Int}(\mathrm{Cl}(A)) \mid A \in \mathscr{S}\}$ is a locally finite regularly open cover of X. □

Theorem 3. *A topological space X is nearly paracompact if and only if each regularly open cover of X has a locally finite pre-open refinement.*

Proof. The necessity follows directly from the fact that open sets are pre-open sets.

To prove sufficiency, let $\mathscr{U}$ be a regularly open cover of X and $\mathscr{S}$ be a locally finite pre-open refinement of $\mathscr{U}$. For each $A \in \mathscr{S}$, there exists a $U \in \mathscr{U}$ such that $A \subset U$. U being regularly open, $\mathrm{Int}(\mathrm{Cl}(A)) \subset U$. So $\{\mathrm{Int}(\mathrm{Cl}(A)) \mid A \in \mathscr{S}\}$ is an open refinement of $\mathscr{U}$. The local finiteness of $\{\mathrm{Int}(\mathrm{Cl}(A)) \mid A \in \mathscr{S}\}$ follows from the fact that $\{\mathrm{Cl}(A) \mid A \in \mathscr{S}\}$

is locally finite as the collection $\mathscr{S}$ of subsets of X is locally finite and for each $A \in \mathscr{S}$, we have $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \subset \mathrm{Cl}(A)$. So X is nearly paracompact. □

Theorem 4. *Let X be a Hausdorff nearly paracompact topological space. If $x \in X$ and F is regularly closed in X such that $x \notin F$, then there exist an open set G and a pre-open set H such that $x \in G, F \subset H$ and $G \cap H = \varnothing$.*

Proof. For $\xi \in F$, there exist open sets U_ξ, V_ξ such that $x \in U_\xi, \xi \in V_\xi$ and $U_\xi \cap V_\xi = \varnothing$. Hence $U_\xi \cap \mathrm{Int}(\mathrm{Cl}(V_\xi)) = \varnothing$. The collection $\mathscr{U} = \{\mathrm{Int}(\mathrm{Cl}(V_\xi)) \mid \xi \in F\} \cup \{X - F\}$ is a regularly open cover of X. By Theorem 3, $\mathscr{U}$ has a locally finite pre-open refinement $\mathscr{W}$. Let $H = \bigcup\{W \in \mathscr{W} \mid W \cap F \neq \varnothing\}$. Here H is pre-open with $F \subset H$. By local finiteness of $\mathscr{W}$, there exists a neighbourhood D of x intersecting a finite number of sets $W_1, W_2, \dots, W_n$ of $\mathscr{W}$ such that $W_k \cap F \neq \varnothing, k \in \{1, 2, \dots, n\}$. Since $\mathscr{W}$ is a refinement of $\mathscr{U}$ and $W_k \cap F \neq \varnothing$ for each $k \in \{1, 2, \dots, n\}$, there exists a $V_{\xi_k} \in \mathscr{U}$ such that $W_k \subset \mathrm{Int}(\mathrm{Cl}(V_{\xi_k}))$ for each $k \in \{1, 2, \dots, n\}$. Putting $G = D \cap (\bigcap_{k=1}^{n} U_{\xi_k})$, we see that $G \cap H = \varnothing$. □

Definition 5 (Singal and Arya [8])**.** A topological space X is called *almost regular* if for each regularly closed set F and each $x \in X - F$, there exist disjoint open sets U, V such that $x \in U, F \subset V$.

Theorem 5. *Let X be an almost regular nearly paracompact topological space. If E and F are disjoint regularly closed sets in X, then there exist an open set G and a pre-open set H such that either $E \subset G, F \subset H$ and $G \cap H = \varnothing$ or $E \subset H, F \subset G$ and $G \cap H = \varnothing$ hold.*

Proof. If we choose $x \in F$ and proceed like in the proof of Theorem 4 using the almost regularity of X, then we obtain an open set G and a pre-open set H such that $E \subset G, F \subset H$ and $G \cap H = \varnothing$.

If we choose $x \in E$ and then proceed like in the proof of Theorem 4 using the almost regularity of X, then we obtain an open set G and a pre-open set H such that $E \subset H, F \subset G$ and $G \cap H = \varnothing$. □

Theorems 4 and 5 endeavor us to introduce the following two separation axioms.

Definition 6. A topological space X is said to be *almost pre-regular* if for each regularly closed set F and each $x \in X - F$, there exist an open set G and a pre-open set H such that $x \in G, F \subset H$ and $G \cap H = \varnothing$.

It is easy to see that a topological space X is almost pre-regular if and only if for any regularly open set G and any $x \in G$, there exists an open set H such that $x \in H \subset \mathrm{pCl}(H) \subset G$. We also see that an almost regular space X is almost pre-regular and vice-versa.

Definition 7. A topological space X is said to be *almost pre-normal* if there exist an open set G and a pre-open set H for disjoint regularly closed sets E, F such that $E \subset G, F \subset H$ and $G \cap H = \varnothing$.

It follows easily that a topological space X is almost pre-normal if and only if for any regularly open set G and any regularly closed set E with $E \subset G$, there exists an open set H such that $E \subset H \subset \mathrm{pCl}(H) \subset G$.

Definition 8 (Mukharjee et al. [6])**.** A topological space X is called *pre-regular* if for each x and each open set U with $x \in U$, there exists a pre-open set V such that $x \in V \subset \mathrm{Cl}(V) \subset U$.

We showed in [6] that a pre-regular nearly compact space is a compact space. In this line, we have Theorem 6. To prove it, we use Theorem 4.1 [4] which states that a topological space X is paracompact if and only if every open cover of X has a locally finite pre-open refinement.

Theorem 6. *A pre-regular nearly paracompact space is a paracompact space.*

Proof. Let $\mathscr{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of X. For each $x \in X$, there exists an $\alpha \in A$ such that $x \in U_\alpha$. By pre-regularity of X, we obtain a pre-open set $G_{\alpha(x)}$ such that $x \in G_{\alpha(x)} \subset \mathrm{Cl}(G_{\alpha(x)}) \subset U_\alpha$. By Lemma 1, $\mathrm{Int}(\mathrm{Cl}(G_{\alpha(x)}))$ is regularly open. So $\mathscr{G} = \{\mathrm{Int}(\mathrm{Cl}(G_{\alpha(x)})) \mid x \in X\}$ is a regularly open cover of X. By near paracompactness of X, we obtain a locally finite pre-open refinement $\mathscr{S}$ of $\mathscr{G}$ by Theorem 3. For each $A \in \mathscr{S}$, there exists a pre-open set B such that $\mathrm{Int}(\mathrm{Cl}(B)) \in \mathscr{G}$ and $A \subset \mathrm{Int}(\mathrm{Cl}(B))$. Now $A \subset \mathrm{Int}(\mathrm{Cl}(B)) \subset \mathrm{Cl}(B) \subset U$ for some $U \in \mathscr{U}$. So $\mathscr{S}$ is also a locally finite pre-open refinement of $\mathscr{U}$. □

Like Lemma 1, we may have the following.

Lemma 2. *If A is pre-closed in a topological space X, then* $\mathrm{Cl}(\mathrm{Int}(A))$ *is regularly closed in X.*

Proof. Easy to prove and hence omitted. □

Theorem 7. *The following statements are equivalent on an almost regular topological space X:*

(i) X is nearly paracompact.

(ii) Each regularly open cover of X has a locally finite pre-open refinement.

(iii) Each regularly open cover of X has a locally finite regularly open refinement.

(iv) Each regularly open cover $\mathscr{G} = \{G_\gamma \mid \gamma \in \Gamma\}$ of X has a locally finite pre-closed refinement $\{A_\alpha \mid \alpha \in \Delta,$ and $x \in \mathrm{Int}(A_\alpha)$ if $x \in A_\alpha$ for some $\alpha \in A\}$.

(v) Each regularly open cover of X has a locally finite regularly closed refinement.

Proof. (i)⇒(ii): Follows by Theorem 3.

(ii)⇒(iii): Let $\mathscr{U}$ be a regularly open cover of X. By (ii), we obtain a locally finite pre-open refinement $\mathscr{S}$ of $\mathscr{U}$. For each $A \in \mathscr{S}$, $\mathrm{Int}(\mathrm{Cl}(A))$ is regularly open by Lemma 1 and there exists a $U \in \mathscr{U}$ such that $\mathrm{Int}(\mathrm{Cl}(A)) \subset U$. As the collection $\mathscr{S}$ of subsets of X is locally finite, the collection $\{\mathrm{Cl}(A) \mid A \in \mathscr{S}\}$ and hence the collection $\{\mathrm{Int}(\mathrm{Cl}(A)) \mid A \in \mathscr{S}\}$ of subsets of X is locally finite. So $\{\mathrm{Int}(\mathrm{Cl}(A)) \mid A \in \mathscr{S}\}$ is a locally finite regularly open refinement of $\mathscr{U}$.

(iii)⇒(iv): Let $\mathscr{U}$ be a regularly open cover of X. By (iii), there exists a locally finite regularly open refinement $\mathscr{V}$ of $\mathscr{U}$. For each $x \in X$, there exists a $G_x \in \mathscr{V}$ such that $x \in G_x$. Since an almost regularly space is also almost pre-regular, we obtain an open set H_x such that $x \in H_x \subset \mathrm{pCl}(H_x) \subset G_x$ by almost pre-regularity of X. So

$$\{\mathrm{pCl}(H_x) \mid x \in X, x \in \mathrm{Int}(\mathrm{pCl}(H_x))\}$$

is a locally finite pre-closed refinement of $\mathscr{U}$.

(iv)⇒(v): Let $\mathscr{G} = \{G_\gamma \mid \gamma \in \Gamma\}$ be a regularly open cover of X. By (iv), we obtain a locally finite pre-closed refinement

$$\mathscr{S} = \big\{A_\alpha \mid \alpha \in \Delta, x \in \mathrm{Int}(A_\alpha) \text{ if } x \in A_\alpha \text{ for some } x \in X\big\}$$

of $\mathscr{U}$. For some $z \in X$, let $z \in A_\beta, \beta \in \Delta$. Then $z \in \text{Int}(A_\beta)$. Also there exists a $\gamma \in \Gamma$ such that $A_\beta \subset G_\gamma$. Then $z \in G_\gamma$. By the almost regularity of X, there exists an open set H_z in X such that $z \in H_z \subset \text{Cl}(H_z) \subset G_\gamma$. It also implies that $z \in H_z \subset \text{pCl}(H_z) \subset \text{Cl}(H_z) \subset G_\gamma$. For $\beta \in \Delta$, there exists a closed set F_β such that $\text{Int}(A_\beta) \subset F_\beta \subset A_\beta$. We put $E_\beta = F_\beta \cap \text{pCl}(H_z)$. Then we have $z \in \text{Int}(A_\beta) \cap H_z \subset E_\beta \subset F_\beta \subset A_\beta$ which implies that $z \in \text{Cl}(\text{Int}(E_\beta)) \subset F_\beta \subset A_\beta \subset G_\gamma$. Since closed sets are obviously pre-closed, E_β is pre-closed in X and so $\text{Cl}(\text{Int}(E_\beta))$ is regularly closed in X for each $\beta \in \Delta$ by Lemma 2. Since $\mathscr{S}$ is locally finite and $\text{Cl}(\text{Int}(E_\beta)) \subset A_\beta$ for each $\beta \in \Delta$ and $A_\beta \subset G_\gamma$ for some $\gamma \in \Gamma$, we see that $\mathscr{E} = \{\text{Cl}(\text{Int}(E_\beta)) \mid \beta \in \Delta\}$ is a locally finite regularly closed refinement of $\mathscr{G}$.

(v)⇒(i): It is same to $(g) \Rightarrow (a)$ of Theorem 1.5 [7]. □

Lemma 3 (Mashhour et al. [5]). *If A and B are subsets of a topological space X such that $A \in PO(B)$ and $B \in PO(X)$ then $A \in PO(X)$.*

For brevity, we agree to use the following notations:

(i) The closure of a subset A in X is denoted by $\text{Cl}_X(A)$.

(ii) We write 'an X-open cover' (resp. 'a collection $\mathscr{C}$ of subsets of X is X-locally finite') to mean 'a cover by open sets in X' (resp. 'each $x \in X$ has a neighbourhood which meets only finitely many members of $\mathscr{C}$'). Likewise, we use the term 'X-open set', 'X-pre-open set', 'X-pre-open cover' etc.

Theorem 8. *Let A be open in a topological space X. Then the subspace A of X is nearly paracompact if and only if each X-open cover $\mathscr{G}$ of A has an A-locally finite X-pre-open weak refinement $\mathscr{H}$ of $\mathscr{G}$ such that $\mathscr{H}$ has an X-open super-cover.*

Proof. Firstly, suppose that A is nearly paracompact and $\mathscr{G}^{(X)}$ is an X-open cover of A. We have noting to prove if there exists a $G \in \mathscr{G}^{(X)}$ such that $A \subset G$. Hence we suppose that $G \subsetneq A$ for all $G \in \mathscr{G}^{(X)}$. For each $G \in \mathscr{G}^{(X)}$, $A \cap G$ is open in A. So $\mathscr{U}^{(A)} = \{A \cap G \mid G \in \mathscr{G}^{(X)}\}$ is an A-open cover of A. Since A is nearly paracompact, $\mathscr{U}^{(A)}$ has an A-locally finite A-pre-open weak refinement $\mathscr{V}^{(A)}$ such that $\mathscr{V}^{(A)}$ has an A-open super-cover $\mathscr{W}^{(A)}$. For each $V^{(A)} \in \mathscr{V}^{(A)}$, there exists a $U^{(A)} \in \mathscr{U}^{(A)}$ such that $V^{(A)} \subset U^{(A)} = A \cap G^{(X)}$ for some $G^{(X)} \in \mathscr{G}^{(X)}$ which implies $V^{(A)} \subset G^{(X)}$. By Lemma 3, each member of $\mathscr{V}^{(A)}$ is pre-open in X also. Hence $\mathscr{V}^{(A)}$ is A-locally finite X-pre-open weak refinement of $\mathscr{G}^{(X)}$. Again for each $W^{(A)} \in \mathscr{W}^{(A)}$, there exists a A-pre-open set $V^{(A)} \in \mathscr{V}^{(A)}$ such that $V^{(A)} \subset W^{(A)} \subset \text{Cl}_A(V^{(A)}) \subset \text{Cl}_X(V^{(A)})$. As A is X-open, $W^{(A)}$ is also X-open. So $\mathscr{W}^{(A)}$ is an X-open super-super cover of $\mathscr{V}^{(A)}$.

To prove the converse, let $\mathscr{G}^{(A)}$ be an A-cover. For each $G^{(A)} \in \mathscr{G}^{(A)}$, we have an X-open set $G^{(X)}$ such that $G^{(A)} = A \cap G^{(X)}$. We construct $\mathscr{G}^{(X)} = \{G^{(X)} \mid G^{(A)} = A \cap G^{(X)}, G^{(A)} \in \mathscr{G}^{(A)}\}$. Obviously, $\mathscr{G}^{(X)}$ is an X-cover of A. Hence we obtain an A-locally finite X-pre-open weak refinement $\mathscr{H}^{(X)}$ of $\mathscr{G}^{(X)}$ such that $\mathscr{H}^{(X)}$ has an X-open super-cover $\mathscr{U}^{(X)}$. Then we have the following:

(i) For each $H^{(X)} \in \mathscr{H}^{(X)}$, there exists a $G^{(X)} \in \mathscr{G}^{(X)}$ such that $H^{(X)} \subset G^{(X)}$ which gives $A \cap H^{(X)} \subset A \cap G^{(X)}$.

(ii) Each $H^{(X)} \in \mathscr{H}^{(X)}$ being X-pre-open and A is open in X, $A \cap H^{(X)}$ is pre-open in X. So we have an X-open set $G^{(X)}$ such that $A \cap H^{(X)} \subset G^{(X)} \subset \text{Cl}_X(A \cap G^{(X)})$. Hence we obtain $A \cap H^{(X)} \subset A \cap G^{(X)} \subset A \cap \text{Cl}_X(A \cap H^{(X)}) = \text{Cl}_A(A \cap H^{(X)})$. So $A \cap H^{(X)}$ is A-pre-open.

(iii) For each $U^{(X)} \in \mathscr{U}^{(X)}$, there exists an $H^{(X)} \in \mathscr{H}^{(X)}$ such that

$$H^{(X)} \subset U^{(X)} \subset \mathrm{Cl}_{(X)}(H^{(X)}) \implies$$
$$A \cap H^{(X)} \subset A \cap U^{(X)} \subset A \cap (A \cap \mathrm{Cl}_{(X)}(H^{(X)})) \subset A \cap \mathrm{Cl}_X(A \cap \mathrm{Cl}_{(X)}(H^{(X)})) \subset$$
$$\subset A \cap \mathrm{Cl}_X((A \cap H^{(X)})) \ \big[\text{as } A \text{ is open in } X\big] = \mathrm{Cl}_A(A \cap H^{(X)}).$$

It is obvious that

$$\bigcup_{U^{(X)} \in \mathscr{U}^{(X)}} A \cap U^{(X)} = A.$$

By (i) and (ii), it follows that $\mathscr{H}^{(A)} = \{A \cap H^{(X)} \mid H^{(X)} \in \mathscr{H}^{(X)}\}$ is an A-locally finite A-pre-open weak refinement of $\mathscr{G}^{(A)}$. By (iii), we see that $\mathscr{H}^{(A)}$ has an A-open super-cover $\{A \cap U^{(X)} \mid U^{(X)} \in \mathscr{U}^{(X)}\}$. □

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