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FRACTAL FUNCTIONS OF EXPONENTIAL TYPE THAT IS GENERATED BY THE $\mathbf{Q_2^*}$-REPRESENTATION OF ARGUMENT

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We consider function f which is depended on the parameters $0 < a \in \mathbb{R}$, $q_{0n} \in (0;1)$, $n \in \mathbb{N}$ and convergent positive series $v_1 + v_2 + ... + v_n + ...$, defined by equality $f(x = \Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_n...}) = a^{\varphi(x)}$, where $\alpha_n \in \{0,1\}$, $\varphi(x = \Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_n...}) = \alpha_1 v_1 + ... + \alpha_n v_n + ...$, $q_{1n} = 1 - q_{0n}$, $\Delta^{Q_2^*}_{\alpha_1...\alpha_n...} = \alpha_1 q_{1-\alpha_1,1} + \sum\limits_{n=2}^{\infty} \big(\alpha_n q_{1-\alpha_n,n} \prod\limits_{i=1}^{n-1} q_{\alpha_i,i}\big)$. In the paper we study structural, variational, integral, differential and fractal properties of the function f.

Introduction. By a fractal (fractal set) we mean the set of space $\mathbb{R}^n$, which has a fractional self-affine dimension (structural fractality) or a fractional dimension of Hausdorff-Bezikovich [13] (further fractal dimension). It is metric fractality.

We say that the function $y = f(x)$ has fractal properties or is a fractal if at least one of the following essential sets for the function is a fractal. This essential set can be a range of the function, a level set, a set of non-differentiability, a set of points of growth or instability, a set of points at which the derivative is not equal to zero, the graph of the function and so on. In this context, among the continuous functions the potential fractals are: nowhere monotonic functions, nowhere differentiable functions, singular functions (continuous functions whose derivative is equal zero almost everywhere in the sense of the Lebesgue measure), and others. Moreover, there exist absolutely continuous functions with fractal properties.

In paper [12], Sendov considered one class of functions with fractal properties. It was defined by the classic binary representation of the argument and absolutely convergent infinite products. Using the Sendov idea of definition of such a function, we described and studied a new class of fractal functions with argument having Q_2^*-representation.

In this paper, the functions of more massive class are studied. The expansion is obtained by usage of Q_2^*-representation of argument, which is a significant generalization of the Q_2-representation of numbers. As a result, we obtained functions with new properties.

1. Main object. Let $a > 0$ be a fixed real number, $A = \{0, 1\}$ be an alphabet, $L = A \times A \times ...$ be a space of sequences of elements of the alphabet (zeros and ones), $\sum_{n=1}^{\infty} v_n$ be an absolutely convergent series with the sum r_0 and a descending sequence of terms v_n and

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let $x = \Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_n...}$ be a Q_2^*-representation of number x [13, 21], that is defined by an infinite two-line stochastic matrix $||q_{in}||$ with positive elements: $q_{0k} + q_{1k} = 1$, $\prod_{k=1}^{\infty} \max\{q_{ik}\} = 0$, i.e.

$$x = \alpha_1 q_{1-\alpha_1,1} + \sum_{n=2}^{\infty} \left(\alpha_n q_{1-\alpha_n,n} \prod_{i=1}^{n-1} q_{\alpha_i,i} \right) \equiv \Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_n...}.$$

In this paper, the main object of investigation is the function defined by the equality

$$f(x = \Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_n...}) \equiv a^{\varphi(x)}, \text{ where } \varphi(x) \equiv \sum_{i=1}^{\infty} \alpha_i v_i. \tag{1}$$

Clearly, that the function $f(x)$ is well-defined on the set of Q_2^*-*unary* numbers (namely that have a single Q_2-representation). The numbers of the countable everywhere dense set in $[0;1]$ have two Q_2^*-representations (these are numbers that have following representations $\Delta^{Q_2^*}_{c_1...c_m1(0)} = \Delta^{Q_2^*}_{c_1...c_m0(1)}$, they are called Q_2^*-*binary*). The definition of function f in Q_2^*-binary point is correct if the definition of function φ is correct. But it is possible when for any $n \in \mathbb{N}$ and set of numbers $(\alpha_1, \alpha_2, ..., \alpha_n)$ the following equation holds $\varphi(\Delta^{Q_2^*}_{\alpha_1...\alpha_n0(1)}) = \varphi(\Delta^{Q_2^*}_{\alpha_1...\alpha_n1(0)})$, which is equivalent $v_n = r_n \equiv v_{n+1} + v_{n+2} + ...$, $n \in \mathbb{N}$. In a general case, the definition of function f is correct if we will use only one of the representation of Q_2^*-binary numbers. Let us use the representation with a period (0).

Interest to the functions of type (1) is generated by the existence of fractal properties of various kinds. Among them there are structural properties (scale invariance of the graph), metric properties (Hausdorff-Besicovith dimension) which are important for a function of set. Also the functions of type (1) have close connection with three modern areas of mathematical research:
1) geometry of numerical series [3, 4, 5, 6, 8, 11];
2) theory of singular functions and distributions of random variables [13];
3) fractal geometry and fractal analyses, in particular, with the theory of transformations (and functions) preserving fractal dimension [1, 20].

Remark that the fractal dimensions of ranges of the functions φ and f coincide, because the exponential function preserves its dimension [1].

Example 1. If $v_n = \frac{1}{2^n}$ and Q_2^*-representation is a classic binary representation, namely $q_{in} = \frac{1}{2}$, then $\varphi(x) = x$ and $f(x) = a^x$.

Example 2. If $v_n = \frac{1}{2^n}$ and $Q_2^* = Q_2$, i.e. $q_{in} = q_i$ for all $n \in N$, and $q_0 \neq \frac{1}{2}$, then $\varphi(x)$ is a singular Salem's function [10, 7]. In this case, the function f is also a singular function, i.e. it is a continuous function whose derivative is equal zero almost everywhere in the sense of the Lebesgue measure.

Example 3. If $v_n = \frac{2}{3^n}$, then the range of the function φ is a classic Cantor set having the fractal dimension of Hausdorff-Bezikovich $\log_3 2 \approx 0,63$.

Remark 1. A class S of functions (1) is wide, because we can vary sequences (v_n) and the parameters defining Q_2^*-representation of numbers, and also a choice of number a.

One of the key concepts in the theory Q_2^*-representation of numbers is the concept of Q_2^*-cylinder. Recall its definition. The set

$$\Delta^{Q_2^*}_{c_1c_2...c_m} = \{x \in [0;1] : x = \Delta^{Q_2^*}_{c_1c_2...c_m\alpha_1\alpha_2...}, (\alpha_i) \in L\}$$

is called Q_2^*-cylinder of rank m with base $c_1c_2...c_m$.

Q_2^*-cylinder have properties:

1) $\Delta^{Q_2^*}_{c_1\ldots c_m} = \Delta^{Q_2^*}_{c_1\ldots c_m 0} \cup \Delta^{Q_2^*}_{c_1\ldots c_m 1}$; $[0;1] = \bigcup\limits_{c_1\in A} \ldots \bigcup\limits_{c_m\in A} \Delta^{Q_2^*}_{c_1\ldots c_m}$;

2) $\max \Delta^{Q_2^*}_{c_1c_2\ldots c_m 0} = \min \Delta^{Q_2^*}_{c_1c_2\ldots c_m 1}$;

3) $\Delta^{Q_2^*}_{c_1c_2\ldots c_m}$ is a segment $[a;b]$, where $a = \beta_{c_1 1} + \sum\limits_{k=2}^{m} (\beta_{c_k k} \prod\limits_{i=1}^{k-1} q_{c_i i})$, $b = a + \prod\limits_{i=1}^{m} q_{c_i i}$;

4) $|\Delta^{Q_2^*}_{c_1c_2\ldots c_m}| = \prod\limits_{i=1}^{m} q_{c_i i}$; $\quad \frac{|\Delta^{Q_2^*}_{c_1c_2\ldots c_m i}|}{|\Delta^{Q_2^*}_{c_1c_2\ldots c_m}|} = q_{i,m+1}$;

5) For $\forall$ (c_n): $\bigcap\limits_{m=1}^{\infty} \Delta^{Q_2^*}_{c_1\ldots c_m} = \Delta^{Q_2^*}_{c_1\ldots c_m\ldots}$.

2. Functional relations and structural properties. The following equalities are obvious: $\varphi(0 = \Delta^{Q_2^*}_{(0)}) = 0$, $\varphi(1 = \Delta^{Q_2^*}_{(1)}) = r_0$, $f(0) = 1$, $f(1) = a^{r_0}$;

$$\varphi(\Delta^{Q_2^*}_{1\alpha_2\ldots\alpha_n\ldots}) = v_1 + \varphi(\Delta^{Q_2^*}_{0\alpha_2\ldots\alpha_n\ldots}),\ f(\Delta^{Q_2^*}_{1\alpha_2\ldots\alpha_n\ldots}) = a^{v_1} f(\Delta^{Q_2^*}_{0\alpha_2\ldots\alpha_n\ldots}). \tag{2}$$

A function $I(x)$, defined by equality $I(x = \Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{Q_2^*}_{[1-\alpha_1][1-\alpha_2]\ldots[1-\alpha_n]\ldots}$, is called *inversor of Q_2^*-representation* of numbers. It is continuous and strictly decreasing. And in view of the matrix $||q_{ik}||$ the function $I(x)$ is either singular or absolutely continuous and is not a combination of the singular part and absolutely continuous part. In particular, it is singular if $0 < \lim\limits_{k\to\infty} q_{0k} = q_0 < 1$, $q_0 \neq \frac{1}{2}$ and it is absolutely continuous when $q_0 = \frac{1}{2}$ [17].

Lemma 1. *The following equations hold*

1) $\dfrac{\varphi(\Delta^{Q_2^*}_{c\alpha_2\alpha_3\ldots})}{\varphi(\Delta^{Q_2^*}_{[1-c]\alpha_2\alpha_3\ldots})} = 1 + \dfrac{v_1(2c-1)}{\varphi(\Delta^{Q_2^*}_{[1-c]\alpha_2\alpha_3\ldots})}$;

2) $\varphi(I(x)) = \varphi(1) - \varphi(x)$, *i.e.* $\varphi(x) + \varphi(I(x)) = \varphi(1)$;

3) $f(x) \cdot f(I(x)) = a^{\varphi(1)}$.

Proof. In fact,

1) $$\frac{\varphi(\Delta^{Q_2^*}_{c\alpha_2\alpha_3\ldots})}{\varphi(\Delta^{Q_2^*}_{[1-c]\alpha_2\alpha_3\ldots})} = \frac{cv_1 + \alpha_2 v_2 + \ldots}{v_1(1-c) + \alpha_2 v_2 + \ldots} = \frac{cv_1 - v_1(1-c) + v_1(1-c) + \alpha_2 v_2 + \ldots}{v_1(1-c) + \alpha_2 v_2 + \ldots} =$$
$$= \frac{(2c-1)v_1}{\varphi(\Delta^{Q_2^*}_{[1-c]\alpha_2\alpha_3\ldots})} + 1;$$

2) $\varphi(I(x)) = \varphi(\Delta^{Q_2^*}_{[1-\alpha_1][1-\alpha_2]\ldots[1-\alpha_n]\ldots}) = v_1(1-\alpha_1) + v_2(1-\alpha_2) + \ldots + v_n(1-\alpha_n) + \ldots =$
$= v_1 + v_2 + \ldots + v_n + \ldots - (v_1\alpha_1 + v_2\alpha_2 + \ldots + v_n\alpha_n + \ldots) = \varphi(1) - \varphi(x)$;

3) $f(x) \cdot f(I(x)) = a^{\varphi(x)} \cdot a^{\varphi(I(x))} = a^{\varphi(x)} \cdot a^{\varphi(1)-\varphi(x)} = a^{\varphi(1)}$. □

Lemma 2. *The graph Γ_f of the function f, defined by the equation (1), is a self-similar set having the structure:*

$$\Gamma_f \equiv \Gamma_0 \cup \Gamma_1, \quad \Gamma_i \equiv \{M(x;y) : x \in \Delta^{Q_2^*}_i, y = f(x)\},$$

where $\Gamma_i = \gamma_i(\Gamma_f)$: $\gamma_i : \begin{cases} x' = \Delta^{Q_2^*}_{i\alpha_2\alpha_3\ldots\alpha_n\ldots}, \\ y' = a^{v_1(i-\alpha_1(x))} f(x). \end{cases}$

Proof. Prove that $\Gamma_i = \gamma_i(\Gamma_f)$.

1. Firstly, we show inclusion: $\Gamma_i \subset \gamma_i(\Gamma_f)$. Let be $M_0(x_0; y_0) \in \Gamma_i$, namely $x_0 = \Delta^{Q_2^*}_{i\alpha_2\alpha_3\ldots\alpha_n\ldots}$, $y_0 = a^{iv_1+\alpha_2 v_2+\cdots}$. Let us show that $M_0 \in \gamma_i(\Gamma_f)$. To do this, we consider the point $M_*(x_*; y_*)$ on the graph of the function f: $x_* = \Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} = \delta_{\alpha_1}(\omega(x))$, $y_* = f(x_*)$ and its image in the transformation γ_i

$$x'_* = \Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} = x_0,\ y'_* = a^{v_1(i-\alpha_1)} f(x_*) = a^{iv_1+\alpha_2 v_2+\ldots+\alpha_n v_n+\cdots} = f(x_0).$$

Therefore, $(x'_*; y'_*) = (x_0; y_0) \in \gamma_i(\Gamma_f)$

2. Now we prove the inclusion $\gamma_i(\Gamma_f) \subset \Gamma_i$. Let $M(x; y) \in \Gamma_f$, $y = f(x)$, $\gamma_i(M) = M'(x'; y')$, i.e. $x' = \Delta^{Q_2^*}_{i\alpha_2\alpha_3\ldots\alpha_n\ldots}$, $y = a^{v_1(i-i)} f(x') = f(x')$. Hence, $M(x; y) \in \Gamma_i$. From the proven inclusions we obtain $\Gamma_i = \gamma_i(\Gamma_f)$. □

3. Continuous functions f.

Theorem 1. *The function f is continuous at every Q_2^*-unary point and is right continuous at every Q_2^*-binary point.*

Proof. Let $x_0 = \Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ be an arbitrary Q_2^*-unary number. We consider $x = \Delta^{Q_2^*}_{c_1\ldots c_n\ldots}$ such that $x \neq x_0$. Then there exists m such that $c_m \neq \alpha_m$ and $c_i = \alpha_i$ when $i < m$, and the conditions $x \to x_0$ and $m \to \infty$ are equivalent. We consider the relationship

$$\frac{f(x)}{f(x_0)} = \prod_{i=1}^{m-1} a^{v_i(c_i-\alpha_i)} \cdot a^{v_m(c_m-\alpha_m)} \cdot \prod_{i=m+1}^{\infty} a^{v_i(c_i-\alpha_i)},$$

and

$$\prod_{i=1}^{m-1} a^{v_i(c_i-\alpha_i)} = 1,\ \lim_{m\to\infty} a^{v_m} = 1 = \lim_{m\to\infty} \prod_{i=m+1}^{\infty} a^{v_i(c_i-\alpha_i)}.$$

Therefore, $\lim\limits_{x\to x_0} f(x) = f(x_0)$, namely the function f is continuous at the point x_0.

We consider Q_2^*-binary point $x_0 = \Delta^{Q_2^*}_{c_1c_2\ldots c_m 1(0)} = \Delta^{Q_2^*}_{c_1c_2\ldots c_m 0(1)}$. Let $x > x_0$ and the point x be close enough to x_0. Then $x = \Delta^{Q_2^*}_{c_1\ldots c_m 1\underbrace{0\ldots0}_{k}\alpha_{m+k+2}\ldots}$, and among the terms of the sequence (α_{m+k+2}) there are units. However, the condition $x \to x_0$ is equivalent to the $k \to \infty$. Since $\lim\limits_{x\to x_0} \frac{f(x)}{f(x_0)} = \lim\limits_{k\to\infty} a^{\sum_{n=m+k+2}^{\infty} v_n\alpha_n} = 1$, f is right continuous at the point x_0. □

Lemma 3. *In order that the function $\varphi(x)$ is continuous at the point $x_0 = \Delta^{Q_2^*}_{c_1c_2\ldots c_{m-1}1(0)}$ it is necessary and sufficient that the following equality $v_m = v_{m+1} + v_{m+2} + \ldots$.*

Proof. In view of the previous theorem, the function f (it is equivalent to φ) is continuous at the point x_0 only if it is left continuous at this point, namely when for any sequences (x_k) such that $x_0 > x_k \to x_0 (k \to \infty)$ the following equality is true $\lim\limits_{x_k\to x_0} \varphi(x_k) = \varphi(x_0)$.

If $x_k < x_0$ and x_k is close enough to x_0, then it has the following Q_2^*-representation $x_k = \Delta^{Q_2^*}_{c_1\ldots c_{m-1}0\underbrace{1\ldots1}_{k}\alpha_{m+k+1}\alpha_{m+k+2}\ldots}$. Then

$$\varphi(x_0) - \varphi(x_k) = v_m - (v_{m+1} + \cdots + v_{m+k} + \alpha_{m+k+1}v_{m+k+1} + \alpha_{m+k+2}v_{m+k+2} + \cdots).$$

Hence, it follows that $\lim\limits_{k\to\infty}[\varphi(x_0) - \varphi(x_k)] = 0$ only if $v_m = r_m$. □

Theorem 2. *The function $\varphi(x)$ is continuous on the segment $[0;1]$ if and only if $v_n = \frac{r_0}{2^n}$ for any $n \in \mathbb{N}$ and some $v \in \mathbb{R}$.*

Proof. Taking into account previous theorems and lemmas, we see that the function f is continuous on the segment $[0;1]$ if and only if it is left continuous at each Q_2^*-binary point, i.e. if $v_n = r_n$ for arbitrary n. And this is valid when $v_1 = r_1 = r_0 - v_1$, i.e.$v_1 = \frac{r_0}{2}$ and $v_n = r_n \equiv r_{n-1} - v_n$. It yields $v_1 = \frac{r_0}{2}$, $v_n = \frac{r_0}{2^n}$. □

Lemma 4. *If at the point $x_0 = \Delta^{Q_2^*}_{c_1c_2\ldots c_{m-1}1(0)}$ the function φ has a discontinuity, i.e. $v_m > r_m$, then the jumps $\delta_\varphi(x_0)$ and $\delta_f(x_0)$ of the functions φ and f accordingly at this point do not depend on the set of digits $c_1, \ldots, c_{m-1}$ and they are calculated by the formulas $\delta_\varphi(x_0) = v_m - r_m$, $\delta_f(x_0) = a^{v_m} - a^{r_m}$. The sum of all jumps of the function φ is equal to*

$$\sum_{m=1}^{\infty} 2^{m-1}(v_m - r_m). \tag{3}$$

Proof. Since the function f at the point x_0 is right continuous then

$$\delta_\varphi(x_0) = \lim_{\varepsilon\to 0}[\varphi(x_0+\varepsilon) - \varphi(x_0-\varepsilon)] = \varphi(x_0) - \lim_{x\to x_0-0}\varphi(x) =$$
$$= \varphi(x_0) - \lim_{k\to\infty}\varphi(\Delta^{Q_2^*}_{c_1\ldots c_{m-1}0\underbrace{1\ldots1}_{k}a_1a_2\ldots}) =$$
$$= v_m - (v_{m+1} + v_{m+2} + \ldots + v_{m+k} + v_{m+k+1} + v_{m+k+2} + \ldots) = v_m - r_m. \qquad \square$$

Since the exponential function is strictly monotonic, then $\delta_f(x_0) = a^{v_m} - a^{r_m}$.

There exist 2^{m-1} Q_2^*-binary points of ranks m (it is equivalent to: sets of zeros and ones $c_1, \ldots, c_{m-1}$ with length $m-1$), at which the function φ has the jump $v_m - r_m$. Then the sum of all jumps of function φ is equal (3).

4. Integral properties. The f function is Lebesgue integrable because it can have no more than a countable set of discontinuities.

Theorem 3. *For the function f from (1) the following equality is satisfied*

$$\int_0^1 f(x)dx = \prod_{k=1}^{\infty}(q_{0k} + a^{v_k}q_{1k}). \tag{4}$$

Proof. Since $f(\Delta^{Q_2^*}_{1\alpha_2\alpha_3\ldots}) = a^{v_1}f(\Delta^{Q_2^*}_{0\alpha_2\alpha_3\ldots})$ we have

$$\int_0^1 f(x)dx = \int\limits_{\Delta_0^{Q_2^*}} f(x)dx + \int\limits_{\Delta_1^{Q_2^*}} f(x)dx = \int\limits_{\Delta_0^{Q_2^*}} f(x)dx + a^{v_1}\int\limits_{\Delta_0^{Q_2^*}} f(t)d\Big(q_{01} + \frac{q_{11}}{q_{01}}t\Big) =$$
$$= \int\limits_{\Delta_0^{Q_2^*}} f(x)dx + a^{v_1}\frac{q_{11}}{q_{01}}\int\limits_{\Delta_0^{Q_2^*}} f(t)dt = \Big(1 + \frac{a^{v_1}q_{11}}{q_{01}}\Big)\int\limits_{\Delta_0^{Q_2^*}} f(x)dx.$$

Since $f(\Delta^{Q_2^*}_{01\alpha_3\alpha_4\ldots}) = a^{v_2}f(\Delta^{Q_2^*}_{00\alpha_3\alpha_4\ldots})$ we deduce

$$\int\limits_{\Delta_0^{Q_2^*}} f(x)dx = \int\limits_{\Delta_{00}^{Q_2^*}} f(x)dx + \int\limits_{\Delta_{01}^{Q_2^*}} f(x)dx = \int\limits_{\Delta_{00}^{Q_2^*}} f(x)dx + a^{v_2}\int\limits_{\Delta_{00}^{Q_2^*}} f(t)d\Big(q_{01}q_{02} + \frac{q_{12}}{q_{02}}t\Big) =$$

$$= \int\limits_{\Delta^{Q_2^*}_{00}} f(x)dx + a^{v_2}\frac{q_{12}}{q_{02}} \int\limits_{\Delta^{Q_2^*}_{00}} f(t)dt = \Big(1+\frac{a^{v_2}q_{12}}{q_{02}}\Big) \int\limits_{\Delta^{Q_2^*}_{00}} f(x)dx.$$

Therefore,

$$\int_0^1 f(x)dx = \Big(1+\frac{a^{v_1}q_{11}}{q_{01}}\Big)\Big(1+\frac{a^{v_2}q_{12}}{q_{02}}\Big) \int\limits_{\Delta^{Q_2^*}_{00}} f(x)dx.$$

Similarly, by k steps we get

$$\int_0^1 f(x)dx = \prod_{i=1}^{k}\Big(1+\frac{a^{v_i}q_{1i}}{q_{0i}}\Big) \cdot \int\limits_{\Delta^{Q_2^*}_{\underbrace{0\cdots0}_{k}}} f(x)dx.$$

Since $a^{v_k} \to 1$ $(k\to\infty)$ and $f(x)\to 1$ as $x \to 0$, for sufficiently large $k \in \mathbb{N}$ one has

$$\int\limits_{\Delta^{Q_2^*}_{\underbrace{0\cdots0}_{k}}} f(x)dx \approx \prod_{i=1}^{k} q_{0i} \text{ and } \int_0^1 f(x)dx \approx \prod_{i=1}^{k} q_{0i}(1+\frac{a^{v_i}q_{1i}}{q_{0i}}) = \prod_{i=1}^{k}(q_{0i}+a^{v_i}q_{1i}).$$

Since k tends to infinity, we obtain equality (4). □

Corollary 1. *If Q_2^*-representation is Q_2-representation, i.e. $q_{0n} = q_0$ for all $n \in \mathbb{N}$, then*

$$\int_0^1 f(x)dx = \prod_{k=1}^{\infty}(q_0 + a^{v_k}q_1).$$

Remark 2. If Q_2^*-representation of numbers and parameter a are fixed and sequence (u_n) and (v_n) define the functions f and g from this class S then a sequence (t_n) with $t_n = u_n + v_n$ defines a function $\phi = f\cdot g$ from the class.

Remark 3. Integral properties of the function $f(x) = a^{\varphi(x)}$ defined by equality (1) depend on the integral properties of the function $\varphi(x)$, for continuous functions f they are defined scale-invariant properties of graph of function φ.

Lemma 5. *If $v_n = \frac{r_0}{2^n}$, $Q_2^* = Q_2$, i.e. $q_{ik} = q_i$, then $\int_0^1 \varphi(x)dx = r_0q_1$.*

Proof. By the additive property of the integral one has

$$I = \int_0^1 \varphi(x)dx = \int\limits_{x\in\Delta^{Q_2}_{0}} \varphi(x)dx + \int\limits_{x\in\Delta^{Q_2}_{0}} \varphi(x)dx.$$

In view of the following properties

1) Q_2-representation: $\Delta^{Q_2}_{0\alpha_2\alpha_3\ldots} = q_0\Delta^{Q_2}_{\alpha_2\alpha_3\ldots}$, $\Delta^{Q_2}_{1\alpha_2\alpha_3\ldots} = q_0 + q_1\Delta^{Q_2}_{\alpha_2\alpha_3\ldots}$;

2) the sequence (v_n) and the function $\varphi(x)$: $\varphi(\Delta^{Q_2}_{\alpha_1\alpha_2\alpha_3\ldots}) = \frac{r_0}{2}\alpha_1 + \frac{1}{2}\varphi(\Delta^{Q_2}_{\alpha_2\alpha_3\ldots\alpha_n\ldots})$, we have

$$I = \int_0^1 \frac{1}{2}\varphi(t)d(q_0t) + \int_0^1 \Big[\frac{r_0}{2}+\frac{1}{2}\varphi(t)\Big]d(q_0t) =$$

$$= \frac{q_0}{2}\int_0^1 \varphi(t)d(t) + \frac{r_0q_1}{2} + \frac{q_1}{2}\int_0^1 \varphi(t)d(t) = \frac{1}{2}\int_0^1 \varphi(t)dt + \frac{r_0q_1}{2}.$$

Hence, $(1-\frac{1}{2})\int_0^1 \varphi(x)dx = \frac{r_0q_1}{2}$. So, $\int_0^1 \varphi(x)dx = r_0q_1$. □

Theorem 4. *If a matrix $||q_{ik}||$ defining the Q_2^*-representation has the property: $q_{ik} = q_i$ as $k > m$ and $v_{m+n} = \frac{\lambda}{2^n}$, $n \in \mathbb{N}$, then*

$$\int_0^1 \varphi(x)dx = \sum_{(c_1,\dots,c_m)\in A^m} [(c_1v_1 + \dots c_mv_m) + \lambda q_1] \prod_{k=1}^{m} q_{c_k k}. \tag{5}$$

Proof. By the additive property of the integral $I = \int_0^1 \varphi(x)dx = \sum\limits_{(c_1,\dots,c_m)\in A^m} \int\limits_{\Delta^{Q_2^*}} \varphi(x)dx$.

In view of the equalities

$$\varphi(x = \Delta^{Q_2^*}_{c_1\dots c_m\alpha_1\alpha_2\dots\alpha_n\dots}) = c_1v_1 + \dots + c_mv_m + \varphi_1(x),$$

where $\varphi_1(x) = \frac{\lambda\alpha_1}{2} + \frac{\lambda\alpha_2}{2^2} + \dots = \frac{\lambda\alpha_1}{2} + \frac{1}{2}\varphi_1(\Delta^{Q_2}_{\alpha_2\alpha_3\dots})$, $\Delta^{Q_2}_{i\alpha_2\alpha_3\dots} = iq_{1-i} + q_i\Delta^{Q_2}_{\alpha_2\alpha_3\dots}$, $i = 0, 1$, we calculate the integral I:

$$\int\limits_{\Delta^{Q_2^*}_{c_1\dots c_m}} \varphi(x)dx = \int_0^1 [c_1v_1 + \dots + c_mv_m + \varphi_1(t)]d(B_m + P_mt) =$$

$$= P_m(c_1v_1 + \dots + c_mv_m) + P_m\int_0^1 \varphi_1(t)dt,$$

where

$$B_m \equiv \beta_{c_1 1} + \sum_{k=2}^{m} \beta_{c_k k} \prod_{i=1}^{k-1} q_{c_i i}, \;\; P_m \equiv \prod_{i=1}^{m} q_{c_i i}.$$

By the previous lemma, we have $\int_0^1 \varphi_1(t)dt = \lambda q_1$. Then equality is fulfilled (5). □

5. Range of the functions φ and f. Given the structure of the function f, namely:

$$[0;1] \ni x \leftrightarrow (\alpha_n) \leftrightarrow \varphi(x) \leftrightarrow f = a^{\varphi},$$

we see that the properties of the function f are determined mainly by the properties of the function φ and Q_2^*-representation. Therefore, we begin the study of the range of the function f by studying the range of the function $\varphi(\Delta^{Q_2^*}_{\alpha_1\alpha_2\dots\alpha_n\dots}) = \alpha_1v_1 + \alpha_2v_2 + \dots + \alpha_nv_n + \dots$.

Clearly the range of the function φ is the set $E(v_n)$ of incomplete sums (subsums) of the series $\sum v_n$, namely the set:

$$E(v_n) = \Big\{x : x = \sum_{n\in M\subset\mathbb{N}} v_n, \;\; M \in 2^{\mathbb{N}}\Big\},$$

where M is a family of all subsets of the set of natural numbers, since

$$x = \sum_{n\in M\subset\mathbb{N}} v_n = \sum_{n=1}^{\infty} \varepsilon_n v_n, \;\; \text{where } \varepsilon_n = \begin{cases} 0, & \text{if } n \notin M, \\ 1, & \text{if } n \in M. \end{cases}$$

It is known that the set of incomplete sums of absolutely convergent series is continual and perfect (closed set without isolated points) [3]. It belongs to one of the three topological types [5, 8]:

1) it is a segment or a finite union of segments; 2) it is a nowhere dense set [3];

3) it is bilateral cantorval, namely the union of a continuous nowhere dense set and a countable set of segments, which has an infinite number of adjacent intervals, i.e. it is a set homeomorphic to a set of sums of the series

$$\frac{3}{4}+\frac{2}{4}+\frac{3}{4^2}+\frac{2}{4^2}+...+\frac{3}{4^n}+\frac{2}{4^n}+....$$

For example, a set of incomplete sums of a series $\sum_{n=1}^{\infty}\frac{b}{c^n}$, where $c \in \mathbb{N} \ni b$ and $2 \leq b < c$, is a set of Cantor type having fractal Hausdorff-Besicovith dimension $\log_c b$.

Clearly the range of the function does not depend on properties of the Q_2^*-representation and is determined by the set of incomplete sums of the series, i.e. the series and the number a.

Lemma 6. *If the sequence* (v_n) *has properties:*

1) $v_n > v_{n+1} > 0$, $n = \overline{1, m-1}$;

2) $v_m > r_m \equiv v_{m+1} + v_{m+2} + ...,$

then the set $E(v_n)$ *of incomplete sums of series* $v_1 + v_2 + ... + v_n + ...$ *does not contain any points from the interval* $(r_m; v_m)$.

Proof. We consider arbitrary number x from the set of incomplete sums of series, namely $x = \sum_{n=1}^{\infty} \alpha_n v_n$, where $(\alpha_n) \in L$.

If $\alpha_k = 1$ for some $k \leq m$, then $x \geq v_k \geq v_m$. Therefore, $x \notin (r_m; v_m)$.

If $\alpha_k = 0$ for all $k \leq m$, but $\alpha_k = 1$ for all $k > m$, then $x = v_{m+1} + v_{m+2} + ... = r_m$, and when $\alpha_k = 0$ for all $k \leq m$ and there are $\alpha_{m+i} = 0$ for some $i \in \mathbb{N}$, then $x < r_m$.

Therefore, $x \notin (r_m; v_m)$. □

Theorem 5. *If* $v_{n-1} > v_n \geq r_n$ *for all* $n \in \mathbb{N}$, *and the inequality* $v_n > r_n$ *is fulfilled an infinite number of times, then the range of the function* f *is a perfect nowhere dense set of zero or positive Lebesgue measure, for which the Hausdorff-Besicovith dimension can be equal depending on the series to all values from the segment* $[0; 1]$.

Proof. In view of the monotonicity of the function $y = a^x$ and the fact that it preserves the Hausdorff-Besicovith dimension (see [1, 20]), it is sufficient to prove the theorem for the range E_φ of functions φ.

Under the conditions of the theorem, the function $\varphi(x)$ is increasing. From the inequality $x_1 < x_2$ it follows $\varphi(x_1) < \varphi(x_2)$ and the image of the cylinder $\Delta^{Q_2^*}_{c_1...c_m}$ is a set

$$f(\Delta^{Q_2^*}_{c_1...c_m}) = \{x : x = c_1 v_1 + ... + c_m v_m + \sum_{k=1}^{\infty} \alpha_k v_{m+k}, (\alpha_k) \in L\} \equiv \Delta_{c_1...c_m},$$

and $\Delta_{a_1...a_n} = \Delta_{a_1...a_n 0} \cup \Delta_{a_1...a_n 1}$, $\max \Delta_{a_1...a_n 0} \leq \min \Delta_{a_1...a_n 1}$. If $v_n > r_n$, then

$$(\Delta_{\alpha_1...\alpha_{n-1}0(1)}; \Delta_{\alpha_1...\alpha_{n-1}1(0)}) \cap E_\varphi = \varnothing.$$

The inequality $v_n > r_n$ is fulfilled an infinite number of times. Therefore, by the previous lemma, in each of the segments $[a_1 v_1 + ... + a_m v_m; a_1 v_1 + ... + a_m v_m + r_m]$ there exists an interval that does not contain points of the set E_φ, i.e.E_φ is a nowhere dense set. It remains to give examples of series with properties specified in the theorem.

The series $\lambda + \lambda^2 + \lambda^3 + ... + \lambda^n + ...$ $(0 < \lambda < \frac{1}{2})$ has a set of incomplete sums of zero Lebesgue measure. Its Hausdorff-Besicovith dimension equals $\log_\lambda 2 \in (0; 1)$.

The Hausdorff-Besicovith dimension of the set of incomplete sums of the Engel series $\frac{1}{a_1} + \frac{1}{a_1 a_2} + ... + \frac{1}{a_1 a_2 ... a_n} + ...$, where $a_n \in N$, $a_{n+1} \geq a_n > 2$, equals 0, i.e. the range E_φ of the function is abnormally fractal.

If the conditions of the theorem are fulfiled and $0 < \delta_n \equiv \frac{a_n}{r_n} \to 1$ $(n \to \infty)$, then E_φ has the fractal dimension 1 (see [13]). □

Remark that the sets of incomplete sums of series of absolutely convergent series are often the sets of Cantor type, that are important in theory of differential equations on time scales [2].

6. Variational properties of function f.

Theorem 6. *If $v_n \geq r_n$ for each $n \in \mathbb{N}$, then the function φ is non-decreasing.*

Proof. It is obvious that the function φ is non-decreasing if and only if it is non-decreasing on each of the Q_2^*-cylinders. Consider an arbitrary cylinder $\Delta^{Q_2^*}_{c_1\ldots c_{m-1}}$ and its two points

$$x_1 = \Delta^{Q_2^*}_{c_1\ldots c_{m-1}0\alpha_1\alpha_2\ldots}, \quad x_2 = \Delta^{Q_2^*}_{c_1\ldots c_{m-1}1\beta_1\beta_2\ldots}.$$
$$\varphi(x_2) - \varphi(x_1) = (v_m + \beta_1 v_{m+1} + \beta_2 v_{m+2} + \ldots) - (\alpha_1 v_{m+1} + \alpha_2 v_{m+2} + \ldots) =$$
$$= v_m + v_{m+1}(\beta_1 - \alpha_1) + v_{m+2}(\beta_2 - \alpha_2) + \ldots > v_m - v_{m+1} - v_{m+2} - \ldots \qquad \square$$

Corollary 2. *If for some $m \in \mathbb{N}$ and for all $n \geq m$ one has $v_n \geq r_n$, then the function φ is a function of bounded variation.*

Theorem 7. *In order that the function f is nowhere monotonic, it is necessary and sufficient that the inequality $v_n < r_n$ is satisfied for an infinite number of values n.*

Proof. Necessity. Let f be a nowhere monotonic function. Assume that the inequality $v_n < r_n$ is satisfied only a finite number of times, and $v_n \geq r_n$ for all $n \geq k$.

We consider a cylinder $\Delta^{Q_2^*}_{\underbrace{0\cdots0}_{k}}$ and two arbitrary points $x_1 = \Delta^{Q_2^*}_{\underbrace{0\cdots0}_{k}c_1\ldots c_{m-1}0\beta_1\beta_2\ldots}$, $x_2 = \Delta^{Q_2^*}_{\underbrace{0\cdots0}_{k}c_1\ldots c_{m-1}1\alpha_1\alpha_2\ldots}$ belonging to the cylinder. Since $x_1 < x_2$, and

$$\varphi(x_2) - \varphi(x_1) = (v_{m+k} + \alpha_1 v_{m+k+1} + \alpha_2 v_{m+k+2}\ldots) - (\beta_1 v_{m+k+1} + \beta_2 v_{m+k+2} + \ldots) \geq$$
$$\geq v_{m+k} - (v_{m+k+1} + v_{m+k+2} + \ldots) \geq 0,$$

then the function φ at the cylinder $\Delta^{Q_2^*}_{\underbrace{0\cdots0}_{k}}$ is non-decreasing. If $a > 1$, then such a function is f. If $0 < a < 1$, then the function f at the given cylinder is non-decreasing. In both cases, we obtain a contradiction with the condition that the function f is nowhere monotonic.

Sufficiency. Clearly for $v_n < r_n$ there exists such a natural $s = s(n)$ that the inequality $v_n < v_{n+1} + v_{n_2} + \ldots + v_{n+s}$ is satisfied. It is obvious that f is nowhere monotonic if and only if it is non-monotonic on each of the Q_2^*-cylinders. Consider an arbitrary Q_2^*-cylinder $\Delta^{Q_2^*}_{c_1\ldots c_m}$ and three points belonging to the cylinder

$$x_1 = \Delta^{Q_2}_{c_1\ldots c_m\underbrace{1\cdots1}_{k}(0)}, x_2 = \Delta^{Q_2}_{c_1..c_m\underbrace{1\cdots1}_{k-1}0\underbrace{1\cdots1}_{p}(0)}, x_3 = \Delta^{Q_2}_{c_1..c_m\underbrace{1\cdots1}_{k-1}0\underbrace{1\cdots1}_{s}(0)},$$

such that $v_{m+k} < r_{m+k}$, $s < p$, $v_{m+k} < v_{m+k+1} + v_{m+k+2} + \ldots + v_{m+k+s}$. Then $x_3 < x_2 < x_1$. At the same time $\varphi(x_2) - \varphi(x_3) = v_{m+k+s+1} + v_{m+k+s+2} + \ldots + v_{m+k+s+p} > 0$, $\varphi(x_1) - \varphi(x_2) = v_{m+k} - (v_{m+k+1} + \ldots + v_{m+k+p}) < 0$. Namely, $(\varphi(x_2) - \varphi(x_3))(\varphi(x_1) - \varphi(x_2)) < 0$. It implies that φ is non-monotonic at the cylinder $\Delta^{Q_2^*}_{c_1\ldots c_m}$. Therefore, the function f is also non-monotonic at the cylinder. Hence, the f is nowhere monotonic. $\square$

7. Final remarks. In this paper we consider positive series with monotonic descending sequences of terms. The rejection of these properties (conditions) significantly complicates the object of study and the corresponding tasks.

Under two-symbol g-encoding (g-representation) of numbers of segment $[a;b]$ we mean surjective mapping g from the space L of $0-1$ sequences into a segment $[a;b]$, i.e.

$$L \ni (\alpha_n) \xrightarrow{g} x = g(\alpha_n) \in [a;b].$$

In this paper, we considered a composite function, the argument of which uses one two-symbol representation of numbers in a domain and the internal function φ also uses specified another two-symbol representation of numbers of another set. Herewith the relationship between two encodings of numbers of different sets is established by projecting one representation into another. This view allows us to take a broader look at functions of this type. The following function deserves independent attention

$$f(x = \Delta^g_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \gamma(\varphi(\Delta^q_{\alpha_1\alpha_2\ldots\alpha_n\ldots})),$$

where g and q are two two-symbols representations of numbers, and γ is a function that is not necessarily monotonic and is nowhere monotonic or non-differentiable [19].

It is an interesting object of investigation when the g-representation and the q-representation are not topologically equivalent representations, in particular, when one of them is a G_2-representation [16]. The operator of the left shift of the previous is continuous, however, in the Q_2-representation it is discontinuous.

REFERENCES

1. S. Albeverio, M. Pratsiovytyi, G. Torbin, *Fractal probability distributions and transformations preserving the Hausdorff-Besicovitch dimension*, Ergod. Th. & Dynam. Sys., **24** (2004), 1–16.
2. L. Bourdin, O. Stanzhytskyi, E. Trelat, *Addendum to Pontryagin's maximum principle for dynamic systems on time scales*, Journal of Difference Equations and Applications, **23** (2017), №10, 1760–1763.
3. S. Kakeya, *On the partial sums of an infinite series*, Tohoku Sci Rep., **3** (1914), №4, 159–164.
4. N.O. Korsun, M.V. Ptaysiovytyi, *About the set of incomplete sums of positive seriess with one condition of homogeneity and generalization of the binary representation of numbers*, Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Dragomanova, Ser 1. Fiz.-Mat. Nauky, **10** (2009), №10, 28–39.
5. J.A. Guthrie, I.E. Nymann, *The topological structure of the set of subsums of an infinite series*, Collog. Math., **55** (1988), №2, 323–327.
6. V.P. Markitan, M.V. Pratsiovytyi, I.O. Savchenko, *Superfractality of the set of incomplete sums of one positive series*, Ukr. Mat. Zh., **70** (2018), №10, 1403–1416.
7. G. Marsaglia, *Random variables with independent binary digits*, Ann. Math. Statist., **42** (1971), №2, 1922–1929.
8. J.E. Nymann, R.A. Saenz, *On the paper of Guthrie and Nymann on subsums of infinite series, The topological structure of the set of subsums of an infinite series*, Collog. Math., **68** (1995), 259–264.
9. M. Pratsiovytyi, O. Makarchuk, D. Karvatsky, *Lebesgue structure of asymmetric Bernoulli convolution based on Jacobsthal–Lucas sequence, Random operators and stochastic equations*, Random Oper. Stoch. Equ., **28** (2020), №2, 123–130.
10. R. Salem, *On some singular monotonic functions which are strigly increasing*, Trans. Amer. Math. Soc., **53** (1943), 423–439.
11. Ya. Vynnyshyn, V. Markitan, M. Pratsiovytyi, I. Savchenko, *Positive series whose sets of incomplete sums are Cantorvals*, Proceedings of the Inter. Geometry Center, **12** (2019), №2, 26–42. (in Ukrainian)

12. Bl.Kh. Sendov, *Binary self-similar fractal functions*, Fundamentalnaya i prikladnaya matematika, **5** (1999), №2, 589–595. (in Russian)
13. M.V. Pratsiovytyi, Fractal approach to the study of singular distributions, Kyiv: Nats. Pedagog. Mykhailo Dragomanov Univ., 1998. (in Ukrainian)
14. M.V. Pratsiovytyi, *Random variables with independent Q_2-symbols, Asymptotic Methods in the Study of Stochastic Models*, Inst. Math. Nation. Acad. Sci. Ukraine, Kyiv, 1987, 92–102. (in Russian)
15. M.V. Pratsiovytyi, Ya.V. Goncharenko, S.O. Dmytrenko, I.M. Lysenko, S.P. Ratushniak, *About one class of function with fractal properties*, Bukovynian Mathematical Journal, **6** (2021), №1, 273–283. (in Ukrainian)
16. M. Pratsiovytyi, I. Lysenko, Yu. Maslova, *Group of continuous transformations of real interval preserving tails of G_2-representation of numbers*, Algebra and Discrete Mathematics, **29** (2020), №1, 99–108.
17. M.V. Pratsiovytyi, S.P. Ratushniak, *Independent digits of Q_2-representation of random variable with a given distribution*, Proceedings of the Institute of Mathematics of the National Academy of Sciences of Ukraine, **16** (2019), №3, 79–91. (in Ukrainian)
18. M.V. Pratsiovytyi, S.P. Ratushniak, *Continuous nowhere monotone nondifferentiable function with fractal properties defined in terms Q_2-representation*, Nonlinear oscillations, **23** (2020), №2, 231–252. (in Ukrainian)
19. M. Pratsiovytyi, N. Vasylenko, *Fractal properties of functions defined in terms of Q-representation*, International Journal of Math. Analysis, **7** (2013), №61–67, 3155–3169.
20. M.V. Pratsiovytyi, G.M. Torbin, *Fractal geometry and transformation preserving the Hausdorff-Besicovith dimension*, Dynam. Sys.: Proceedings of the Ukrainian Mathematical Congress, 2001, K.: Inst. Math. Nation. Acad. Sci. Ukraine, 2003, 77–93. (in Ukrainian)
21. G.M. Torbin, M.V. Pratsiovytyi, *Random variables with independent Q_2^*-digits* (in Russian), in: Random Evolutions: Theoretical and Applied Problems, Inst. Math. Natl. Acad. Sci. Ukraine, Kyiv, (1992), 95–104. (in Ukrainian)

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