N. U. Rehman, H. M. Alnoghashi

$\mathscr{T}$-COMMUTING GENERALIZED DERIVATIONS ON IDEALS AND SEMI-PRIME IDEAL-II

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The study's primary purpose is to investigate the $\mathscr{A}/\mathscr{T}$ structure of a quotient ring, where $\mathscr{A}$ is an arbitrary ring and $\mathscr{T}$ is a semi-prime ideal of $\mathscr{A}$. In more details, we look at the differential identities in a semi-prime ideal of an arbitrary ring using $\mathscr{T}$-commuting generalized derivation. The article proves a number of statements. A characteristic representative of these assertions is, for example, the following Theorem 3: Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ,ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions: 1) $\lambda([a,b]) \pm [a,\psi(b)] \in \mathscr{T}$, 2) $\lambda(a\circ b) \pm a\circ\psi(b) \in \mathscr{T}$, $\forall$ $a,b\in\mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.

Furthermore, examples are provided to demonstrate that the constraints placed on the hypothesis of the various theorems were not unnecessary.

1. Introduction. Throughout this paper, $\mathscr{A}$ will represent an associative ring not necessarily to be commutative with center $Z(\mathscr{A})$. The symbols $a\circ b$ and $[a,b]$, where $a,b\in\mathscr{A}$, stand for the anti-commutator $ab+ba$ and commutator $ab-ba$ respectively. An ideal $\mathscr{T}$ is said to be a prime ideal of $\mathscr{A}$ if $\mathscr{T}\neq\mathscr{A}$ and $\forall$ $a,b\in\mathscr{A}$, whenever $a\mathscr{A}b\subseteq\mathscr{T}$ implies $a\in\mathscr{T}$ or $b\in\mathscr{T}$ and $\mathscr{A}$ is a prime ring if $\mathscr{T}=0$ is a prime ideal of $\mathscr{A}$, and $\mathscr{T}$ is a semi-prime ideal if $\mathscr{T}\neq\mathscr{A}$ and $\forall$ $a\in\mathscr{A}$, $a\mathscr{A}a\subseteq\mathscr{T}$ implies $a\in\mathscr{T}$ and $\mathscr{A}$ is a semi-prime ring if $\mathscr{T}=0$ is a semi-prime ideal of $\mathscr{A}$. For any $\mathscr{S}\subseteq\mathscr{A}$ and a ring $\mathscr{D}$, a map $f:\mathscr{D}\to\mathscr{A}$ is called a $\mathscr{S}$-commuting map on $\mathscr{D}$ if $[f(a),a]\in\mathscr{S}$ $\forall$ $a\in\mathscr{D}$. In particular, if $\mathscr{S}=\{0\}$, then f is called a commuting map on $\mathscr{D}$. Note that every commuting map is a $\mathscr{S}$-commuting map (put $\{0\}=\mathscr{S}$). But converse is not true in general (let $\mathscr{S}$ be a set of $\mathscr{A}$ such that it has no zero and $[f(a),a]\in\mathscr{S}$, then f is a $\mathscr{S}$-commuting map, but it is not a commuting map. An additive map $\psi:\mathscr{A}\to\mathscr{A}$ is called a derivation of $\mathscr{A}$ if $\psi(ab)=\psi(a)b+a\psi(b)$ holds $\forall$ $a,b\in\mathscr{A}$. An additive map $\lambda:\mathscr{A}\to\mathscr{A}$ associated with a derivation $\psi:\mathscr{A}\to\mathscr{A}$ is called a generalized derivation of $\mathscr{A}$ if $\lambda(ab)=\lambda(a)b+a\psi(b)$ holds $\forall$ $a,b\in\mathscr{A}$.

During last three decades, many authors have proved a significant amount of results on suitably constrained additive mappings such as automorphisms, derivations, skew derivations etc. acting on appropriate subsets of prime and semi-prime rings. Posner [14] was the first to study centralizing derivation, demonstrating that a prime ring $\mathscr{A}$ that admits a non-zero centralizing derivation is commutative. Bell and Martindale [4] discovered that if $\mathscr{A}$ is a semi-prime ring and $\mathscr{I}$ is a non-zero left ideal of $\mathscr{A}$, $\mathscr{A}$ has a non-zero central ideal if it admits a non-zero derivation ψ such that $\psi(\mathscr{I})=0$ and centralizing on $\mathscr{I}$. Mayne [7] shown that centralizing automorphisms had a similar effect.

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A number of authors have extended Posner and Mayne's theorems in various ways. In the year 1988 [6] Lanski generalizes the result of Posner by considering a derivation ψ such that $[\psi(x), x] \in Z(\mathscr{A})$ for all x in a nonzero Lie ideal of $\mathscr{A}$. Hongan [5] proved that if a 2-torsion free semiprime ring A admits a derivation ψ such that $\psi([a,u]) \pm [a,u] \in Z(\mathscr{A})$ for all $a, u \in \mathscr{A}$, then $\mathscr{A}$ is commutative. In [3] Ashraf and Rehman prove that if $\mathscr{A}$ is a 2-torsion free prime ring and L a nonzero Lie ideal of $\mathscr{A}$ such that $u^2 \in L$ for all $u \in L$ and ψ a derivation which satisfies $\psi(u \circ v) - u \circ v$ for all $u, v \in L$, then $L \subseteq Z(\mathscr{A})$. Later, Quadri [15] has extended the mentioned result by considering a generalized derivation λ acting on a nonzero ideal $\mathscr{T}$ of $\mathscr{A}$ and without 2-torsion freeness hypothesis. Further, in [16] Dhara et al. showed that, a prime ring $\mathscr{A}$ must be commutative if it admits two generalized derivations λ, Θ associated with derivations ψ and ξ respectively and satisfies the properties $\Lambda(x)\Theta(y) \pm \Theta(xy) \pm yx \in Z(\mathscr{A})$ for all $x, y \in \mathscr{T}$, where $\mathscr{T}$ is a nonzero two-sided ideal of $\mathscr{A}$. For more details of such studies we refer the readers to [2], [8], [10], [13], [17], [18] and references therein. One may observe that the main focus of these studies is to indicate how the global structure of a ring is often tightly connected with the behavior of such additive mappings defined on it.

In order to extend the standard theory of "derivations in rings" recently, Almahdi [1] et al. initiated the study of derivations of an arbitrary ring $\mathscr{A}$ satisfying some $\mathscr{T}$-valued conditions, where $\mathscr{T}$ is a prime ideal of $\mathscr{A}$. Specifically, they improved the well-known Posner's Second Theorem as follows: *If $\mathscr{T}$ is a prime ideal of a ring $\mathscr{A}$ and ψ a derivation of $\mathscr{A}$ such that $[[\psi(x), x], y] \in \mathscr{T}$ $\forall\, x, y \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is a commutative ring.* Further Mamouni et al. [10] investigated many $\mathscr{T}$-valued differential identities such as: (i) $[\psi_1(x), \psi_2(y)] \in \mathscr{T}$, (ii) $\psi_1(x) \circ \psi_2(y) \in \mathscr{T}$, (iii) $[\psi_1(x), y] + [x, \psi_2(y)] \in \mathscr{T}$, (iv) $[\psi_1(x), y] + [x, \psi_2(y)] - [x, y] \in \mathscr{T}$, (v) $[\psi_1(x), y] + [x, \psi_2(y)] - [y, \psi_1(x)] \in \mathscr{T}$ for all x, y in a prime ring $\mathscr{A}$ and ψ_1, ψ_2 are the derivations of $\mathscr{A}$. The authors also examined some particular cases of these identities in semi-prime rings. In the successive paper Mamouni et al. [12] extended this theory to the class of generalized derivations and obtained the commutativity of the quotient rings. Some further developments have also been appeared in the direction, for instance see [20]. Very recently, Idrissi and Oukhtite [9] introduced the study of I-centralizing and I-commuting mappings in rings, where I is a nonzero ideal of a ring R. They proved the following: *Let $\mathscr{T}$ be a prime ideal of a ring $\mathscr{A}$ and λ be a generalized derivation of $\mathscr{A}$ associated with a derivation ψ. If λ is $\mathscr{T}$-centralizing, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is a commutative integral domain.* Apart from this, the authors have proved many commutativity theorem in $\mathscr{A}/\mathscr{T}$ and finally discussed some applications of their results.

Lemma 1 ([4])**.** *Suppose $\mathscr{I}$ is an ideal of a semi-prime ring $\mathscr{A}$. If $\mathscr{A}$ admits a non-zero derivation ψ such that $[a, \psi(a)] = 0$ $\forall\, a \in \mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.*

Lemma 2 ([1], Lemma 2.1)**.** *Suppose $\mathscr{T}$ is a prime ideal of a ring $\mathscr{A}$. If $\mathscr{A}$ admits a derivation ψ such that $[a, \psi(a)] \in \mathscr{T}$ $\forall\, a \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

2. The main results. Since every prime ideal is semi-prime but the converse is not true in general, therefore, in view of the above discussion it seems appropriate to examine identities involving derivations in semi-prime ideals rather. Our purpose in this paper is to examine some $\mathscr{T}$-valued differential identities, where $\mathscr{T}$ is a semi-prime ideal of a ring $\mathscr{A}$ and then observe the structural properties of $\mathscr{A}$. We will undertake a novel investigation in this study that is both an extension and a generalization of current literature findings. We will use generalized derivation to look at the differential identities in a semi-prime ideal of an arbitrary ring.

Theorem 1. *Suppose $\mathscr{I}$ is an ideal of a ring $\mathscr{A}$ with $\mathscr{T}$ a semi-prime ideal. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $[\lambda(a), \psi(b)] \pm ba \in \mathscr{T}$,
2. $[\lambda(a), \psi(b)] \pm ab \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.

Proof. (1) Assume that

$$[\lambda(a), \psi(b)] \pm ba \in \mathscr{T} \tag{1}$$

$\forall\ a, b \in \mathscr{I}$. Replacing a by at in (1), where $t \in \mathscr{A}$, we have $[\lambda(at), \psi(b)] \pm bat \in \mathscr{T}$. This implies that $[\lambda(a)t + a\psi(t), \psi(b)] \pm bat \in \mathscr{T}$ that is $[\lambda(a)t, \psi(b)] + [a\psi(t), \psi(b)] \pm bat \in \mathscr{T}$. Hence,

$$([\lambda(a), \psi(b)] \pm ba)t + \lambda(a)[t, \psi(b)] + [a\psi(t), \psi(b)] \in \mathscr{T}. \tag{2}$$

Since $[\lambda(a), \psi(b)] \pm ba \in \mathscr{T}$ (from (1)) and since $t \in \mathscr{A}$ and $\mathscr{T}$ is a prime ideal of $\mathscr{A}$, we get $([\lambda(a), \psi(b)] \pm ba)t \in \mathscr{T}$. Subtracting the last relation from (2), we have

$$\lambda(a)[t, \psi(b)] + [a\psi(t), \psi(b)] \in \mathscr{T}.$$

This implies that

$$\lambda(a)[t, \psi(b)] + [a, \psi(b)]\psi(t) + a[\psi(t), \psi(b)] \in \mathscr{T} \tag{3}$$

$\forall\ a, b \in \mathscr{I}$ and $t \in \mathscr{A}$. Substituting ua for a in (3), where $u \in \mathscr{A}$, we get $\lambda(ua)[t, \psi(b)] + [ua, \psi(b)]\psi(t) + ua[\psi(t), \psi(b)] \in \mathscr{T}$ this implies that

$$\lambda(ua)[t, \psi(b)] + u[a, \psi(b)]\psi(t) + [u, \psi(b)]a\psi(t) + ua[\psi(t), \psi(b)] \in \mathscr{T}.$$

By using the definition of λ in the last expression, we get

$$(\lambda(u)a + u\psi(a))[t, \psi(b)] + u[a, \psi(b)]\psi(t) + [u, \psi(b)]a\psi(t) + ua[\psi(t), \psi(b)] \in \mathscr{T} \tag{4}$$

$\forall\ a, b \in \mathscr{I}$ and $u, t \in \mathscr{A}$. Left multiplying (3) by u, we obtain

$$u\lambda(a)[t, \psi(b)] + ua[\psi(t), \psi(b)] + u[a, \psi(b)]\psi(t) \in \mathscr{T} \tag{5}$$

$\forall\ a, b \in \mathscr{I}$ and $u, t \in \mathscr{A}$. Comparing (4) and (5), this gives

$$(\lambda(u)a + u\psi(a) - u\lambda(a))[t, \psi(b)] + [u, \psi(b)]a\psi(t) \in \mathscr{T} \tag{6}$$

$\forall\ a, b \in \mathscr{I}$ and $u, t \in \mathscr{A}$. Putting $t = \lambda(s)$ and $u = \lambda(c)$ in (6), where $s, c \in \mathscr{I}$, we conclude

$$\begin{aligned}&(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[\lambda(s), \psi(b)] \\ &+[\lambda(c), \psi(b)]a\psi(\lambda(s)) \in \mathscr{T}.\end{aligned} \tag{7}$$

Replacing a by s in (1) and then left multiplying it by $(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))$, we get

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[\lambda(s), \psi(b)] \tag{8}$$

$$\pm(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))bs \in \mathscr{T}.$$

Replacing a by c in (1) and then right multiplying it by $a\psi(\lambda(s))$, we have

$$[\lambda(c), \psi(b)]a\psi(\lambda(s)) \pm bc(a\psi(\lambda(s))) \in \mathscr{T}. \tag{9}$$

Comparing (7), (8) and (9), we find that

$$\mp(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))bs \mp bc(a\psi(\lambda(s))) \in \mathscr{T}.$$

Hence

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))bs + bc(a\psi(\lambda(s))) \in \mathscr{T} \tag{10}$$

$\forall\ a, b, c, s \in \mathscr{I}$. Writing rb instead of b in (10) where $r \in \mathscr{A}$, we get

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))rbs + rbc(a\psi(\lambda(s))) \in \mathscr{T} \tag{11}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $r \in \mathscr{A}$. Left multiplying (10) by r this gives

$$r(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))bs + rbc(a\psi(\lambda(s))) \in \mathscr{T} \tag{12}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $r \in \mathscr{A}$. Comparing (11) and (12), we have

$$[\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a), r]bs \in \mathscr{T} \tag{13}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $r \in \mathscr{A}$. Replacing b by kb in (13), where $k \in \mathscr{A}$, we get

$$[\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a), r]kbs \in \mathscr{T} \tag{14}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $r, k \in \mathscr{A}$. Taking a by ak in (13), we have

$$[\lambda(\lambda(c))ak + \lambda(c)\psi(ak) - \lambda(c)\lambda(ak), r]bs \in \mathscr{T}$$

that is

$$[(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))k, r]bs \in \mathscr{T}$$

hence

$$\begin{aligned}&[\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a), r]kbs\\ &+(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[k, r]bs \in \mathscr{T}.\end{aligned}$$

By using (14) in the last relation, we have

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[k, r]bs \in \mathscr{T}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $r, k \in \mathscr{A}$. Putting k by t and r by $\psi(b)$ in the last expression, where $t \in \mathscr{A}$, we conclude that

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[t, \psi(b)]bs \in \mathscr{T}. \tag{15}$$

Taking u by $\lambda(c)$ in (6)

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[t, \psi(b)] + [\lambda(c), \psi(b)]a\psi(t) \in \mathscr{T}$$

$\forall\ a, b, c \in \mathscr{I}$ and $t \in \mathscr{A}$. Right multiplying the last relation by bs, we get

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[t, \psi(b)]bs + [\lambda(c), \psi(b)]a\psi(t)bs \in \mathscr{T}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $t \in \mathscr{A}$. By using (15) in the last expression, we have

$$[\lambda(c), \psi(b)]a\psi(t)bs \in \mathscr{T} \tag{16}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $t \in \mathscr{A}$. Putting a by c in (1) and then right multiplying (1) by $a\psi(t)bs$, we see that $[\lambda(c), \psi(b)]a\psi(t)bs \pm bca\psi(t)bs \in \mathscr{T}$. By using (16) in the last expression, we get $\pm bca\psi(t)bs \in \mathscr{T}$ and so $bca\psi(t)bs \in \mathscr{T}$. Taking c by s in the last relation, we find that $bsa\psi(t)bs \in \mathscr{T}$. Left multiplying the last expression by $\psi(t)$, we get $(\psi(t)bs)a(\psi(t)bs) \in \mathscr{T}$ and so $\psi(t)bs \in \mathscr{T}$. Putting b by $b\psi(s)$ in the last relation and then right multiplying the last expression by $\psi(s)$ and then subtracting one of them from the other, we have $\psi(t)b[\psi(s), s] \in \mathscr{T}$. Replacing t by s in the last relation, we get $\psi(s)b[\psi(s), s] \in \mathscr{T}$. Taking b by sb in the last expression and then left multiplying the last relation by s and then subtracting one of them from the other, we obtain $[\psi(s), s]b[\psi(s), s] \in \mathscr{T}$ and so $[\psi(s), s] \in \mathscr{T}$ $\forall\ s \in \mathscr{I}$.

(2) We obtain the desired result by employing the same approaches as in the proof of (1). □

By using Lemma 2 and Theorem 1, we easily get the following corollary:

Corollary 1. *Suppose $\mathscr{T}$ is a prime ideal of a ring $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $[\lambda(a), \psi(b)] \pm ba \in \mathscr{T}$,
2. $[\lambda(a), \psi(b)] \pm ab \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.

$\mathscr{A}$ has a non-zero central ideal, according to Lemma 1 and Theorem 1. As a result, we arrive to the following corollary.

Corollary 2. *Suppose $\mathscr{I}$ is an ideal of a semi-prime ring $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $[\lambda(a), \psi(b)] \pm ba = 0$,
2. $[\lambda(a), \psi(b)] \pm ab = 0$,

$\forall\ a, b \in \mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.

Theorem 2. *Suppose $\mathscr{I}$ is an ideal of a ring $\mathscr{A}$ with $\mathscr{T}$ a semi-prime ideal. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda(ab) - \lambda(b)\lambda(a) \in \mathscr{T}$,
2. $\lambda(ab) - \lambda(a)\lambda(b) \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.

Proof. (1) Assume that

$$\lambda(ab) - \lambda(b)\lambda(a) \in \mathscr{T} \tag{17}$$

$\forall\, a, b \in \mathscr{I}$. By using the definition of λ in (17), we obtain

$$\lambda(a)b + a\psi(b) - \lambda(b)\lambda(a) \in \mathscr{T} \tag{18}$$

$\forall\, a, b \in \mathscr{I}$. Writing ab instead of a in (18), we have $\lambda(ab)b + ab\psi(b) - \lambda(b)\lambda(ab) \in \mathscr{T}$ hence $\lambda(ab)b + ab\psi(b) - \lambda(b)\lambda(a)b - \lambda(b)a\psi(b) \in \mathscr{T}$ that is

$$(\lambda(ab) - \lambda(b)\lambda(a))b + ab\psi(b) - \lambda(b)a\psi(b) \in \mathscr{T} \tag{19}$$

$\forall\, a, b \in \mathscr{I}$. Right multiplying (17) by b, we get

$$(\lambda(ab) - \lambda(b)\lambda(a))b \in \mathscr{T} \tag{20}$$

$\forall\, a, b \in \mathscr{I}$. Subtracting (20) from (19), we see that

$$ab\psi(b) - \lambda(b)a\psi(b) \in \mathscr{T} \tag{21}$$

$\forall\, a, b \in \mathscr{I}$. Replacing a by $\lambda(t)a$ in (21), where $t \in \mathscr{A}$, we get

$$\lambda(t)ab\psi(b) - \lambda(b)\lambda(t)a\psi(b) \in \mathscr{T} \tag{22}$$

$\forall\, a, b \in \mathscr{I}$ and $t \in \mathscr{A}$. Left multiplying (21) by $\lambda(t)$, we obtain

$$\lambda(t)ab\psi(b) - \lambda(t)\lambda(b)a\psi(b) \in \mathscr{T} \tag{23}$$

$\forall\, a, b \in \mathscr{I}$ and $t \in \mathscr{A}$. Comparing (22) and (23), this gives $(\lambda(b)\lambda(t) - \lambda(t)\lambda(b))a\psi(b) \in \mathscr{T}$. Putting t by c in lat relation, where $c \in \mathscr{I}$, we get

$$(\lambda(b)\lambda(c) - \lambda(c)\lambda(b))a\psi(b) \in \mathscr{T} \tag{24}$$

$\forall\, a, b, c \in \mathscr{I}$. Replacing a by c in (17) and then right multiplying (17) by $a\psi(b)$, we find that

$$(\lambda(cb) - \lambda(b)\lambda(c))a\psi(b) \in \mathscr{T} \tag{25}$$

$\forall\, a, b, c \in \mathscr{I}$. Taking b by c and a by b in (17) and then right multiplying (17) by $a\psi(b)$, we see that

$$(\lambda(bc) - \lambda(c)\lambda(b))a\psi(b) \in \mathscr{T} \tag{26}$$

$\forall\, a, b, c \in \mathscr{I}$. Subtracting (25) from (26), we get

$$(\lambda(b)\lambda(c) - \lambda(c)\lambda(b))a\psi(b) + (\lambda(bc) - \lambda(cb))a\psi(b) \in \mathscr{T}.$$

By using (24) in the last expression, we have

$$(\lambda(bc) - \lambda(cb))a\psi(b) \in \mathscr{T}.$$

Hence, $(\lambda(bc - cb))a\psi(b) \in \mathscr{T}$ that is

$$\lambda([b, c])a\psi(b) \in \mathscr{T} \tag{27}$$

$\forall\ a, b, c \in \mathscr{I}$. Putting a by ba in (27), we conclude

$$\lambda([b,c])ba\psi(b) \in \mathscr{T} \tag{28}$$

$\forall\ a, b, c \in \mathscr{I}$. Substituting cb for c in (27), we have $\lambda([b, cb])a\psi(b) \in \mathscr{T}$. This implies that $\lambda([b, c]b)a\psi(b) \in \mathscr{T}$. Hence, $\lambda([b, c])ba\psi(b) + [b, c]\psi(b)a\psi(b) \in \mathscr{T}$. By using (28) in the last relation, we get $[b, c]\psi(b)a\psi(b) \in \mathscr{T}$. Taking a by $a[b, c]$ in the last expression, we get $[b, c]\psi(b)a[b, c]\psi(b) \in \mathscr{T}$ and so $[b, c]\psi(b) \in \mathscr{T}$. Writing tc instead of c in the last relation and using it, where $t \in \mathscr{A}$, we obtain $[b, t]c\psi(b) \in \mathscr{T}$. Putting $t = \psi(b)$ in the last expression, we see that

$$[b, \psi(b)]c\psi(b) \in \mathscr{T} \tag{29}$$

$\forall\ b, c \in \mathscr{I}$ and $t \in \mathscr{A}$. Replacing c by cb in (29) and then right multiplying (29) by b and then subtracting one of them from the other, we have $[b, \psi(b)]c[b, \psi(b)] \in \mathscr{T}$ and so $[b, \psi(b)] \in \mathscr{T}$.

(2) Assume that

$$\lambda(ab) - \lambda(a)\lambda(b) \in \mathscr{T} \tag{30}$$

$\forall\ a, b \in \mathscr{I}$. By using the definition of λ in (30), we obtain $\lambda(a)b + a\psi(b) - \lambda(a)\lambda(b) \in \mathscr{T}$ that is

$$\lambda(a)(b - \lambda(b)) + a\psi(b) \in \mathscr{T} \tag{31}$$

$\forall\ a, b \in \mathscr{I}$. Substituting bc for b in (31), where $c \in \mathscr{I}$, we have $\lambda(a)(bc-\lambda(bc))+a\psi(bc) \in \mathscr{T}$. By using the definitions of λ and ψ in the last relation, we get $\lambda(a)(bc - \lambda(b)c - b\psi(c)) + a\psi(b)c + ab\psi(c) \in \mathscr{T}$. That is $(\lambda(a)(b - \lambda(b)) + a\psi(b))c - \lambda(a)b\psi(c) + ab\psi(c) \in \mathscr{T}$. Right multiplying 31) by c then using it in the last expression, we obtain $-\lambda(a)b\psi(c)+ab\psi(c) \in \mathscr{T}$ and so $\lambda(a)b\psi(c) - ab\psi(c) \in \mathscr{T}$ that is

$$(\lambda(a) - a)b\psi(c) \in \mathscr{T} \tag{32}$$

$\forall\ a, b, c \in \mathscr{I}$. Writing ub instead of b in (32), where $u \in \mathscr{I}$, we get

$$(\lambda(a) - a)ub\psi(c) \in \mathscr{T} \tag{33}$$

$\forall\ a, b, c, u \in \mathscr{I}$. Putting a by au in (32), where $u \in \mathscr{I}$, we have $(\lambda(au)-au)b\psi(c) \in \mathscr{T}$. This implies that $(\lambda(a)u + a\psi(u) - au)b\psi(c) \in \mathscr{T}$ that is $(\lambda(a) - a)ub\psi(c) + a\psi(u)b\psi(c) \in \mathscr{T}$. By using (33) in the last relation, we obtain $a\psi(u)b\psi(c) \in \mathscr{T}\ \forall\ a, b, c, u \in \mathscr{I}$. Taking u by c in the last expression, we see that $a\psi(c)b\psi(c) \in \mathscr{T}\ \forall\ a, b, c \in \mathscr{I}$. Replacing b by ba in the last relation, we find that $(a\psi(c))b(a\psi(c)) \in \mathscr{T}$ and so $a\psi(c) \in \mathscr{T}$. Putting a by ac in the last expression and then right multiplying the last relation by c and then subtracting one of them from the other, we get $a[\psi(c), c] \in \mathscr{T}$. Left multiplying the last expression by $[\psi(c), c]$, we have $[\psi(c), c]a[\psi(c), c] \in \mathscr{T}$ and so $[\psi(c), c] \in \mathscr{T}\ \forall\ c \in \mathscr{I}$. □

Corollary 3. *Suppose $\mathscr{T}$ is a prime ideal of a ring $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda(ab) - \lambda(a)\lambda(b) \in \mathscr{T}$,
2. $\lambda(ab) - \lambda(b)\lambda(a) \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{A}$, *then* $\psi(\mathscr{A}) \subseteq \mathscr{T}$ *or* $\mathscr{A}/\mathscr{T}$ *is commutative.*

Corollary 4. *Suppose* $\mathscr{I}$ *is an ideal of a semi-prime ring* $\mathscr{A}$. *If* (λ, ψ) *is a non-zero generalized derivation of* $\mathscr{A}$ *and the derivation satisfies any one of the conditions*

1. $\lambda(ab) = \lambda(a)\lambda(b)$,
2. $\lambda(ab) = \lambda(b)\lambda(a)$,

$\forall\ a, b \in \mathscr{I}$, *then* $\mathscr{A}$ *has a non-zero central ideal.*

Theorem 3. *Let* $\mathscr{A}$ *be a ring with* $\mathscr{T}$ *a semi-prime ideal and* $\mathscr{I}$ *an ideal of* $\mathscr{A}$. *If* (λ, ψ) *is a non-zero generalized derivation of* $\mathscr{A}$ *and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm [a, \psi(b)] \in \mathscr{T}$,
2. $\lambda(a \circ b) \pm a \circ \psi(b) \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{I}$, *then* ψ *is* $\mathscr{T}$*-commuting on* $\mathscr{I}$.

Proof. (1) Assume that

$$\lambda([a, b]) \pm [a, \psi(b)] \in \mathscr{T} \tag{34}$$

$\forall\ a, b \in \mathscr{I}$. Putting $b = a$ in (34), we have $[a, \psi(a)] \in \mathscr{T}$.

(2) Assume that

$$\lambda(a \circ b) \pm a \circ \psi(b) \in \mathscr{T} \tag{35}$$

$\forall\ a, b \in \mathscr{I}$. Writing ab for a by in (35), we have $\lambda(ab \circ b) \pm ab \circ \psi(b) \in \mathscr{T}$. This implies that $\lambda((a \circ b)b) \pm ab \circ \psi(b) \in \mathscr{T}$ that is $\lambda((a \circ b)b) \pm (a \circ \psi(b))b \pm a[b, \psi(b)] \in \mathscr{T}$. By using definition of λ in the last relation, we get $\lambda(a \circ b)b + (a \circ b)\psi(b) \pm (a \circ \psi(b))b \pm a[b, \psi(b)] \in \mathscr{T}$. That is $(\lambda(a \circ b) \pm a \circ \psi(b))b + (a \circ b)\psi(b) \pm a[b, \psi(b)] \in \mathscr{T}$. Right multiplying (35) by b then using it in the last expression, we obtain

$$(a \circ b)\psi(b) \pm a[b, \psi(b)] \in \mathscr{T} \tag{36}$$

$\forall\ a, b \in \mathscr{I}$. Substituting $\psi(b)a$ for a in (36), we see that $(\psi(b)a \circ b)\psi(b) \pm \psi(b)a[b, \psi(b)] \in \mathscr{T}$. This implies that $\psi(b)(a \circ b)\psi(b) - [\psi(b), b]a\psi(b) \pm \psi(b)a[b, \psi(b)] \in \mathscr{T}$. Hence, $\psi(b)((a \circ b)\psi(b) \pm a[b, \psi(b)]) - [\psi(b), b]a\psi(b) \in \mathscr{T}$. Left multiplying (36) by $\psi(b)$ and then using it in the last relation, we find that $-[\psi(b), b]a\psi(b) \in \mathscr{T}$ and so $[\psi(b), b]a\psi(b) \in \mathscr{T}$. Putting a by ab in the last expression and then right multiplying the last relation by b and then subtracting one of them from the other, we get $[\psi(b), b]a[\psi(b), b] \in \mathscr{T}$ and so $[\psi(b), b] \in \mathscr{T}$. □

Corollary 5. *Let* $\mathscr{A}$ *be a ring and* $\mathscr{T}$ *a prime ideal. If* (λ, ψ) *is a non-zero generalized derivation of* $\mathscr{A}$ *and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm [a, \psi(b)] \in \mathscr{T}$,
2. $\lambda(a \circ b) \pm a \circ \psi(b) \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{A}$, *then* $\psi(\mathscr{A}) \subseteq \mathscr{T}$ *or* $\mathscr{A}/\mathscr{T}$ *is commutative.*

Corollary 6. *Let* $\mathscr{A}$ *be a semi-prime ring and* $\mathscr{I}$ *an ideal of* $\mathscr{A}$. *If* (λ, ψ) *is a non-zero generalized derivation of* $\mathscr{A}$ *and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) = \pm[a, \psi(b)]$,

2. $\lambda(a \circ b) = \pm a \circ \psi(b)$,

$\forall\ a, b \in \mathscr{I}$, *then* $\mathscr{A}$ *has a non-zero central ideal.*

Theorem 4. *Let* $\mathscr{A}$ *be a ring with* $\mathscr{T}$ *a semi-prime ideal and* $\mathscr{I}$ *an ideal of* $\mathscr{A}$. *If* (λ, ψ) *is a non-zero generalized derivation of* $\mathscr{A}$ *and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm \lambda(b)a \in \mathscr{T}$,
2. $\lambda([a, b]) \pm \lambda(a)b \in \mathscr{T}$,
3. $\lambda(a \circ b) \pm \lambda(b)a \in \mathscr{T}$,
4. $\lambda(a \circ b) \pm \lambda(a)b \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{I}$, *then* ψ *is* $\mathscr{T}$*-commuting on* $\mathscr{I}$.

Proof. (1) Assume that

$$\lambda([a, b]) \pm \lambda(b)a \in \mathscr{T} \tag{37}$$

$\forall\ a, b \in \mathscr{I}$. Replacing b by ba in (37), we find that $\lambda([a, ba]) \pm \lambda(ba)a \in \mathscr{T}$. This implies that $\lambda([a, b]a) \pm \lambda(ba)a \in \mathscr{T}$. By using the definition of λ in the last expression, we get $\lambda([a, b])a + [a, b]\psi(a) \pm \lambda(b)a^2 \pm b\psi(a)a \in \mathscr{T}$ that is $(\lambda([a, b]) \pm \lambda(b)a)a + [a, b]\psi(a) \pm b\psi(a)a \in \mathscr{T}$. Right multiplying (37) by a and then using it in the last relation, we have

$$[a, b]\psi(a) \pm b\psi(a)a \in \mathscr{T} \tag{38}$$

$\forall\ a, b \in \mathscr{I}$. Taking b by $\psi(a)b$ in (38), we get $[a, \psi(a)b]\psi(a) \pm \psi(a)b\psi(a)a \in \mathscr{T}$ that is

$$\psi(a)[a, b]\psi(a) + [a, \psi(a)]b\psi(a) \pm \psi(a)b\psi(a)a \in \mathscr{T}.$$

This implies that $\psi(a)([a, b]\psi(a) \pm b\psi(a)a) + [a, \psi(a)]b\psi(a) \in \mathscr{T}$. Left multiplying (38) by $\psi(a)$ and then using it in the last expression, we see that

$$[a, \psi(a)]b\psi(a) \in \mathscr{T} \tag{39}$$

$\forall\ a, b \in \mathscr{I}$. Putting b by ba in (39) and then right multiplying (39) by a and then subtracting one of them from the other, we find that $[a, \psi(a)]b[a, \psi(a)] \in \mathscr{T}$ and so $[a, \psi(a)] \in \mathscr{T}$.

(2) We acquire the appropriate outcome by continuing along the same lines with the necessary changes.

(3) Assume that

$$\lambda(a \circ b) \pm \lambda(b)a \in \mathscr{T} \tag{40}$$

$\forall\ a, b \in \mathscr{I}$. Substituting ba for b in (40), we have $\lambda(a \circ ba) \pm \lambda(ba)a \in \mathscr{T}$ that is $\lambda((a \circ b)a) \pm \lambda(ba)a \in \mathscr{T}$. By using definition of λ in the last relation, we get $\lambda(a \circ b)a + (a \circ b)\psi(a) \pm \lambda(b)a^2 \pm b\psi(a)a \in \mathscr{T}$. Hence $(\lambda(a \circ b) \pm \lambda(b)a)a + (a \circ b)\psi(a) \pm b\psi(a)a \in \mathscr{T}$. Right multiplying (40) by a and then using it in the last expression, we obtain

$$(a \circ b)\psi(a) \pm b\psi(a)a \in \mathscr{T} \tag{41}$$

$\forall\ a, b \in \mathscr{I}$. Taking b by $\psi(a)b$ in (41), we get $(a \circ \psi(a)b)\psi(a) \pm \psi(a)b\psi(a)a \in \mathscr{T}$ that is $\psi(a)(a \circ b)\psi(a) + [a, \psi(a)]b\psi(a) \pm \psi(a)b\psi(a)a \in \mathscr{T}$. Hence, $\psi(a)((a \circ b)\psi(a) \pm b\psi(a)a) + [a, \psi(a)]b\psi(a) \in \mathscr{T}$. Left multiplying (41) by $\psi(a)$ and then using it in the last relation, we see that $[a, \psi(a)]b\psi(a) \in \mathscr{T}$. Now, the same as in (39), we get $[a, \psi(a)] \in \mathscr{T}$.

(4) The same as in (3). $\square$

Corollary 7. *Let $\mathscr{A}$ be a ring and $\mathscr{T}$ a prime ideal. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a,b]) \pm \lambda(b)a \in \mathscr{T}$,
2. $\lambda([a,b]) \pm \lambda(a)b \in \mathscr{T}$,
3. $\lambda(a \circ b) \pm \lambda(b)a \in \mathscr{T}$,
4. $\lambda(a \circ b) \pm \lambda(a)b \in \mathscr{T}$,

$\forall\, a, b \in \mathscr{A}$, *then* $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ *is commutative.*

Corollary 8. *Let $\mathscr{A}$ be a semi-prime ring and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a,b]) \pm \lambda(b)a = 0$,
2. $\lambda([a,b]) \pm \lambda(a)b = 0$,
3. $\lambda(a \circ b) \pm \lambda(b)a = 0$,
4. $\lambda(a \circ b) \pm \lambda(a)b = 0$,

$\forall\, a, b \in \mathscr{I}$, *then $\mathscr{A}$ has a non-zero central ideal.*

Theorem 5. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a,b]) \pm ab \in \mathscr{T}$,
2. $\lambda([a,b]) \pm ba \in \mathscr{T}$,
3. $\lambda(a \circ b) \pm ab \in \mathscr{T}$,
4. $\lambda(a \circ b) \pm ba \in \mathscr{T}$,

$\forall\, a, b \in \mathscr{I}$, *then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.*

Proof. (1) Assume that

$$\lambda([a,b]) \pm ab \in \mathscr{T} \tag{42}$$

$\forall\, a, b \in \mathscr{I}$. Writing ba instead of b in (42), $\lambda([a,ba]) \pm aba \in \mathscr{T}$. This implies that $\lambda([a,b]a) \pm aba \in \mathscr{T}$. Hence, $\lambda([a,b])a + [a,b]\psi(a) \pm aba \in \mathscr{T}$. That is $(\lambda([a,b]) \pm ab)a + [a,b]\psi(a) \in \mathscr{T}$. Right multiplying (42) by a then using it in the last expression, we get

$$[a,b]\psi(a) \in \mathscr{T} \tag{43}$$

$\forall\, a, b \in \mathscr{I}$. Substituting $\psi(a)b$ for b in (43), we get $[a, \psi(a)b]\psi(a) \in \mathscr{T}$ that is $\psi(a)[a,b]\psi(a) + [a, \psi(a)]b\psi(a) \in \mathscr{T}$. Left multiplying (43) by $\psi(a)$ then using it in the last relation, we obtain $[a, \psi(a)]b\psi(a) \in \mathscr{T}$. Now, the same as in (39), we get $[a, \psi(a)] \in \mathscr{T}$.

(2) Assume that

$$\lambda([a,b]) \pm ba \in \mathscr{T} \tag{44}$$

$\forall\, a, b \in \mathscr{I}$. Substituting ba for b in (44), we obtain $\lambda([a,ba]) \pm ba^2 \in \mathscr{T}$ that is $\lambda([a,b]a) \pm ba^2 \in \mathscr{T}$. Hence, $\lambda([a,b])a + [a,b]\psi(a) \pm ba^2 \in \mathscr{T}$. This implies that $(\lambda([a,b]) \pm ba)a + [a,b]\psi(a) \in \mathscr{T}$. Right multiplying (44) by a and then using it in the last expression, we obtain $[a,b]\psi(a) \in \mathscr{T}$. Now, the same as in (43), we get $[a, \psi(a)] \in \mathscr{T}$.

(3) Assume that

$$\lambda(a \circ b) \pm ab \in \mathscr{T} \tag{45}$$

$\forall\, a, b \in \mathscr{I}$. Writing ba instead of b in (45), we get $\lambda(a \circ ba) \pm aba \in \mathscr{T}$ that is $\lambda((a \circ b)a) \pm aba \in \mathscr{T}$. Hence, $\lambda(a \circ b)a + (a \circ b)\psi(a) \pm aba \in \mathscr{T}$. This implies that $(\lambda(a \circ b) \pm ab)a + (a \circ b)\psi(a) \in \mathscr{T}$. Right multiplying (45) by a and then using it in the last relation, we get

$$(a \circ b)\psi(a) \in \mathscr{T} \tag{46}$$

$\forall\, a, b \in \mathscr{I}$. Substituting $\psi(a)b$ for b in (46), we obtain $(a \circ \psi(a)b)\psi(a) \in \mathscr{T}$ that is $\psi(a)(a \circ b)\psi(a) + [a, \psi(a)]b\psi(a) \in \mathscr{T}$. Left multiplying (46) by $\psi(a)$ then using it in the last expression, we obtain $[a, \psi(a)]b\psi(a) \in \mathscr{T}$. Now, the same as in (39), we get $[a, \psi(a)] \in \mathscr{T}$.

(4) Assume that

$$\lambda(a \circ b) \pm ba \in \mathscr{T} \tag{47}$$

$\forall\, a, b \in \mathscr{I}$. Replacing b by ba in (47), we obtain $\lambda(a \circ ba) \pm ba^2 \in \mathscr{T}$ that is $\lambda((a \circ b)a) \pm ba^2 \in \mathscr{T}$. Hence, $\lambda(a \circ b)a + (a \circ b)\psi(a) \pm ba^2 \in \mathscr{T}$. This implies that $(\lambda(a \circ b) \pm ba)a + (a \circ b)\psi(a) \in \mathscr{T}$. Right multiplying (47) by a then using it in the last relation, we see that $(a \circ b)\psi(a) \in \mathscr{T}$. Now, the same as in (46), we get $[a, \psi(a)] \in \mathscr{T}$. □

Corollary 9. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a prime ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm ab \in \mathscr{T}\ \forall\, a, b \in \mathscr{A}$,
2. $\lambda([a, b]) \pm ba \in \mathscr{T}\ \forall\, a, b \in \mathscr{A}$,
3. $\lambda(a \circ b) \pm ab \in \mathscr{T}\ \forall\, a, b \in \mathscr{A}$,
4. $\lambda(a \circ b) \pm ba \in \mathscr{T}\ \forall\, a, b \in \mathscr{A}$,

then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.

Corollary 10. *Let $\mathscr{A}$ be a semi-prime ring and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm ab = 0\ \forall\, a, b \in \mathscr{I}$,
2. $\lambda([a, b]) \pm ba = 0\ \forall\, a, b \in \mathscr{I}$,
3. $\lambda(a \circ b) \pm ab = 0\ \forall\, a, b \in \mathscr{I}$,
4. $\lambda(a \circ b) \pm ba = 0\ \forall\, a, b \in \mathscr{I}$,

then $\mathscr{A}$ has a non-zero central ideal.

Theorem 6. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm (a \circ b) \in \mathscr{T}$,
2. $\lambda([a, b]) \in \mathscr{T}$,

$\forall\, a, b \in \mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.

proof. (1) Assume that

$$\lambda([a,b]) \pm (a \circ b) \in \mathscr{T} \tag{48}$$

$\forall\ a, b \in \mathscr{I}$. Substituting ba for b in (48), we obtain $\lambda([a, ba]) \pm (a \circ ba) \in \mathscr{T}$ that is $\lambda([a,b]a) \pm (a \circ b)a \in \mathscr{T}$. Hence, $\lambda([a,b])a + [a,b]\psi(a) \pm (a \circ b)a \in \mathscr{T}$. This implies that $(\lambda([a,b]) \pm (a \circ b))a + [a,b]\psi(a) \in \mathscr{T}$. Right multiplying (48) by a and then using it in the last expression, we get $[a,b]\psi(a) \in \mathscr{T}$. Now, the same as in (43), we get $[a, \psi(a)] \in \mathscr{T}$.

(2) the proof is follows as (1). □

Corollary 11. *Let $\mathscr{A}$ be a ring and $\mathscr{T}$ a prime ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a,b]) \pm (a \circ b) \in \mathscr{T}$,
2. $\lambda([a,b]) \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.

Corollary 12. *Let $\mathscr{A}$ be a semi-prime ring and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a,b]) = \pm(a \circ b)$,
2. $\lambda([a,b]) = 0$,

$\forall\ a, b \in \mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.

Now we present an example which prove that the primeness of above corollaries is essential.

Example 1. Let $\mathscr{A} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$, $\mathscr{T} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$. Define additive maps λ and ψ of $\mathscr{A}$ as follows:
$\lambda = \psi \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Here, λ is a non-zero generalized derivation associated with a derivation ψ. The fact that $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathscr{A} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \subseteq \mathscr{T}$ implies that $\mathscr{T}$ is not a prime ideal. Also, we have $\psi(\mathscr{A}) \nsubseteq \mathscr{T}$ and $\mathscr{A}/\mathscr{T}$ is not commutative. Here, we see that (λ, ψ) satisfies the following conditions: (i) $\lambda(ab) \pm \lambda(b)\lambda(a) \in \mathscr{T}$, (ii) $\lambda(ab) \pm \lambda(a)\lambda(b) \in \mathscr{T}$, (iii) $\lambda([a,b]) \pm [a, \psi(b)] \in \mathscr{T}$, (iv) $\lambda(a \circ b) \pm (a \circ \psi(b)) \in \mathscr{T}$, (v) $\lambda([a,b]) \pm \lambda(b)a \in \mathscr{T}$, (vi) $\lambda([a,b]) \pm (\lambda(a)b \in \mathscr{T}$, (vii) $\lambda(a \circ b) \pm \lambda(b)a \in \mathscr{T}$, $(viii)$ $\lambda(a \circ b) \pm \lambda(a)b \in \mathscr{T}$, and (ix) $\lambda([a,b]) \in \mathscr{T}\ \forall\ a, b \in \mathscr{A}$. The hypothesis of primeness in the various corollaries is not superfluous.

Conclusion. In this article, the main focus has been to develop the relationship between the structure of the semiprime ring $\mathscr{A}/\mathscr{T}$ and the behavior of generalized derivations defined on $\mathscr{A}$ that satisfy certain $\mathscr{T}$ valued identities over $\mathscr{A}$. Further an investigation, the $\mathscr{A}/\mathscr{T}$ structure of quotient ring, where $\mathscr{A}$ is an arbitrary ring and $\mathscr{T}$ is a semiprime ideal on some additive mappings defined on $\mathscr{A}$ and some applications of their results.

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Department of Mathematics, Aligarh Muslim University
Aligarh-202002, India
rehman100@gmail.com
nu.rehman.mm@amu.ac.in
halnoghashi@gmail.com

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