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NOTE TO THE BEHAVIOR OF THE MAXIMAL TERM OF DIRICHLET SERIES ABSOLUTELY CONVERGENT IN HALF-PLANE

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Denote by $S_0(\Lambda)$ the class of Dirichlet series $F(s)=\sum_{n=0}^{\infty}a_n\exp\{s\lambda_n\}$ $(s=\sigma+it)$ with an increasing to $+\infty$ sequence $\Lambda=(\lambda_n)$ of exponents $(\lambda_0=0)$ and the abscissa of absolute convergence $\sigma_a=0$. We say that $F\in S_0^*(\Lambda)$ if $F\in S_0(\Lambda)$ and $\ln\lambda_n=o(\ln|a_n|)$ $(n\to\infty)$. Let $\mu(\sigma,F)=\max\{|a_n|\exp(\sigma\lambda_n)\colon n\geq 0\}$ be the maximal term of Dirichlet series. It is proved that in order that

$$\ln(1/|\sigma|)=o(\ln\mu(\sigma))\ (\sigma\uparrow 0)$$

for every function $F\in S_0^*(\Lambda)$ it is necessary and sufficient that

$$\varlimsup_{n\to\infty}\frac{\ln\lambda_{n+1}}{\ln\lambda_n}<+\infty.$$

For an analytic in the disk $\{z\colon |z|<1\}$ function $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $r\in(0,1)$ we put $M_f(r)=\max\{|f(z)|\colon |z|=r<1\}$ and $\mu_f(r)=\max\{|a_n|r^n\colon n\geq 0\}$. As a corollary we get the following statement: *if there exists a sequence* (n_j) *such that*

$$\ln n_{j+1}=O(\ln n_j)\ \textit{and}\ \ln n_j=o(\ln|a_{n_j}|)\ \textit{as}\ j\to\infty,$$

then the functions $\ln\mu_f(r)$ *and* $\ln M_f(r)$ *are or are not slowly increasing simultaneously.*

1. Introduction. For an analytic in the disk $\{z\colon |z|<1\}$ function

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,\quad z=re^{i\theta},\tag{1}$$

let $M_f(r)=\max\{|f(z)|\colon |z|=r<1\}$ and $\mu_f(r)=\max\{|a_n|r^n\colon n\geq 0\}$ be the maximal term. A positive continuous and increasing to $+\infty$ on $[0,1)$ function l is called slowly increasing if $l((x+1)/2))\sim l(x)$ as $x\uparrow 1$. It is known [1] that if

$$\ln\frac{1}{1-r}=o(\ln\mu_f(r)),\quad r\uparrow 1,\tag{2}$$

then $\ln\mu_f(r)$ and $\ln M_f(r)$ are or are not slowly increasing simultaneously. If the condition (2) does not hold then [1] the slow growth of $\ln M_f(r)$ does not follow from the slow growth of $\ln\mu_f(r)$, and vice versa [2]. The following *question* arises: under which conditions on a_n the relation (2) is true?

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If $\ln|a_n| \le K \ln n$ $(n \ge n_0)$ then

$$\ln \mu_f(r) \le \max\{K \ln t + t \ln r\colon t \ge 1\} + O(1) = K \ln \frac{1}{-\ln r} + O(1) = K \ln \frac{1}{1-r} + O(1)$$

as $r \uparrow 1$. Therefore, in order that (2) holds, it is necessary that $\ln n_k = o(\ln|a_{n_k}|)$ $(k \to +\infty)$ for some increasing sequence (n_k) of integers, and our question is reduced to finding of conditions on this sequence (n_k). A result proved below for Dirichlet series absolutely convergent in half-plane implies that such condition is $\ln n_{k+1} = O(\ln n_k)$ $(k \to \infty)$.

2. Main result. So, let $\Lambda = (\lambda_n)$ be an increasing to $+\infty$ sequence of positive numbers $(\lambda_0 = 0)$, and Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{3}$$

has the abscissa of absolute convergence $\sigma_a = 0$. For $\sigma < 0$ let

$$\mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n)\colon n \ge 0\}$$

be the maximal term of series (3). We investigate conditions on (a_n) and (λ_n), under which

$$\ln \frac{1}{|\sigma|} = o(\ln \mu(\sigma)), \quad \sigma \uparrow 0. \tag{4}$$

To that end we denote by $S_0^*(\Lambda)$ the class of Dirichlet series (3) absolutely convergent in the half-plane $\{s\colon \mathrm{Re}\,\sigma < 0\}$ such that $\ln \lambda_n = o(\ln|a_n|)$ $(n \to \infty)$.

Theorem 1. *In order that* (4) *holds for every function $F \in S_0^*(\Lambda)$, it is necessary and sufficient that*

$$\varlimsup_{n\to\infty} \frac{\ln \lambda_{n+1}}{\ln \lambda_n} < +\infty. \tag{5}$$

Proof. Let us start with the sufficiency. Let $\Omega(0)$ be the class of positive unbounded on $(-\infty, 0)$ functions Φ such that the derivative Φ' is positive, continuously differentiable and increasing to $+\infty$ on $(-\infty, 0)$. We denote by φ the inverse function to Φ', and let $\Psi(x) = x - \Phi(x)/\Phi'(x)$ be the function associated with Φ in the sense of Newton. It is clear that the function φ is continuously differentiable and increasing to 0 on $(0, +\infty)$. The function Ψ is ([3, 4], [5, p.30]) continuously differentiable and increasing to 0 on $(-\infty, 0)$.

For $\Phi \in \Omega(0)$ and $0 \le a < b < +\infty$ we put

$$G_1(a, b, \Phi) = \frac{ab}{b-a} \int_a^b \frac{\Phi(\varphi(t))}{t^2} dt, \quad G_2(a, b, \Phi) = \Phi\Big(\frac{1}{b-a} \int_a^b \varphi(t) dt\Big).$$

Then ([6], [5, p.34]) $G_1(a, b, \Phi) < G_2(a, b, \Phi)$. It is clear that $G_2(\lambda_n, \lambda_{n+1}, \Phi) = \Phi(\varkappa_n)$, where

$$\varkappa_n = \frac{1}{\lambda_{n+1} - \lambda_n} \int_{\lambda_n}^{\lambda_{n+1}} \varphi(t) dt.$$

Theorem 3.1 in [4], [5, p. 34-35] implies that if $\ln|a_n| \ge -\lambda_n \Psi(\varphi(\lambda_n))$ $(n \ge n_0)$ then

$$\ln \mu(\sigma, F) \ge \Phi(\sigma) \frac{G_1(\lambda_n, \lambda_{n+1}, \Phi)}{G_2(\lambda_n, \lambda_{n+1}, \Phi)} \tag{6}$$

for all $\sigma \in [\varphi(\lambda_n), \varphi(\lambda_{n+1})]$ and $n \geq n_0$. We remark also that if a function f is positive, continuous and increasing to $+\infty$ on $[0, +\infty)$ such that $f(x) > x$ and $\lambda_{n+1} \leq f(\lambda_n)$ then ([7], [5, p. 34])

$$\frac{G_1(\lambda_n, \lambda_{n+1}, \Phi)}{G_2(\lambda_n, \lambda_{n+1}, \Phi)} \geq \frac{G_1(\lambda_n, f(\lambda_n), \Phi)}{G_2(\lambda_n, f(\lambda_n), \Phi)}. \tag{7}$$

Now, let $T > 0$ be an arbitrary number and $\Phi(\sigma) = T \ln \frac{1}{|\sigma|}$. Then $\varphi(x) = -\frac{T}{x}$, $\Psi(\sigma) = -|\sigma| \ln \frac{e}{|\sigma|}$, and $\Psi(\varphi(x)) = -\frac{T}{x} \ln \frac{ex}{T}$. Therefore,

$$G_1(\lambda_n, \lambda_{n+1}, \Phi) = T \frac{\lambda_{n+1} \ln \lambda_n - \lambda_n \ln \lambda_{n+1}}{\lambda_{n+1} - \lambda_n} + T \ln \frac{e}{T}$$

and

$$G_2(\lambda_n, \lambda_{n+1}, \Phi) = T \ln \frac{\lambda_{n+1} - \lambda_n}{\ln \lambda_{n+1} - \ln \lambda_n} - T \ln T.$$

From the definition of $S^*(\Lambda)$ we have $\ln |a_n| \geq T \ln \lambda_n \geq T \ln(e\lambda_n/T) = -\lambda_n \Psi(\varphi(\lambda_n))$ for arbitrary $T \geq e$ and all $n \geq n_0(T)$, and from condition (5) it follows that there exists a number $\beta > 0$ such that $\lambda_{n+1} \leq \lambda_n^{1+\beta}$ $(n \geq n_0)$. Therefore, (7) implies

$$\frac{G_1(\lambda_n, \lambda_{n+1}, \Phi)}{G_2(\lambda_n, \lambda_{n+1}, \Phi)} \geq \frac{G_1(\lambda_n, \lambda_n^{1+\beta}, \Phi)}{G_2(\lambda_n, \lambda_n^{1+\beta}, \Phi)} = \frac{\frac{\lambda_n^{1+\beta} \ln \lambda_n - (1+\beta)\lambda_n \ln \lambda_n}{\lambda_n^{1+\beta} - \lambda_n} - \ln \frac{T}{e}}{\ln \frac{\lambda_n^{1+\beta} - \lambda_n}{\beta \ln \lambda_n} - \ln T} = \frac{1 + o(1)}{1 + \beta}$$

as $n \to \infty$ and, thus, from (6) we get

$$\ln \mu(\sigma, F) \geq \frac{(1 + o(1))T}{1 + \beta} \ln \frac{1}{|\sigma|}, \quad \sigma \uparrow 0.$$

i. e. in view of the arbitrariness of T we obtain (4).

Now we prove the necessity. Suppose that condition (5) does not hold, i. e. there exists an increasing to $+\infty$ sequence of integers such that $\ln \lambda_{n_k+1}/\ln \lambda_{n_k} \to \infty$, $k \to \infty$. We choose a slowly increasing to $+\infty$ on $[0, +\infty)$ continuously differentiable function α such that $\alpha(\ln \lambda_{n_{k+1}}) \leq \frac{\ln \lambda_{n_k+1}}{\ln \lambda_{n_k}}$ $(k \geq k_0)$ and the function $\Phi(\sigma) = \alpha(\ln \frac{1}{|\sigma|}) \ln \frac{1}{|\sigma|}$ belongs to $\Omega(0)$. We choose the coefficients of Dirichlet series such that $\ln |a_n| = -\lambda_n \Psi(\varphi(\lambda_n))$. Then $\varkappa_n = \frac{\ln |a_n| - \ln |a_{n+1}|}{\lambda_{n+1} - \lambda_n}$, because $(x\Psi(\varphi(x)))' = \varphi(x)$. Since the function α is slowly increasing, we have $x\alpha'(x)/\alpha(x) \to 0$ as $x \to +\infty$. Therefore,

$$\Phi'(\sigma) = \frac{1}{|\sigma|} \Big\{ \alpha'\Big(\ln \frac{1}{|\sigma|}\Big) \ln \frac{1}{|\sigma|} + \alpha\Big(\ln \frac{1}{|\sigma|}\Big) \Big\} = \frac{1 + o(1)}{|\sigma|} \alpha\Big(\ln \frac{1}{|\sigma|}\Big), \quad \sigma \uparrow 0,$$

and in order to find the asymptotical behaviour of φ it is necessary to solve the equation

$$\ln \frac{1}{|\sigma|} + \ln \alpha\Big(\ln \frac{1}{|\sigma|}\Big) = \ln x + o(1), \quad x \to +\infty. \tag{8}$$

We find a solution $\sigma = \sigma(x)$ of (8) in the form

$$\ln \frac{1}{|\sigma|} = \ln x - \beta, \quad \beta = \beta(x) = o(\ln x), \quad x \to +\infty. \tag{9}$$

Substituting (9) in (8) we obtain $\beta = \ln\alpha(\ln x - \beta) + o(1), x \to +\infty$. But for some $\xi \in (\ln x - \beta, \ln x)$ we have $\alpha(\ln x) - \alpha(\ln x - \beta) = \alpha'(\xi)\beta = o(\xi\alpha'(\xi)) = o(\alpha(\xi)) = o(\alpha(\ln x))$ $(x \to +\infty)$, i.e. $\beta(x) = \ln\alpha(\ln x) + o(1), x \to +\infty$, and, therefore, from (9) we obtain $\ln\frac{1}{|\sigma|} = \ln x - \ln\alpha(\ln x) + o(1),\ x \to +\infty$. Thus,

$$\varphi(x) = -\frac{(1+o(1))\alpha(\ln x)}{x}, \quad x \to +\infty. \tag{10}$$

Using L'Hôspitale rule and relation (10) we see that Dirichlet series (3) with choosen coefficients belongs to $S_0^*(\Lambda)$. From (10) it follows also that

$$\varkappa_{n_k} = \frac{1}{\lambda_{n_k+1} - \lambda_{n_k}} \int\limits_{\lambda_{n_k}}^{\lambda_{n_k+1}} \varphi(x)dx = -\frac{1+o(1)}{\lambda_{n_k+1} - \lambda_{n_k}} \int\limits_{\lambda_{n_k}}^{\lambda_{n_k+1}} \frac{\alpha(\ln x)}{x} dx \geq$$

$$\geq -\frac{(1+o(1))\alpha(\ln\lambda_{n_k+1})(\ln\lambda_{n_k+1} - \ln\lambda_{n_k})}{\lambda_{n_k+1} - \lambda_{n_k}} = -\frac{(1+o(1))\alpha(\ln\lambda_{n_k+1})\ln\lambda_{n_k+1}}{\lambda_{n_k+1}},$$

$$\ln\frac{1}{|\varkappa_{n_k}|} \geq \ln\frac{\lambda_{n_k+1}}{\alpha(\ln\lambda_{n_k+1})\ln\lambda_{n_k+1}} + o(1) = (1+o(1))\ln\lambda_{n_k+1} \tag{11}$$

as $k \to +\infty$. On the other hand, since [4]

$$\ln\mu(\varkappa_n, F) = -\lambda_n\Psi(\varphi(\lambda_n)) + \varkappa_n\lambda_n = G_1(\lambda_n, \lambda_{n+1}, \Phi)$$

and in view of (10)

$$\Phi(\varphi(x)) = \alpha(\ln x - \ln\alpha(\ln x) + o(1))(\ln x - \ln\alpha(\ln x) + o(1)) = (1+o(1))\alpha(\ln x)\ln x$$

as $x \to +\infty$, we get

$$\ln\mu(\varkappa_{n_k}, F) = (1+o(1))\lambda_{n_k} \int\limits_{\lambda_{n_k}}^{\lambda_{n_k+1}} \frac{\Phi(\varphi(x))}{x^2} dx =$$

$$= (1+o(1))\lambda_{n_k} \int\limits_{\lambda_{n_k}}^{\lambda_{n_k+1}} \frac{\alpha(\ln x)\ln x}{x^2} dx \leq (1+o(1))\lambda_{n_k}\alpha(\ln\lambda_{n_k+1}) \int\limits_{\lambda_{n_k}}^{\lambda_{n_k+1}} \frac{\ln x}{x^2} dx =$$

$$= (1+o(1))\lambda_{n_k}\alpha(\ln\lambda_{n_k+1})\Big(\frac{\ln\lambda_{n_k}+1}{\lambda_{n_k}} - \frac{\ln\lambda_{n_k+1}+1}{\lambda_{n_k+1}}\Big) =$$

$$= (1+o(1))\alpha(\ln\lambda_{n_{k+1}})\ln\lambda_{n_k}, \quad k \to \infty. \tag{12}$$

From (11) and (12) it follows that

$$\frac{\ln\mu(\varkappa_{n_k}, F)}{\ln(1/|\varkappa_{n_k}|} \leq (1+o(1))\frac{\alpha(\ln\lambda_{n_{k+1}})\ln\lambda_{n_k}}{\ln\lambda_{n_k+1}} \leq 1+o(1), \quad k \to \infty,$$

i. e. relation (4) does not hold. The necessity of condition (5) is proved. □

3. Corollaries. Since $\max\{|a_n|\exp(\sigma\lambda_n)\colon n \geq 0\} \geq \max\{|a_{n_j}|\exp(\sigma\lambda_{n_j})\colon j \geq 1\}$ for any sequence (n_j), Theorem 1 implies the following statement.

Corollary 1. *If there exists a subsequence* (λ_{n_j}) *of the sequence* (λ_n) *such that* $\ln\lambda_{n_{j+1}} = O(\ln\lambda_{n_j})$ *and* $\ln\lambda_{n_j} = o(\ln|a_{n_j}|)$ *as* $j\to\infty$ *then* (4) *holds.*

If in power series (1) we make the substitution $z = e^s$ then we obtain Dirichlet series (3) with $\lambda_n = n$, $|\sigma| = |\ln r| = (1 + o(1))(1 - r)$, $r \uparrow 1$, and $\mu_f(r, F) = \mu(\ln r, F)$. Therefore, if there exists a sequence (n_j) such that $\ln n_{j+1} = O(\ln n_j)$ and $\ln n_j = o(\ln|a_{n_j}|)$ as $j \to \infty$ then (2) holds. Hence and from above-mentioned result in [1] the following corollary follows.

Corollary 2. *If there exists a sequence* (n_j) *such that*

$$\ln n_{j+1} = O(\ln n_j) \text{ and } \ln n_j = o(\ln|a_{n_j}|) \text{ as } j \to +\infty,$$

then the functions $\ln\mu_f(r)$ *and* $\ln M_f(r)$ *are or are not slowly increasing simultaneously. In particular, if* $\ln n = o(\ln|a_n|)$ *as* $n\to\infty$ *then the functions* $\ln\mu_f(r)$ *and* $\ln M_f(r)$ *are or are not slowly increasing simultaneously.*

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