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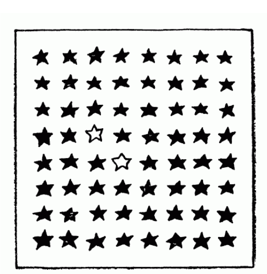
A COUNTEREXAMPLE TO HENRY E. DUDENEY'S STAR PUZZLE

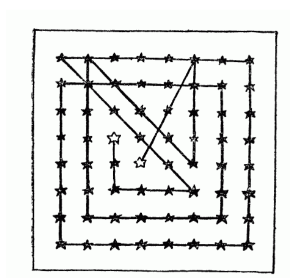
A. V. Ravsky. *A counterexample to Henry E. Dudeney's star puzzle*, Mat. Stud. **56** (2021), 215–217.

We found a solution of Henry E. Dudeney's star puzzle (a path on a chessboard from $c5$ to $d4$ in 14 straight strokes) in 14 queen moves, which was claimed impossible by the puzzle author. Generalizing this result to other board sizes, we obtained bounds on minimal number of moves in a board filling queen path with given source and destination.

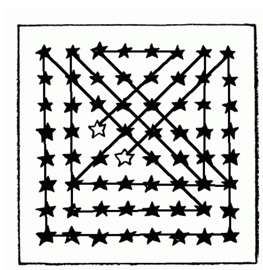
About ten years ago I spent many nice hours with the old book [3] of puzzles by Henry E. Dudeney. I still remember one of them. This is the star puzzle [1] (X36 = AM329, according to the list [5] by Donald E. Knuth), published more than a century ago in The Strand Magazine. It was based in London from 1891 to 1950 and is famous as the place where Arthur Conan Doyle's Sherlock Holmes stories were published. The task of the puzzle is the following



On this field of stars, draw a path from one light star to the other, crossing all stars and constituted by 14 straight strokes.



This is the proposed solution and there was claimed that there is no required paths constituted by 14 queen moves.



But, surprisingly, I found this path constituted by 14 queen moves: c5-f8-c8-h3-b3-g8-g3-b8-b2-g2-a8-a1-h1-h8-d4.

I was searching a solution by hand, as in the good old days. But later Carlos Rivera posed a related puzzle, see [6] for this and next results from this paragraph. Answering

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the questions of Carlos Rivera (and also of Janos Barat and Konstantin Knop), Giovanni Resta wrote a program. Assuming its correctness, he found five other solutions (which are twins of mine), verified that for each pair of cells it is not possible to cover all the cells with a path of length 13 (or less) going from one cell to the other, and analyzed all the source-destination pairs of cells, to see which admitted a board filling 14-moves path joining them. He provided a detailed and graphical description of this analysis, based on which I showed that any pair of cells can be connected by a board filling queen path constituted by at most 15 moves.

We also considered related problems and discussed a generalization of the above results to other board sizes. Let $\mathcal{B}$ be an $m\times n$ board, that is a rectangular board with m rows and n columns and $t(n,m)$ be the minimum number of moves of a *closed* queen path, which covers each cell of $\mathcal{B}$. In the solution of Puzzle 416 ("Sinking the fishing-boats") from [2] is shown that $t(7,7)\le 12$. In its comment is claimed that $t(m,m)=2m-2$ when $m\ge 7$.

For rectangular boards, simple bounds for $t(m,n)$ are provided by the following

Proposition 1. *Let $m\le n$ be natural numbers. Then $t(m,n)\le 2m$ if m is even or $m=n$, and $t(m,n)\le 2m+1$ if m is odd.*

Proof. We use rook paths to provide the required bounds. For this purpose, place a rook at a top corner of an $m\times n$ board and move it along a sweeping path from the top to the bottom, constituted by horizontal moves from one board side to the other, interleaved by moves by one cell down. When the constructing path will fill the bottom row, the rook will need one or two moves (depending on the parity of m) to return to the start cell. Note that when $m=n$ then the rook can return to the start cell by one queen move anyway. □

For any cells P and Q of $\mathcal{B}$, let $f(P,Q)$ be the minimum number of moves of a board filling queen path from P to Q.

Proposition 2. *Let $\mathcal{B}$ be an $m\times n$ board and P, Q be any cells of $\mathcal{B}$, connected by a queen move. Then $t(m,n)\le f(P,P)\le t(m,n)+1$ and $t(m,n)-1\le f(P,Q)\le t(m,n)+2$.*

Proof. Since any board filling queen path from P to P is closed, we have $t(m,n)\le f(P,P)$. On the other hand, let $\mathcal{P}$ be any closed board filling queen path on $\mathcal{B}$, constituted of $t(m,n)$ moves. Then there exists a move M of $\mathcal{P}$, which covers the cell P. Let $M=P'P''$ be a move from a cell P' to a cell P''. If either P' or P'' equals P then choosing this cell as a new start cell of $\mathcal{P}$, provides us a board filling queen path from P to P, constituted of $t(m,n)$ moves. Otherwise we split the move $P'P''$ into moves $P'P$ and $P'P''$, and put P as the start cell of the split path. This provides us a board filling queen path from P to P, constituted of $t(m,n)+1$ moves.

Let $\mathcal{P}$ be an arbitrary board filling queen path from P to Q, constituted of $f(P,Q)$ moves. Extending $\mathcal{P}$ by a move from Q to P, we obtain a closed board filling queen path on $\mathcal{B}$, constituted of $f(P,Q)$ moves. On the other hand, let $\mathcal{P}$ be any closed board filling queen path on $\mathcal{B}$, constituted of $t(m,n)$ moves. Then there exists a move M of $\mathcal{P}$, which covers the cell P. Let $M=P'P''$ be a move from a cell P' to a cell P''. Replacing the move $P'P''$ by moves $P'P$, QP, and PP'' (if some of these moves are empty then we skip them) and adjusting the order of moves, we obtain a board filling queen path from P to Q, constituted of at most $t(m,n)+2$ moves. $\square$

In fact, $t(m,n) = \min\{f(P,Q) : P,Q$ are cells of $\mathcal{B}$ connected by a queen move$\} + 1 =$ $= \min\{f(P,P) : P$ is a cell of $\mathcal{B}\}$. The above bounds suggest the following

Problem 1. *Find minimum and maximum values of $f(P,Q)$ when P and Q are cells of $\mathcal{B}$.*

Proposition 3. *Let $m \le n$ be natural numbers, $\mathcal{B}$ be an $m \times n$ board, and P and Q be any cells of $\mathcal{B}$. Then $f(P,Q) \le 2m+1$ if P and Q lie in distinct rows or m is even, and $f(P,Q) \le 2m+2$ otherwise.*

Proof. The idea of the proof is similar to that of Proposition 1. Place a rook at a P and move it to the leftmost cell of the row. Then move the rook, traversing rows from one board side to the other, switching rows in arbitrary order, but ending at the row containing the cell Q, and finishing the path by moving the rook to Q. Counting the number of moves of the path, we obtain the required bounds. Note that when P is not at the left of Q and in the same row and m is even then we do not need to traverse this row, so we can save one move. $\square$

On the other hand, similarly to the proof by John Selfridge from [4], we can show the following

Proposition 4. *Let $m \le n$ be natural numbers, $\mathcal{B}$ be an $m \times n$ board, and P and Q be any cells of $\mathcal{B}$. Then $f(P,Q) \ge 2m-1$ if $m < n$ and $f(P,Q) \ge 2m-2$ if $m = n$. Moreover, if $P = Q$ and $n \ge 2m$ then $f(P,Q) \ge 2m$.*

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