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BOUNDS ON THE EXTENT OF A TOPOLOGICAL SPACE

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The extent $e(X)$ of a topological space X is a superemum of sizes of closed discrete subspaces of X. Assuming that X belongs to some class of topological spaces, we bound $e(X)$ by other cardinal characteristics of X, for instance Lindelöf number, spread or density.

By a "space" in the present paper we mean a topological space. All spaces considered in the paper are *not* supposed to satisfy any of the separation axioms, if otherwise is not stated. We recall that a family of subsets of a space is discrete if each point of the space has a neighborhood intersecting at most one set of the family. For a cover $\mathcal{U}$ of a space X and a set $A \subset X$ we put

$$\mathcal{S}t(A;\mathcal{U}) = \bigcup\{U \in \mathcal{U} \colon U \cap \mathcal{A} \neq \varnothing\}.$$

We recall the following cardinal characteristics of a space X.

- $w(X) = \min\{|\mathcal{B}| \colon \mathcal{B}$ is a base of the topology of $X\}$ is the *weight* of X;
- $nw(X) = \min\{|\mathcal{N}| \colon \mathcal{N}$ is a network of the topology of $X\}$ is the *network weight* of X;
- $d(X) = \min\{|A| \colon A \subset X,\ \overline{A} = X\}$ is the *density* of X;
- $l(X)$, the *Lindelöf number* of X, is the smallest cardinal κ such that each open cover $\mathcal{U}$ of X has a subcover $\mathcal{V} \subset \mathcal{U}$ of cardinality $|\mathcal{V}| \le \kappa$;
- $s(X) = \sup\{|D| \colon D$ is a discrete subspace of $X\}$ is the *spread* of X;
- $e(X) = \sup\{|D| \colon D$ is a closed discrete subspace of $X\}$ is the *extent* of X;
- $c(X) = \sup\{|\mathcal{U}| \colon \mathcal{U}$ is a disjoint family of non-empty open sets in $X\}$ is the *cellularity* of X.
- $de(X) = \sup\{|\mathcal{A}| \colon \mathcal{A}$ is a discrete family of non-empty subsets in $X\}$ is the *discrete extent* of X;
- $we(X)$, the *weak extent* of X, is the smallest cardinal κ such that for every open cover $\mathcal{U}$ of X there is a subset $A \subset X$ of cardinality $|A| \le \kappa$ such that $\mathcal{S}t(A;\mathcal{U}) = X$;
- $wl(X)$, the *weak Lindelöf number* of X, is the smallest cardinal κ such that for every open cover $\mathcal{U}$ of X there is a subfamily $\mathcal{V} \subset \mathcal{U}$ of cardinality $|\mathcal{V}| \le \kappa$ such that $\bigcup \mathcal{V}$ is dense in X.

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It is easy to see that each space X has

$$e(X) \leq de(X) \leq l(X).$$

If X is a T_1-space then $de(X) = e(X)$. For each space X, $we(X) \leq de(X)$, by Proposition 1.1 from [3], see also Proposition 75 from [19]. For more inequalities between the above cardinal characteristics see the right part of the diagram before Proposition 1.1 from [3].

A space X is *collectively Hausdorff*, if for each discrete family $\mathcal{F}$ of finite subsets of X there is a discrete family $(U_F)_{F\in\mathcal{F}}$ of open sets such that $F \subset U_F$ for all $F \in \mathcal{F}$. A space X is a *developable*, if it has a sequence of open covers $(\mathcal{U}_n)_{n\in\omega}$ such that the family $\{\mathcal{S}t(\{x\};\mathcal{U}_n)\}_{n\in\omega}$ is a neighborhood base at each point $x \in X$. By [10, 4.5], each T_1 regular developable space has a σ-discrete network.

By the proof of Proposition 1.2 from [3],

- if X is a perfectly normal space then $c(X) \leq de(X)$;
- if X is a collectively Hausdorff space then $e(X) = dc(X)$;
- if X is a T_1-space with a σ-discrete network then $e(X) = nw(X)$;
- if X is a developable space then $we(X) = d(X)$.

Now we shall detect spaces X satisfying the equality $l(X) = de(X)$. First we recall the necessary definitions.

A space X is

- *meta-Lindelöf* (resp. *metacompact*) if every open cover $\mathcal{U}$ of X has a point-countable (resp. point-finite) open refinement;
- *submeta-Lindelöf* (resp. *submetacompact*) if for every open cover $\mathcal{U}$ there is a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X refining $\mathcal{U}$ such that for every $x \in X$ there is $n \in \omega$ such that the family $\mathcal{U}_n(x) = \{U \in \mathcal{U}_n : x \in X\}$ is at most countable (resp. finite);
- *weakly submeta-Lindelöf* if for every open cover $\mathcal{U}$ there is a sequence $(\mathcal{U}_n)_{n\in\omega}$ of families of open sets in X such that each family $\mathcal{U}_n$ refines $\mathcal{U}$ and for every $x \in X$ there is $n \in \omega$ such that $1 \leq |\mathcal{U}_n(x)| \leq \omega$;

More information on these covering properties can be found in the survey [5].

A space X is *irreducible*, if each open cover $\mathcal{U}$ of X has a minimal open refinement $\mathcal{V}$, that is for each member V of $\mathcal{V}$, $\mathcal{V}\setminus\{V\}$ does not cover X. It is easy to show that if (iff when X is T_1) the cover $\mathcal{U}$ has this property then there exists a discrete family $\mathcal{A}$ of non-empty subsets of X and a neighborhood $U_A \in \mathcal{U}$ for each $A \in \mathcal{A}$ such that

$$\bigcup_{A\in\mathcal{A}} U_A = X.$$

So $de(X) = l(X)$ for each irreducible space X. A special irreducible space is a *D-space* X, that is a T_1 space such that if given a neighborhood U_x of each $x \in X$ then there is a closed discrete subset D of X such that $\bigcup_{x\in D} U_x = X$. See Theorem 4.1 from [11], for a list of known classes of regular T_1 spaces which are D-spaces. It includes, semistratifiable spaces, $\Sigma^\sharp$-spaces, subspaces of symmetrizable spaces, and spaces with a point-countable (weak) base. Also $e(X) = l(X)$ when X belong to a class of weakly $a\,D$-spaces, considered in [1], which contains all D-spaces.

Mashburn in [17] showed that each submeta-Lindelöf (or weakly $\overline{\delta\theta}$-refinable) T_1 space X is irreducible, so $e(X) = de(X) = l(X)$. Yu and Yun in [23] improved this by showing that any finite T_1 union of submeta-Lindelöf spaces is irreducible. On the other hand, in [17] is noted that it is easy to construct a Lindelöf space which is neither T_1 nor irreducible. Nevertheless, we have the following statement.

Proposition 1. *If a space X is submeta-Lindelöf, then*

$$de(X) \le l(X) \le \omega \cdot de(X).$$

Proof. Since the inequality $de(X) \le l(X)$ is trivial, it suffices to check that $l(X) \le \omega \cdot de(X)$. We lose no generality assuming that the space X is not empty. Fix an open cover $\mathcal{U}$ of X. Since X is submeta-Lindelöf, there exists a sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers of X refining $\mathcal{U}$, such that for any point $x \in X$ there is a number $n \in \omega$ such that the family $\mathcal{U}_n(x) = \{U \in \mathcal{U}_n \colon x \in U\}$ is at most countable. Let $n(x) \in \omega$ be the smallest number with $|\mathcal{U}_{n(x)}(x)| \le \omega$ and put $U(x) = \bigcup \mathcal{U}_{n(x)}(x)$. Fix a well-ordering $\le$ on the set X such that for any points $x, y \in X$ with $n_x < n_y$ we get $x < y$. Let x_0 be the smallest element of the well-ordered set $(X, \le)$. For a non-empty subset $A \subset X$ by $\min(A)$ we shall denote the smallest element of A with respect to the well-order $\le$. For the empty subset $A = \varnothing \subset X$ the point $\min(A)$ is not defined but it will be convenient to define $\min\varnothing = x_0$. By transfinite induction, for every ordinal α let $x_\alpha = \min(X \setminus \bigcup_{\beta<\alpha} U(x_\beta))$. Let λ be the smallest ordinal such that $X = \bigcup_{\alpha<\lambda} U(x_\alpha)$. Observe that for every $\alpha < \beta < \lambda$ we get $x_\alpha < x_\beta$. Indeed, assuming that $x_\beta < x_\alpha$ and taking into account that $x_\beta \notin \bigcup_{\gamma<\alpha} U(x_\gamma)$, we get a contradiction with the minimality of x_α.

We claim that the family of singletons $\mathcal{D} = \big\{\{x_\alpha\} : \alpha < \lambda\big\}$ is discrete in X. Given any point $x \in X$ find the smallest ordinal $\alpha < \lambda$ such that $x \in U(x_\alpha)$. For every $k \le n(x_\alpha)$ consider the open neighborhood $U_k(x) = \bigcup \mathcal{U}_k(x)$ of x and put $O_x = U(x_\alpha) \cap \bigcap_{k\le n} U_k(x)$. We claim that $\{x_\beta\}_{\beta<\lambda} \cap O_x \subset \{x_\alpha\}$. Assuming that this implication does not hold, we can find an ordinal $\beta < \lambda$ such that $\beta \ne \alpha$ and $x_\beta \in O_x$. The choice of the points $x_\gamma \notin U(x_\alpha)$, $\gamma > \alpha$, guarantees that $\beta < \alpha$ and hence $x_\beta < x_\alpha$ and $n(x_\beta) \le n(x_\alpha)$. Then for $k = n(x_\beta)$ the inclusion $x_\beta \in O_x \subset U_k(x)$ implies $x \in U_k(x_\beta) = U(x_\beta)$, which contradicts the choice of α as the smallest ordinal with $x \in U(x_\alpha)$. This contradiction shows that $\{x_\beta\}_{\beta<\lambda} \cap O_x \subset \{x_\alpha\}$ and hence the family $\big\{\{x_\alpha\}\big\}_{\alpha<\lambda}$ is discrete in X and has cardinality $\lambda \le de(X)$. It follows that the subfamily $\mathcal{V} = \bigcup_{\alpha<\lambda} \mathcal{U}_{n(x_\alpha)}(x_\alpha) \subset \mathcal{U}$ is a cover of X of cardinality $|\mathcal{V}| \le \omega \cdot \lambda = \omega \cdot de(X)$, witnessing that $l(X) \le \omega \cdot de(X)$. □

For weakly submeta-Lindelöf spaces we can prove a weaker statement.

Proposition 2. *If a space X is weakly submeta-Lindelöf, then $l(X) \le \omega \cdot s(X)$.*

Proof. Fix an open cover $\mathcal{U}$ of X. Since X is weakly submeta-Lindelöf, there exists a sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of families of open sets refining $\mathcal{U}$, such that for any point $x \in X$ there is a number $n \in \omega$ such that $1 \le |\mathcal{U}_n(x)| \le \omega$. Here $\mathcal{U}_n(x) = \{U \in \mathcal{U}_n : x \in U\}$. Let $n(x) \in \omega$ be the smallest number such that $1 \le |\mathcal{U}_{n(x)}(x)| \le \omega$. For every $n \in \omega$ consider the subset $X_n = \{x \in X : n(x) = n\}$ and let $Y_n \subset X_n$ be a maximal subset of X_n such that $y \notin \bigcup \mathcal{U}_n(x) \ne \varnothing$ for every distinct points $x, y \in Y_n$. It is clear that Y_n is a discrete subspace of X and hence $|Y_n| \le s(X)$. By the maximality of Y_n, we get $X_n \cap \bigcup \mathcal{U}_n \subset \bigcup_{y\in Y_n} (\bigcup \mathcal{U}_n(y))$. Then $\mathcal{V} = \bigcup_{n\in\omega} \bigcup_{y\in Y_n} \mathcal{U}_n(y)$ is a subcover of $\mathcal{U}$ of cardinality $|\mathcal{V}| \le \sum_{n\in\omega} \sum_{y\in Y_n} |\mathcal{U}_n(y)| \le \omega \cdot s(X)$. This witnesses that $l(X) \le \omega \cdot s(X)$. □

According to [11], it is an old and apparently open question whether a countably metacompact, weakly submetacompact T_1 regular space is irreducible (see also [1, Problem 1.18]). This suggests the following question (see also [1, Problem 1.20]).

Question 1. *Whether $e(X) = l(X)$ for a countably metacompact, weakly submetacompact T_1 (and regular) space X?*

For meta-Lindelöf spaces the upper bound $l(X) \le \omega \cdot de(X)$ proved in Proposition 1 can be improved to $l(X) \le \omega \cdot \min\{d(X), de(X)\}$.

Proposition 3. *If a space X is meta-Lindelöf, then*

$$de(X) \le l(X) \le \omega \cdot \min\{d(X), de(X)\}.$$

Proof. By Proposition 1, the meta-Lindelöf space X satisfies the inequality $l(X) \le \omega \cdot de(X)$. So, it suffices to prove that $l(X) \le \omega \cdot d(X)$. Since X is meta-Lindelöf, every open cover $\mathcal{U}$ of X can be refined by a point-countable open cover $\mathcal{V}$. Take any dense subset $D \subset X$ of cardinality $|D| = d(X)$ and observe that $\mathcal{V}' = \{V \in \mathcal{V} : V \cap D \neq \varnothing\}$ is a subcover of $\mathcal{V}$ of cardinality $|\mathcal{V}'| \le \omega \cdot |D| = \omega \cdot d(X)$. For every $V \in \mathcal{V}'$ choose a set $U_V \in \mathcal{U}$ containing V and observe that $\mathcal{U}' = \{U_V : V \in \mathcal{V}'\} \subset \mathcal{U}$ is a subcover of cardinality $|\mathcal{U}'| \le |\mathcal{V}'| \le \omega \cdot d(X)$, witnessing that $l(X) \le \omega \cdot d(X)$. □

Proposition 1 cannot be generalized to weakly submeta-Lindelöf spaces, because there exists a Hausdorff space X which is locally compact, locally countable, separable, submetrizable, σ-discrete (and so weakly submeta-Lindelöf), realcompact, and has $\omega = e(X) < l(X)$, see [8]. Also there is a consistent example of a T_1 normal σ-discrete space X with $e(X) = \omega < l(X)$, see [6].

Example 1. Let $\mathbb{S}$ be *Sorgenfrey line* that is the set $\mathbb{R}$ endowed with a topology generated by a base consisting of half-intervals $[a, b)$, $a < b$. Let $X = \mathbb{S} \times \mathbb{S}$. Then, $c(X) = d(X) = \omega < e(X) = l(X) = \mathfrak{c}$. So, by Proposition 3, the space X is not meta-Lindelöf. On the other hand, the space X is subparacompact, see [15].

Example 2 ([3], Remark 1.3)**.** Let X be the ordinal segment $[0, \omega_1)$ endowed with the order topology. Then X is a normal space with $e(X) = de(X) = \omega < \omega_1 = c(X) = l(X)$. Space X is not weakly submeta-Lindelöf, because each regular countably compact weakly submeta-Lindelöf space is compact, see [22, 6.2].

Example 3. Let X be a Mrówka space, see [21], [9, Exercise 3.6.I.a]. Then X is a Tychonoff non-normal first countable locally compact space, $d(X) = \omega$, but $e(X)$ can be equal to $\mathfrak{c}$.

For every normal T_1 space X and every closed discrete subspace A of X we have $2^{|A|} \le 2^{d(X)}$, see [13]. Thus under $\mathfrak{c} < 2^{\omega_1}$ a normal separable space has countable extent. From the other hand, there are consistent examples of normal spaces X with $d(X) = \omega$ and $e(X) = \mathfrak{c}$, see [14].

In [18] is shown that if a Tychonoff space X has countable weak extent then $e(X)$ can be arbitrarily big, but if X is normal then $e(X) \le \mathfrak{c}$. Moreover, it is not known whether there exists under ZFC a normal space X with $we(X) = \omega < e(X)$, see [4] or [14]. A problem when a space from a special class with countable weak extent has countable extent was also considered in [2].

If X is a Tychonoff space with $wl(X) \le \omega$ then $e(X)$ can be arbitrarily big. Namely, we can put $X = (\beta D \times (\omega+1)) \setminus ((\beta D \setminus D) \times \{\omega\})$, where D is an arbitrarily big discrete space and a set $\omega + 1$ is endowed with the order topology, see Example 4 in [16]. In Example 1.17 of [2] is constructed a Hausdorff space X such that $wl(X) = \omega < we(X)$. By Theorem 1.29 from [2], for any uncountable cardinal κ, a Cantor cube $\{0,1\}^\kappa$ contains a dense subspace X such that $we(X) = \omega$, X contains a dense σ-compact subspace Y (so $wl(X) = \omega$), and $X \setminus Y$ is a closed discrete subset of X of cardinality κ, so $e(X) \ge \kappa$. On the other hand, if X is a T_1 σ-para-Lindelöf space with $wl(X) \le \omega$ then $l(X) \le \omega$, see [12]. Also, similarly to the proof of Basic property 2 from [16] we can show that if X is a paracompact space then $wl(X) = l(X)$.

Under $\mathrm{cf}(\mathfrak{c}) = \mathfrak{c} < 2^{\omega_1}+$ there is no inner model with a measurable cardinal, we have $e(X) \le \omega$ for each separable countably paracompact space X, see [20, Corollary 3.10]. On the other hand, if Y is a subset of $\mathbb{R}$ with $|Y| = \omega_1 < \mathfrak{p}$ then the Moore space $M(Y)$ derived from Y is a separable normal countably paracompact space with uncountable extent [20]. Remark that there is a separable orthocompact countably metacompact space with a closed discrete subset of size $\mathfrak{c}$, see Proposition 4.1 from [20].

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