N. U. Rehman, H. M. Alnoghashi

$\mathscr{T}$-COMMUTING GENERALIZED DERIVATIONS ON IDEALS AND SEMI-PRIME IDEAL-II

N. U. Rehman, H. M. Alnoghashi. *$\mathscr{T}$-commuting generalized derivations on ideals and semi-prime ideal-II*, Mat. Stud. **57** (2022), 98–110.

The study's primary purpose is to investigate the $\mathscr{A}/\mathscr{T}$ structure of a quotient ring, where $\mathscr{A}$ is an arbitrary ring and $\mathscr{T}$ is a semi-prime ideal of $\mathscr{A}$. In more details, we look at the differential identities in a semi-prime ideal of an arbitrary ring using $\mathscr{T}$-commuting generalized derivation. We prove a number of statements. A characteristic representative of these assertions is, for example, the following Theorem 3: Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions: 1) $\lambda([a,b]) \pm [a, \psi(b)] \in \mathscr{T}$, 2) $\lambda(a \circ b) \pm a \circ \psi(b) \in \mathscr{T}$, $\forall\, a, b \in \mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.

Furthermore, examples are provided to demonstrate that the constraints placed on the hypothesis of the various theorems were not unnecessary.

1. Introduction. Throughout this paper, $\mathscr{A}$ will represent an associative ring not necessarily to be commutative with center $Z(\mathscr{A})$. The symbols $a \circ b$ and $[a, b]$, where $a, b \in \mathscr{A}$, stand for the anti-commutator $ab + ba$ and commutator $ab - ba$, respectively. An ideal $\mathscr{T}$ is said to be a prime ideal of $\mathscr{A}$ if $\mathscr{T} \neq \mathscr{A}$ and $\forall\, a, b \in \mathscr{A}$, whenever $a\mathscr{A}b \subseteq \mathscr{T}$ implies $a \in \mathscr{T}$ or $b \in \mathscr{T}$ and $\mathscr{A}$ is a prime ring if $\mathscr{T} = 0$ is a prime ideal of $\mathscr{A}$, and $\mathscr{T}$ is a semi-prime ideal if $\mathscr{T} \neq \mathscr{A}$ and $\forall\, a \in \mathscr{A}$, $a\mathscr{A}a \subseteq \mathscr{T}$ implies $a \in \mathscr{T}$ and $\mathscr{A}$ is a semi-prime ring if $\mathscr{T} = 0$ is a semi-prime ideal of $\mathscr{A}$. For any $\mathscr{S} \subseteq \mathscr{A}$ and a ring $\mathscr{D}$, a map $f\colon \mathscr{D} \to \mathscr{A}$ is called a $\mathscr{S}$-commuting map on $\mathscr{D}$ if $[f(a), a] \in \mathscr{S}$ $\forall\, a \in \mathscr{D}$. In particular, if $\mathscr{S} = \{0\}$, then f is called a commuting map on $\mathscr{D}$. Note that every commuting map is a $\mathscr{S}$-commuting map (put $\{0\} = \mathscr{S}$). But converse is not true in general (let $\mathscr{S}$ be a set of $\mathscr{A}$ such that it has no zero and $[f(a), a] \in \mathscr{S}$, then f is a $\mathscr{S}$-commuting map, but it is not a commuting map. An additive map $\psi\colon \mathscr{A} \to \mathscr{A}$ is called a derivation of $\mathscr{A}$ if $\psi(ab) = \psi(a)b + a\psi(b)$ holds $\forall$ $a, b \in \mathscr{A}$. An additive map $\lambda\colon \mathscr{A} \to \mathscr{A}$ associated with a derivation $\psi\colon \mathscr{A} \to \mathscr{A}$ is called a generalized derivation of $\mathscr{A}$ if $\lambda(ab) = \lambda(a)b + a\psi(b)$ holds $\forall\, a, b \in \mathscr{A}$.

During last three decades, many authors have proved a significant amount of results on suitably constrained additive mappings such as automorphisms, derivations, skew derivations etc. acting on appropriate subsets of prime and semi-prime rings. Posner [14] was the first to study centralizing derivation, demonstrating that a prime ring $\mathscr{A}$ that admits a non-zero centralizing derivation is commutative. Bell and Martindale [4] discovered that if $\mathscr{A}$ is a semi-prime ring and $\mathscr{I}$ is a non-zero left ideal of $\mathscr{A}$, $\mathscr{A}$ has a non-zero central ideal if it admits a non-zero derivation ψ such that $\psi(\mathscr{I}) = 0$ and centralizing on $\mathscr{I}$. Mayne [7] shown that centralizing automorphisms had a similar effect.

2010 *Mathematics Subject Classification:* 16N60, 16W20, 16W25.
Keywords: semi-prime ideal; generalized derivations; commutativity.
doi:10.30970/ms.57.1.98-110

© N. U. Rehman, H. M. Alnoghashi, 2022

A number of authors have extended Posner and Mayne's theorems in various ways. In 1988 Lanski [6] generalizes the result of Posner by considering a derivation ψ such that $[\psi(x), x] \in Z(\mathscr{A})$ for all x in a nonzero Lie ideal of $\mathscr{A}$. Hongan [5] proved that if a 2-torsion free semiprime ring A admits a derivation ψ such that $\psi([a, u]) \pm [a, u] \in Z(\mathscr{A})$ for all $a, u \in \mathscr{A}$, then $\mathscr{A}$ is commutative. In [3] Ashraf and Rehman prove that if $\mathscr{A}$ is a 2-torsion free prime ring and L a nonzero Lie ideal of $\mathscr{A}$ such that $u^2 \in L$ for all $u \in L$ and ψ a derivation which satisfies $\psi(u \circ v) - u \circ v$ for all $u, v \in L$, then $L \subseteq Z(\mathscr{A})$. Later, Quadri [15] has extended the mentioned result by considering a generalized derivation λ acting on a nonzero ideal $\mathscr{T}$ of $\mathscr{A}$ and without 2-torsion freeness hypothesis. Further, in [16] Dhara et al. showed that, a prime ring $\mathscr{A}$ must be commutative if it admits two generalized derivations λ, Θ associated with derivations ψ and ξ respectively and satisfies the properties $\Lambda(x)\Theta(y) \pm \Theta(xy) \pm yx \in Z(\mathscr{A})$ for all $x, y \in \mathscr{T}$, where $\mathscr{T}$ is a nonzero two-sided ideal of $\mathscr{A}$. For more details of such studies we refer the readers to [2], [8], [10], [13], [17], [18] and references therein. One may observe that the main focus of these studies is to indicate how the global structure of a ring is often tightly connected with the behavior of such additive mappings defined on it.

In order to extend the standard theory of "derivations in rings" recently, Almahdi [1] et al. initiated the study of derivations of an arbitrary ring $\mathscr{A}$ satisfying some $\mathscr{T}$-valued conditions, where $\mathscr{T}$ is a prime ideal of $\mathscr{A}$. Specifically, they improved the well-known Posner's Second Theorem as follows: *If $\mathscr{T}$ is a prime ideal of a ring $\mathscr{A}$ and ψ a derivation of $\mathscr{A}$ such that $[[\psi(x), x], y] \in \mathscr{T}$ $\forall$ $x, y \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is a commutative ring.* Further Mamouni et al. [10] investigated many $\mathscr{T}$-valued differential identities such as: (i) $[\psi_1(x), \psi_2(y)] \in \mathscr{T}$, (ii) $\psi_1(x) \circ \psi_2(y) \in \mathscr{T}$, (iii) $[\psi_1(x), y] + [x, \psi_2(y)] \in \mathscr{T}$, (iv) $[\psi_1(x), y] + [x, \psi_2(y)] - [x, y] \in \mathscr{T}$, (v) $[\psi_1(x), y] + [x, \psi_2(y)] - [y, \psi_1(x)] \in \mathscr{T}$ for all x, y in a prime ring $\mathscr{A}$ and ψ_1, ψ_2 are the derivations of $\mathscr{A}$. The authors also examined some particular cases of these identities in semi-prime rings. In the successive paper Mamouni et al. [12] extended this theory to the class of generalized derivations and obtained the commutativity of the quotient rings. Some further developments have also been appeared in the direction, for instance see [20]. Very recently, Idrissi and Oukhtite [9] introduced the study of I-centralizing and I-commuting mappings in rings, where I is a nonzero ideal of a ring R. They proved the following: *Let $\mathscr{T}$ be a prime ideal of a ring $\mathscr{A}$ and λ be a generalized derivation of $\mathscr{A}$ associated with a derivation ψ. If λ is $\mathscr{T}$-centralizing, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is a commutative integral domain.* Apart from this, the authors have proved many commutativity theorem in $\mathscr{A}/\mathscr{T}$ and finally discussed some applications of their results.

Lemma 1 ([4])**.** *Suppose $\mathscr{I}$ is an ideal of a semi-prime ring $\mathscr{A}$. If $\mathscr{A}$ admits a non-zero derivation ψ such that $[a, \psi(a)] = 0$ $\forall$ $a \in \mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.*

Lemma 2 ([1], Lemma 2.1)**.** *Suppose $\mathscr{T}$ is a prime ideal of a ring $\mathscr{A}$. If $\mathscr{A}$ admits a derivation ψ such that $[a, \psi(a)] \in \mathscr{T}$ $\forall$ $a \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

2. The main results. Since every prime ideal is semi-prime but the converse is not true in general, therefore, in view of the above discussion it seems appropriate to examine identities involving derivations in semi-prime ideals rather. Our purpose in this paper is to examine some $\mathscr{T}$-valued differential identities, where $\mathscr{T}$ is a semi-prime ideal of a ring $\mathscr{A}$ and then observe the structural properties of $\mathscr{A}$. We will undertake a novel investigation in this study that is both an extension and a generalization of current literature findings. We will use generalized derivation to look at the differential identities in a semi-prime ideal of an arbitrary ring.

Theorem 1. *Suppose $\mathscr{I}$ is an ideal of a ring $\mathscr{A}$ with $\mathscr{T}$ a semi-prime ideal. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $[\lambda(a), \psi(b)] \pm ba \in \mathscr{T}$,
2. $[\lambda(a), \psi(b)] \pm ab \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.

Proof. (1) Assume that

$$[\lambda(a), \psi(b)] \pm ba \in \mathscr{T} \tag{1}$$

$\forall\ a, b \in \mathscr{I}$. Replacing a by at in (1), where $t \in \mathscr{A}$, we have $[\lambda(at), \psi(b)] \pm bat \in \mathscr{T}$. This implies that $[\lambda(a)t + a\psi(t), \psi(b)] \pm bat \in \mathscr{T}$ that is $[\lambda(a)t, \psi(b)] + [a\psi(t), \psi(b)] \pm bat \in \mathscr{T}$. Hence,

$$([\lambda(a), \psi(b)] \pm ba)t + \lambda(a)[t, \psi(b)] + [a\psi(t), \psi(b)] \in \mathscr{T}. \tag{2}$$

Since $[\lambda(a), \psi(b)] \pm ba \in \mathscr{T}$ (from (1)) and since $t \in \mathscr{A}$ and $\mathscr{T}$ is a prime ideal of $\mathscr{A}$, we get $([\lambda(a), \psi(b)] \pm ba)t \in \mathscr{T}$. Subtracting the last relation from (2), we have

$$\lambda(a)[t, \psi(b)] + [a\psi(t), \psi(b)] \in \mathscr{T}.$$

This implies that

$$\lambda(a)[t, \psi(b)] + [a, \psi(b)]\psi(t) + a[\psi(t), \psi(b)] \in \mathscr{T} \tag{3}$$

$\forall\ a, b \in \mathscr{I}$ and $t \in \mathscr{A}$. Substituting ua for a in (3), where $u \in \mathscr{A}$, we deduce $\lambda(ua)[t, \psi(b)] + [ua, \psi(b)]\psi(t) + ua[\psi(t), \psi(b)] \in \mathscr{T}$ this implies that

$$\lambda(ua)[t, \psi(b)] + u[a, \psi(b)]\psi(t) + [u, \psi(b)]a\psi(t) + ua[\psi(t), \psi(b)] \in \mathscr{T}.$$

By using the definition of λ in the last expression, we get

$$(\lambda(u)a + u\psi(a))[t, \psi(b)] + u[a, \psi(b)]\psi(t) + [u, \psi(b)]a\psi(t) + ua[\psi(t), \psi(b)] \in \mathscr{T} \tag{4}$$

$\forall\ a, b \in \mathscr{I}$ and $u, t \in \mathscr{A}$. Left multiplying (3) by u, we obtain

$$u\lambda(a)[t, \psi(b)] + ua[\psi(t), \psi(b)] + u[a, \psi(b)]\psi(t) \in \mathscr{T} \tag{5}$$

$\forall\ a, b \in \mathscr{I}$ and $u, t \in \mathscr{A}$. Comparing (4) and (5), this gives

$$(\lambda(u)a + u\psi(a) - u\lambda(a))[t, \psi(b)] + [u, \psi(b)]a\psi(t) \in \mathscr{T} \tag{6}$$

$\forall\ a, b \in \mathscr{I}$ and $u, t \in \mathscr{A}$. Putting $t = \lambda(s)$ and $u = \lambda(c)$ in (6), where $s, c \in \mathscr{I}$, we conclude

$$\begin{aligned}&(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[\lambda(s), \psi(b)] \\ &+[\lambda(c), \psi(b)]a\psi(\lambda(s)) \in \mathscr{T}.\end{aligned} \tag{7}$$

Replacing a by s in (1) and then left multiplying it by $(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))$, we get

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[\lambda(s), \psi(b)] \tag{8}$$

$$\pm(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))bs \in \mathscr{T}.$$

Replacing a by c in (1) and then right multiplying it by $a\psi(\lambda(s))$, we have

$$[\lambda(c), \psi(b)]a\psi(\lambda(s)) \pm bc(a\psi(\lambda(s))) \in \mathscr{T}. \tag{9}$$

Comparing (7), (8) and (9), we find that

$$\mp(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))bs \mp bc(a\psi(\lambda(s))) \in \mathscr{T}.$$

Hence

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))bs + bc(a\psi(\lambda(s))) \in \mathscr{T} \tag{10}$$

$\forall\ a, b, c, s \in \mathscr{I}$. Writing rb instead of b in (10) where $r \in \mathscr{A}$, we get

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))rbs + rbc(a\psi(\lambda(s))) \in \mathscr{T} \tag{11}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $r \in \mathscr{A}$. Left multiplying (10) by r this gives

$$r(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))bs + rbc(a\psi(\lambda(s))) \in \mathscr{T} \tag{12}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $r \in \mathscr{A}$. Comparing (11) and (12), we have

$$[\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a), r]bs \in \mathscr{T} \tag{13}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $r \in \mathscr{A}$. Replacing b by kb in (13), where $k \in \mathscr{A}$, we get

$$[\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a), r]kbs \in \mathscr{T} \tag{14}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $r, k \in \mathscr{A}$. Taking a by ak in (13), we have

$$[\lambda(\lambda(c))ak + \lambda(c)\psi(ak) - \lambda(c)\lambda(ak), r]bs \in \mathscr{T}$$

that is

$$[(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))k, r]bs \in \mathscr{T}$$

hence

$$[\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a), r]kbs$$
$$+(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[k, r]bs \in \mathscr{T}.$$

By using (14) in the last relation, we have

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[k, r]bs \in \mathscr{T}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $r, k \in \mathscr{A}$. Putting k by t and r by $\psi(b)$ in the last expression, where $t \in \mathscr{A}$, we conclude that

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[t, \psi(b)]bs \in \mathscr{T}. \tag{15}$$

Taking u by $\lambda(c)$ in (6)

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[t, \psi(b)] + [\lambda(c), \psi(b)]a\psi(t) \in \mathscr{T}$$

$\forall\ a, b, c \in \mathscr{I}$ and $t \in \mathscr{A}$. Right multiplying the last relation by bs, we get

$$(\lambda(\lambda(c))a + \lambda(c)\psi(a) - \lambda(c)\lambda(a))[t, \psi(b)]bs + [\lambda(c), \psi(b)]a\psi(t)bs \in \mathscr{T}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $t \in \mathscr{A}$. By using (15) in the last expression, we have

$$[\lambda(c), \psi(b)]a\psi(t)bs \in \mathscr{T} \tag{16}$$

$\forall\ a, b, c, s \in \mathscr{I}$ and $t \in \mathscr{A}$. Putting a by c in (1) and then right multiplying (1) by $a\psi(t)bs$, we see that $[\lambda(c), \psi(b)]a\psi(t)bs \pm bca\psi(t)bs \in \mathscr{T}$. By using (16) in the last expression, we get $\pm bca\psi(t)bs \in \mathscr{T}$ and so $bca\psi(t)bs \in \mathscr{T}$. Taking c by s in the last relation, we find that $bsa\psi(t)bs \in \mathscr{T}$. Left multiplying the last expression by $\psi(t)$, we get $(\psi(t)bs)a(\psi(t)bs) \in \mathscr{T}$ and so $\psi(t)bs \in \mathscr{T}$. Putting b by $b\psi(s)$ in the last relation and then right multiplying the last expression by $\psi(s)$ and then subtracting one of them from the other, we have $\psi(t)b[\psi(s), s] \in \mathscr{T}$. Replacing t by s in the last relation, we get $\psi(s)b[\psi(s), s] \in \mathscr{T}$. Taking b by sb in the last expression and then left multiplying the last relation by s and then subtracting one of them from the other, we obtain $[\psi(s), s]b[\psi(s), s] \in \mathscr{T}$ and so $[\psi(s), s] \in \mathscr{T}$ $\forall\ s \in \mathscr{I}$.

(2) We obtain the desired result by employing the same approaches as in the proof of (1). □

By using Lemma 2 and Theorem 1, we easily get the following corollary:

Corollary 1. *Suppose $\mathscr{T}$ is a prime ideal of a ring $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $[\lambda(a), \psi(b)] \pm ba \in \mathscr{T}$,
2. $[\lambda(a), \psi(b)] \pm ab \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.

$\mathscr{A}$ has a non-zero central ideal, according to Lemma 1 and Theorem 1. As a result, we arrive to the following corollary.

Corollary 2. *Suppose $\mathscr{I}$ is an ideal of a semi-prime ring $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $[\lambda(a), \psi(b)] \pm ba = 0$,
2. $[\lambda(a), \psi(b)] \pm ab = 0$,

$\forall\ a, b \in \mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.

Theorem 2. *Suppose $\mathscr{I}$ is an ideal of a ring $\mathscr{A}$ with $\mathscr{T}$ a semi-prime ideal. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda(ab) - \lambda(b)\lambda(a) \in \mathscr{T}$,
2. $\lambda(ab) - \lambda(a)\lambda(b) \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.

Proof. (1) Assume that

$$\lambda(ab) - \lambda(b)\lambda(a) \in \mathscr{T} \tag{17}$$

$\forall\ a, b \in \mathscr{I}$. By using the definition of λ in (17), we obtain

$$\lambda(a)b + a\psi(b) - \lambda(b)\lambda(a) \in \mathscr{T} \tag{18}$$

$\forall\ a, b \in \mathscr{I}$. Writing ab instead of a in (18), we have $\lambda(ab)b + ab\psi(b) - \lambda(b)\lambda(ab) \in \mathscr{T}$ hence $\lambda(ab)b + ab\psi(b) - \lambda(b)\lambda(a)b - \lambda(b)a\psi(b) \in \mathscr{T}$ that is

$$(\lambda(ab) - \lambda(b)\lambda(a))b + ab\psi(b) - \lambda(b)a\psi(b) \in \mathscr{T} \tag{19}$$

$\forall\ a, b \in \mathscr{I}$. Right multiplying (17) by b, we get

$$(\lambda(ab) - \lambda(b)\lambda(a))b \in \mathscr{T} \tag{20}$$

$\forall\ a, b \in \mathscr{I}$. Subtracting (20) from (19), we see that

$$ab\psi(b) - \lambda(b)a\psi(b) \in \mathscr{T} \tag{21}$$

$\forall\ a, b \in \mathscr{I}$. Replacing a by $\lambda(t)a$ in (21), where $t \in \mathscr{A}$, we get

$$\lambda(t)ab\psi(b) - \lambda(b)\lambda(t)a\psi(b) \in \mathscr{T} \tag{22}$$

$\forall\ a, b \in \mathscr{I}$ and $t \in \mathscr{A}$. Left multiplying (21) by $\lambda(t)$, we obtain

$$\lambda(t)ab\psi(b) - \lambda(t)\lambda(b)a\psi(b) \in \mathscr{T} \tag{23}$$

$\forall\ a, b \in \mathscr{I}$ and $t \in \mathscr{A}$. Comparing (22) and (23), this gives $(\lambda(b)\lambda(t) - \lambda(t)\lambda(b))a\psi(b) \in \mathscr{T}$. Putting t by c in lat relation, where $c \in \mathscr{I}$, we get

$$(\lambda(b)\lambda(c) - \lambda(c)\lambda(b))a\psi(b) \in \mathscr{T} \tag{24}$$

$\forall\ a, b, c \in \mathscr{I}$. Replacing a by c in (17) and then right multiplying (17) by $a\psi(b)$, we find that

$$(\lambda(cb) - \lambda(b)\lambda(c))a\psi(b) \in \mathscr{T} \tag{25}$$

$\forall\ a, b, c \in \mathscr{I}$. Taking b by c and a by b in (17) and then right multiplying (17) by $a\psi(b)$, we see that

$$(\lambda(bc) - \lambda(c)\lambda(b))a\psi(b) \in \mathscr{T} \tag{26}$$

$\forall\ a, b, c \in \mathscr{I}$. Subtracting (25) from (26), we get

$$(\lambda(b)\lambda(c) - \lambda(c)\lambda(b))a\psi(b) + (\lambda(bc) - \lambda(cb))a\psi(b) \in \mathscr{T}.$$

By using (24) in the last expression, we have

$$(\lambda(bc) - \lambda(cb))a\psi(b) \in \mathscr{T}.$$

Hence, $(\lambda(bc - cb))a\psi(b) \in \mathscr{T}$ that is

$$\lambda([b, c])a\psi(b) \in \mathscr{T} \tag{27}$$

$\forall\ a, b, c \in \mathscr{I}$. Putting a by ba in (27), we conclude

$$\lambda([b,c])ba\psi(b) \in \mathscr{T} \tag{28}$$

$\forall\ a, b, c \in \mathscr{I}$. Substituting cb for c in (27), we have $\lambda([b, cb])a\psi(b) \in \mathscr{T}$. This implies that $\lambda([b,c]b)a\psi(b) \in \mathscr{T}$. Hence, $\lambda([b,c])ba\psi(b) + [b,c]\psi(b)a\psi(b) \in \mathscr{T}$. By using (28) in the last relation, we get $[b,c]\psi(b)a\psi(b) \in \mathscr{T}$. Taking a by $a[b,c]$ in the last expression, we get $[b,c]\psi(b)a[b,c]\psi(b) \in \mathscr{T}$ and so $[b,c]\psi(b) \in \mathscr{T}$. Writing tc instead of c in the last relation and using it, where $t \in \mathscr{A}$, we obtain $[b,t]c\psi(b) \in \mathscr{T}$. Putting $t = \psi(b)$ in the last expression, we see that

$$[b, \psi(b)]c\psi(b) \in \mathscr{T} \tag{29}$$

$\forall\ b, c \in \mathscr{I}$ and $t \in \mathscr{A}$. Replacing c by cb in (29) and then right multiplying (29) by b and then subtracting one of them from the other, we have $[b,\psi(b)]c[b,\psi(b)] \in \mathscr{T}$ and so $[b,\psi(b)] \in \mathscr{T}$.

(2) Assume that

$$\lambda(ab) - \lambda(a)\lambda(b) \in \mathscr{T} \tag{30}$$

$\forall\ a, b \in \mathscr{I}$. By using the definition of λ in (30), we obtain $\lambda(a)b + a\psi(b) - \lambda(a)\lambda(b) \in \mathscr{T}$ that is

$$\lambda(a)(b - \lambda(b)) + a\psi(b) \in \mathscr{T} \tag{31}$$

$\forall\ a, b \in \mathscr{I}$. Substituting bc for b in (31), where $c \in \mathscr{I}$, we have $\lambda(a)(bc-\lambda(bc))+a\psi(bc) \in \mathscr{T}$. By using the definitions of λ and ψ in the last relation, we get $\lambda(a)(bc - \lambda(b)c - b\psi(c)) + a\psi(b)c + ab\psi(c) \in \mathscr{T}$. That is $(\lambda(a)(b - \lambda(b)) + a\psi(b))c - \lambda(a)b\psi(c) + ab\psi(c) \in \mathscr{T}$. Right multiplying 31) by c then using it in the last expression, we obtain $-\lambda(a)b\psi(c)+ab\psi(c) \in \mathscr{T}$ and so $\lambda(a)b\psi(c) - ab\psi(c) \in \mathscr{T}$ that is

$$(\lambda(a) - a)b\psi(c) \in \mathscr{T} \tag{32}$$

$\forall\ a, b, c \in \mathscr{I}$. Writing ub instead of b in (32), where $u \in \mathscr{I}$, we get

$$(\lambda(a) - a)ub\psi(c) \in \mathscr{T} \tag{33}$$

$\forall\ a, b, c, u \in \mathscr{I}$. Putting a by au in (32), where $u \in \mathscr{I}$, we have $(\lambda(au)-au)b\psi(c) \in \mathscr{T}$. This implies that $(\lambda(a)u + a\psi(u) - au)b\psi(c) \in \mathscr{T}$ that is $(\lambda(a) - a)ub\psi(c) + a\psi(u)b\psi(c) \in \mathscr{T}$. By using (33) in the last relation, we obtain $a\psi(u)b\psi(c) \in \mathscr{T}\ \forall\ a, b, c, u \in \mathscr{I}$. Taking u by c in the last expression, we see that $a\psi(c)b\psi(c) \in \mathscr{T}\ \forall\ a, b, c \in \mathscr{I}$. Replacing b by ba in the last relation, we find that $(a\psi(c))b(a\psi(c)) \in \mathscr{T}$ and so $a\psi(c) \in \mathscr{T}$. Putting a by ac in the last expression and then right multiplying the last relation by c and then subtracting one of them from the other, we get $a[\psi(c), c] \in \mathscr{T}$. Left multiplying the last expression by $[\psi(c), c]$, we have $[\psi(c), c]a[\psi(c), c] \in \mathscr{T}$ and so $[\psi(c), c] \in \mathscr{T}\ \forall\ c \in \mathscr{I}$. □

Corollary 3. *Suppose $\mathscr{T}$ is a prime ideal of a ring $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda(ab) - \lambda(a)\lambda(b) \in \mathscr{T}$,
2. $\lambda(ab) - \lambda(b)\lambda(a) \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{A}$, *then* $\psi(\mathscr{A}) \subseteq \mathscr{T}$ *or* $\mathscr{A}/\mathscr{T}$ *is commutative.*

Corollary 4. *Suppose* $\mathscr{I}$ *is an ideal of a semi-prime ring* $\mathscr{A}$. *If* (λ, ψ) *is a non-zero generalized derivation of* $\mathscr{A}$ *and the derivation satisfies any one of the conditions*

1. $\lambda(ab) = \lambda(a)\lambda(b)$,
2. $\lambda(ab) = \lambda(b)\lambda(a)$,

$\forall\ a, b \in \mathscr{I}$, *then* $\mathscr{A}$ *has a non-zero central ideal.*

Theorem 3. *Let* $\mathscr{A}$ *be a ring with* $\mathscr{T}$ *a semi-prime ideal and* $\mathscr{I}$ *an ideal of* $\mathscr{A}$. *If* (λ, ψ) *is a non-zero generalized derivation of* $\mathscr{A}$ *and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm [a, \psi(b)] \in \mathscr{T}$,
2. $\lambda(a \circ b) \pm a \circ \psi(b) \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{I}$, *then* ψ *is* $\mathscr{T}$*-commuting on* $\mathscr{I}$.

Proof. (1) Assume that

$$\lambda([a, b]) \pm [a, \psi(b)] \in \mathscr{T} \tag{34}$$

$\forall\ a, b \in \mathscr{I}$. Putting $b = a$ in (34), we have $[a, \psi(a)] \in \mathscr{T}$.

(2) Assume that

$$\lambda(a \circ b) \pm a \circ \psi(b) \in \mathscr{T} \tag{35}$$

$\forall\ a, b \in \mathscr{I}$. Writing ab for a by in (35), we have $\lambda(ab \circ b) \pm ab \circ \psi(b) \in \mathscr{T}$. This implies that $\lambda((a \circ b)b) \pm ab \circ \psi(b) \in \mathscr{T}$ that is $\lambda((a \circ b)b) \pm (a \circ \psi(b))b \pm a[b, \psi(b)] \in \mathscr{T}$. By using definition of λ in the last relation, we get $\lambda(a \circ b)b + (a \circ b)\psi(b) \pm (a \circ \psi(b))b \pm a[b, \psi(b)] \in \mathscr{T}$. That is $(\lambda(a \circ b) \pm a \circ \psi(b))b + (a \circ b)\psi(b) \pm a[b, \psi(b)] \in \mathscr{T}$. Right multiplying (35) by b then using it in the last expression, we obtain

$$(a \circ b)\psi(b) \pm a[b, \psi(b)] \in \mathscr{T} \tag{36}$$

$\forall\ a, b \in \mathscr{I}$. Substituting $\psi(b)a$ for a in (36), we see that $(\psi(b)a \circ b)\psi(b) \pm \psi(b)a[b, \psi(b)] \in \mathscr{T}$. This implies that $\psi(b)(a \circ b)\psi(b) - [\psi(b), b]a\psi(b) \pm \psi(b)a[b, \psi(b)] \in \mathscr{T}$. Hence, $\psi(b)((a \circ b)\psi(b) \pm a[b, \psi(b)]) - [\psi(b), b]a\psi(b) \in \mathscr{T}$. Left multiplying (36) by $\psi(b)$ and then using it in the last relation, we find that $-[\psi(b), b]a\psi(b) \in \mathscr{T}$ and so $[\psi(b), b]a\psi(b) \in \mathscr{T}$. Putting a by ab in the last expression and then right multiplying the last relation by b and then subtracting one of them from the other, we get $[\psi(b), b]a[\psi(b), b] \in \mathscr{T}$ and so $[\psi(b), b] \in \mathscr{T}$. □

Corollary 5. *Let* $\mathscr{A}$ *be a ring and* $\mathscr{T}$ *a prime ideal. If* (λ, ψ) *is a non-zero generalized derivation of* $\mathscr{A}$ *and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm [a, \psi(b)] \in \mathscr{T}$,
2. $\lambda(a \circ b) \pm a \circ \psi(b) \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{A}$, *then* $\psi(\mathscr{A}) \subseteq \mathscr{T}$ *or* $\mathscr{A}/\mathscr{T}$ *is commutative.*

Corollary 6. *Let* $\mathscr{A}$ *be a semi-prime ring and* $\mathscr{I}$ *an ideal of* $\mathscr{A}$. *If* (λ, ψ) *is a non-zero generalized derivation of* $\mathscr{A}$ *and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) = \pm[a, \psi(b)]$,

2. $\lambda(a \circ b) = \pm a \circ \psi(b)$,

$\forall\ a, b \in \mathscr{I}$, *then* $\mathscr{A}$ *has a non-zero central ideal.*

Theorem 4. *Let* $\mathscr{A}$ *be a ring with* $\mathscr{T}$ *a semi-prime ideal and* $\mathscr{I}$ *an ideal of* $\mathscr{A}$. *If* (λ, ψ) *is a non-zero generalized derivation of* $\mathscr{A}$ *and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm \lambda(b)a \in \mathscr{T}$,
2. $\lambda([a, b]) \pm \lambda(a)b \in \mathscr{T}$,
3. $\lambda(a \circ b) \pm \lambda(b)a \in \mathscr{T}$,
4. $\lambda(a \circ b) \pm \lambda(a)b \in \mathscr{T}$,

$\forall\ a, b \in \mathscr{I}$, *then* ψ *is* $\mathscr{T}$*-commuting on* $\mathscr{I}$.

Proof. (1) Assume that

$$\lambda([a, b]) \pm \lambda(b)a \in \mathscr{T} \tag{37}$$

$\forall\ a, b \in \mathscr{I}$. Replacing b by ba in (37), we find that $\lambda([a, ba]) \pm \lambda(ba)a \in \mathscr{T}$. This implies that $\lambda([a, b]a) \pm \lambda(ba)a \in \mathscr{T}$. By using the definition of λ in the last expression, we get $\lambda([a, b])a + [a, b]\psi(a) \pm \lambda(b)a^2 \pm b\psi(a)a \in \mathscr{T}$ that is $(\lambda([a, b]) \pm \lambda(b)a)a + [a, b]\psi(a) \pm b\psi(a)a \in \mathscr{T}$. Right multiplying (37) by a and then using it in the last relation, we have

$$[a, b]\psi(a) \pm b\psi(a)a \in \mathscr{T} \tag{38}$$

$\forall\ a, b \in \mathscr{I}$. Taking b by $\psi(a)b$ in (38), we get $[a, \psi(a)b]\psi(a) \pm \psi(a)b\psi(a)a \in \mathscr{T}$ that is

$$\psi(a)[a, b]\psi(a) + [a, \psi(a)]b\psi(a) \pm \psi(a)b\psi(a)a \in \mathscr{T}.$$

This implies that $\psi(a)([a, b]\psi(a) \pm b\psi(a)a) + [a, \psi(a)]b\psi(a) \in \mathscr{T}$. Left multiplying (38) by $\psi(a)$ and then using it in the last expression, we see that

$$[a, \psi(a)]b\psi(a) \in \mathscr{T} \tag{39}$$

$\forall\ a, b \in \mathscr{I}$. Putting b by ba in (39) and then right multiplying (39) by a and then subtracting one of them from the other, we find that $[a, \psi(a)]b[a, \psi(a)] \in \mathscr{T}$ and so $[a, \psi(a)] \in \mathscr{T}$.

(2) We acquire the appropriate outcome by continuing along the same lines with the necessary changes.

(3) Assume that

$$\lambda(a \circ b) \pm \lambda(b)a \in \mathscr{T} \tag{40}$$

$\forall\ a, b \in \mathscr{I}$. Substituting ba for b in (40), we have $\lambda(a \circ ba) \pm \lambda(ba)a \in \mathscr{T}$ that is $\lambda((a \circ b)a) \pm \lambda(ba)a \in \mathscr{T}$. By using definition of λ in the last relation, we get $\lambda(a \circ b)a + (a \circ b)\psi(a) \pm \lambda(b)a^2 \pm b\psi(a)a \in \mathscr{T}$. Hence $(\lambda(a \circ b) \pm \lambda(b)a)a + (a \circ b)\psi(a) \pm b\psi(a)a \in \mathscr{T}$. Right multiplying (40) by a and then using it in the last expression, we obtain

$$(a \circ b)\psi(a) \pm b\psi(a)a \in \mathscr{T} \tag{41}$$

$\forall\ a, b \in \mathscr{I}$. Taking b by $\psi(a)b$ in (41), we get $(a \circ \psi(a)b)\psi(a) \pm \psi(a)b\psi(a)a \in \mathscr{T}$ that is $\psi(a)(a \circ b)\psi(a) + [a, \psi(a)]b\psi(a) \pm \psi(a)b\psi(a)a \in \mathscr{T}$. Hence, $\psi(a)((a \circ b)\psi(a) \pm b\psi(a)a) + [a, \psi(a)]b\psi(a) \in \mathscr{T}$. Left multiplying (41) by $\psi(a)$ and then using it in the last relation, we see that $[a, \psi(a)]b\psi(a) \in \mathscr{T}$. Now, the same as in (39), we get $[a, \psi(a)] \in \mathscr{T}$.

(4) The same as in (3). □

Corollary 7. *Let $\mathscr{A}$ be a ring and $\mathscr{T}$ a prime ideal. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm \lambda(b)a \in \mathscr{T}$,
2. $\lambda([a, b]) \pm \lambda(a)b \in \mathscr{T}$,
3. $\lambda(a \circ b) \pm \lambda(b)a \in \mathscr{T}$,
4. $\lambda(a \circ b) \pm \lambda(a)b \in \mathscr{T}$,

$\forall\, a, b \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.

Corollary 8. *Let $\mathscr{A}$ be a semi-prime ring and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm \lambda(b)a = 0$,
2. $\lambda([a, b]) \pm \lambda(a)b = 0$,
3. $\lambda(a \circ b) \pm \lambda(b)a = 0$,
4. $\lambda(a \circ b) \pm \lambda(a)b = 0$,

$\forall\, a, b \in \mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.

Theorem 5. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm ab \in \mathscr{T}$,
2. $\lambda([a, b]) \pm ba \in \mathscr{T}$,
3. $\lambda(a \circ b) \pm ab \in \mathscr{T}$,
4. $\lambda(a \circ b) \pm ba \in \mathscr{T}$,

$\forall\, a, b \in \mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.

Proof. (1) Assume that

$$\lambda([a, b]) \pm ab \in \mathscr{T} \tag{42}$$

$\forall\, a, b \in \mathscr{I}$. Writing ba instead of b in (42), $\lambda([a, ba]) \pm aba \in \mathscr{T}$. This implies that $\lambda([a, b]a) \pm aba \in \mathscr{T}$. Hence, $\lambda([a, b])a + [a, b]\psi(a) \pm aba \in \mathscr{T}$. That is $(\lambda([a, b]) \pm ab)a + [a, b]\psi(a) \in \mathscr{T}$. Right multiplying (42) by a then using it in the last expression, we get

$$[a, b]\psi(a) \in \mathscr{T} \tag{43}$$

$\forall\, a, b \in \mathscr{I}$. Substituting $\psi(a)b$ for b in (43), we get $[a, \psi(a)b]\psi(a) \in \mathscr{T}$ that is $\psi(a)[a, b]\psi(a) + [a, \psi(a)]b\psi(a) \in \mathscr{T}$. Left multiplying (43) by $\psi(a)$ then using it in the last relation, we obtain $[a, \psi(a)]b\psi(a) \in \mathscr{T}$. Now, the same as in (39), we get $[a, \psi(a)] \in \mathscr{T}$.

(2) Assume that

$$\lambda([a, b]) \pm ba \in \mathscr{T} \tag{44}$$

$\forall a, b \in \mathscr{I}$. Substituting ba for b in (44), we obtain $\lambda([a, ba]) \pm ba^2 \in \mathscr{T}$ that is $\lambda([a, b]a) \pm ba^2 \in \mathscr{T}$. Hence, $\lambda([a, b])a + [a, b]\psi(a) \pm ba^2 \in \mathscr{T}$. This implies that $(\lambda([a, b]) \pm ba)a + [a, b]\psi(a) \in \mathscr{T}$. Right multiplying (44) by a and then using it in the last expression, we obtain $[a, b]\psi(a) \in \mathscr{T}$. Now, the same as in (43), we get $[a, \psi(a)] \in \mathscr{T}$.

(3) Assume that

$$\lambda(a \circ b) \pm ab \in \mathscr{T} \tag{45}$$

$\forall\, a, b \in \mathscr{I}$. Writing ba instead of b in (45), we get $\lambda(a\circ ba)\pm aba \in \mathscr{T}$ that is $\lambda((a\circ b)a)\pm aba \in \mathscr{T}$. Hence, $\lambda(a\circ b)a+(a\circ b)\psi(a)\pm aba \in \mathscr{T}$. This implies that $(\lambda(a\circ b)\pm ab)a+(a\circ b)\psi(a) \in \mathscr{T}$. Right multiplying (45) by a and then using it in the last relation, we get

$$(a \circ b)\psi(a) \in \mathscr{T} \tag{46}$$

$\forall\, a, b \in \mathscr{I}$. Substituting $\psi(a)b$ for b in (46), we obtain $(a \circ \psi(a)b)\psi(a) \in \mathscr{T}$ that is $\psi(a)(a \circ b)\psi(a)+[a, \psi(a)]b\psi(a) \in \mathscr{T}$. Left multiplying (46) by $\psi(a)$ then using it in the last expression, we obtain $[a, \psi(a)]b\psi(a) \in \mathscr{T}$. Now, the same as in (39), we get $[a, \psi(a)] \in \mathscr{T}$.

(4) Assume that

$$\lambda(a \circ b) \pm ba \in \mathscr{T} \tag{47}$$

$\forall\, a, b \in \mathscr{I}$. Replacing b by ba in (47), we obtain $\lambda(a\circ ba)\pm ba^2 \in \mathscr{T}$ that is $\lambda((a\circ b)a)\pm ba^2 \in \mathscr{T}$. Hence, $\lambda(a\circ b)a+(a\circ b)\psi(a)\pm ba^2 \in \mathscr{T}$. This implies that $(\lambda(a\circ b)\pm ba)a+(a\circ b)\psi(a) \in \mathscr{T}$. Right multiplying (47) by a then using it in the last relation, we see that $(a \circ b)\psi(a) \in \mathscr{T}$. Now, the same as in (46), we get $[a, \psi(a)] \in \mathscr{T}$. □

Corollary 9. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a prime ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm ab \in \mathscr{T}\ \forall\, a, b \in \mathscr{A}$,
2. $\lambda([a, b]) \pm ba \in \mathscr{T}\ \forall\, a, b \in \mathscr{A}$,
3. $\lambda(a \circ b) \pm ab \in \mathscr{T}\ \forall\, a, b \in \mathscr{A}$,
4. $\lambda(a \circ b) \pm ba \in \mathscr{T}\ \forall\, a, b \in \mathscr{A}$,

then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.

Corollary 10. *Let $\mathscr{A}$ be a semi-prime ring and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm ab = 0\ \forall\, a, b \in \mathscr{I}$,
2. $\lambda([a, b]) \pm ba = 0\ \forall\, a, b \in \mathscr{I}$,
3. $\lambda(a \circ b) \pm ab = 0\ \forall\, a, b \in \mathscr{I}$,
4. $\lambda(a \circ b) \pm ba = 0\ \forall\, a, b \in \mathscr{I}$,

then $\mathscr{A}$ has a non-zero central ideal.

Theorem 6. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ, ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a, b]) \pm (a \circ b) \in \mathscr{T}$,
2. $\lambda([a, b]) \in \mathscr{T}$,

$\forall\, a, b \in \mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.

Proof. (1) Assume that

$$\lambda([a,b]) \pm (a \circ b) \in \mathscr{T} \tag{48}$$

$\forall\ a,b \in \mathscr{I}$. Substituting ba for b in (48), we obtain $\lambda([a,ba]) \pm (a \circ ba) \in \mathscr{T}$ that is $\lambda([a,b]a) \pm (a \circ b)a \in \mathscr{T}$. Hence, $\lambda([a,b])a + [a,b]\psi(a) \pm (a \circ b)a \in \mathscr{T}$. This implies that $(\lambda([a,b]) \pm (a \circ b))a + [a,b]\psi(a) \in \mathscr{T}$. Right multiplying (48) by a and then using it in the last expression, we get $[a,b]\psi(a) \in \mathscr{T}$. Now, the same as in (43), we get $[a,\psi(a)] \in \mathscr{T}$.

(2) the proof is follows as (1). □

Corollary 11. *Let $\mathscr{A}$ be a ring and $\mathscr{T}$ a prime ideal of $\mathscr{A}$. If (λ,ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a,b]) \pm (a \circ b) \in \mathscr{T}$,
2. $\lambda([a,b]) \in \mathscr{T}$,

$\forall\ a,b \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.

Corollary 12. *Let $\mathscr{A}$ be a semi-prime ring and $\mathscr{I}$ an ideal of $\mathscr{A}$. If (λ,ψ) is a non-zero generalized derivation of $\mathscr{A}$ and the derivation satisfies any one of the conditions*

1. $\lambda([a,b]) = \pm(a \circ b)$,
2. $\lambda([a,b]) = 0$,

$\forall\ a,b \in \mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.

Now we present an example which prove that the primeness of above corollaries is essential.

Example 1. Let $\mathscr{A} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a,b,c \in \mathbb{Z} \right\}$, $\mathscr{T} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$. Define additive maps λ and ψ of $\mathscr{A}$ as follows:
$\lambda = \psi \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Here, λ is a non-zero generalized derivation associated with a derivation ψ. The fact that $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathscr{A} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \subseteq \mathscr{T}$ implies that $\mathscr{T}$ is not a prime ideal. Also, we have $\psi(\mathscr{A}) \nsubseteq \mathscr{T}$ and $\mathscr{A}/\mathscr{T}$ is not commutative. Here, we see that (λ,ψ) satisfies the following conditions: (i) $\lambda(ab) \pm \lambda(b)\lambda(a) \in \mathscr{T}$, (ii) $\lambda(ab) \pm \lambda(a)\lambda(b) \in \mathscr{T}$, (iii) $\lambda([a,b]) \pm [a,\psi(b)] \in \mathscr{T}$, (iv) $\lambda(a \circ b) \pm (a \circ \psi(b)) \in \mathscr{T}$, (v) $\lambda([a,b]) \pm \lambda(b)a \in \mathscr{T}$, (vi) $\lambda([a,b]) \pm (\lambda(a)b \in \mathscr{T}$, (vii) $\lambda(a \circ b) \pm \lambda(b)a \in \mathscr{T}$, $(viii)$ $\lambda(a \circ b) \pm \lambda(a)b \in \mathscr{T}$, and (ix) $\lambda([a,b]) \in \mathscr{T}$ $\forall\ a,b \in \mathscr{A}$. The hypothesis of primeness in the various corollaries is not superfluous.

Conclusion. In this paper, the main focus is to develop the relationship between the structure of the semiprime ring $\mathscr{A}/\mathscr{T}$ and the behavior of generalized derivations defined on $\mathscr{A}$ that satisfy certain $\mathscr{T}$ valued identities over $\mathscr{A}$. Further an investigation, the $\mathscr{A}/\mathscr{T}$ structure of quotient ring, where $\mathscr{A}$ is an arbitrary ring and $\mathscr{T}$ is a semiprime ideal on some additive mappings defined on $\mathscr{A}$ and some applications of their results.

Acknowledgement. The authors are greatly indebted to the referee for his/her constructive comments and suggestions, which improves the quality of the paper. For the first author, this research is supported by the Council of Scientific and Industrial Research (CSIR-HRDG), India, Grant No. 25(0306)/20/EMR-II.

REFERENCES

1. F.A.A. Almahdi, A. Mamouni, M. Tamekkante, *A generalization of Posner's theorem on derivations in rings*, Indian J. Pure Appl. Math., **51** (2020), №1, 187–194.
2. M. Ashraf, A. Ali, S. Ali, *Some commtativity theorems for rings with generalized derivations*, Southeast Asian Bull. Math., **31** (2007), 415–421.
3. M. Ashraf, N. Rehman, *On commutativity of rings with dervations*, Results Math., **42** (2002), №1–2, 3–8.
4. H.E. Bell, W.S. Martindale III, *Centralizing mappings of semiprime rings*, Canad. Math. Bull., **30** (1987), №1, 92–101.
5. M. Hongan, *A note on semiprime rings with derivations*, Internat. J. Math. Math. Sci., **20** (1997), №2, 413–415.
6. C. Lanski, *Differential identities, Lie ideals and Posner's theorems*, Pacific J. Math., **134** (1988), №2, 275–297.
7. J. Mayne, *Centralizing automorphisms of prime rings*, Canad. Math. Bull., **19** (1976), 113–115.
8. H.El Mir, A. Mamouni, L. Oukhtite, *Commutativity with algebraic identities involving prime ideals*, Communications of the Korean Mathematical Society, **35**, №3, (2020), 723–731.
9. M.A. Idrissi, L. Oukhtite, *Structure of a quotient ring R/P with generalized derivations acting on prime ideal P and some applications*, Indian J. Pure Appl. Math. doi: 10.1007/s13226-021-00173-x
10. A. Mamouni, B. Nejjar, L. Oukhtite, *Differential identities on prime rings with involution*, J. Algebra & Appl., **17** (2018), №9, 11 p.
11. A. Mamouni, L. Oukhtite, M., Zerra, *On derivations involving prime ideals and commutativity in rings*, São Paulo Journal of Mathematical Sciences, **14** (2020), 675–688.
12. A. Mamouni, L. Oukhtite, M. Zerra, *Prime ideals and generalized derivations with central values on rings*, Rendiconti del Circolo Matematico di Palermo Series 2. doi: 10.1007/s12215-020-00578-3
13. H. Nabiel, *Ring subsets that be center-like subsets*, J. Algebra Appl., **17** (2018), №3, 8 p.
14. E.C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., **8** (1957), №6, 1093–1100.
15. M.A. Quadri, M.S. Khan, N. Rehman, *Generalized derivations and commutativity of prime rings*, Indian J. Pure Appl. Math., **34** (2003), №9, 1393–1396.
16. S.K. Tiwari, R.K. Sharma, B. Dhara, *Identities related to generalized derivation on ideal in prime rings*, Beitr. Algebra Geom., **57** (2016), №4, 809–821.
17. N. Rehman, M.A. Raza, *On m-commuting mappings with skew derivations in prime rings*, St. Petersburg Math. J., **27** (2016), 641–650.
18. N. Rehman, *On Lie ideals and automorphisms in prime rings*, Math. Notes, **107** (2020), №1, 140–144.
19. N. Rehman, E.K. Sögütcü, H.M. Alnoghashi, *A generalization of Posner's theorem on generalized derivations in rings*, preprint.
20. N. Rehman, M. Hongan, H.M. Alnoghashi, *On generalized derivations involving prime ideals*, Rendiconti del Circolo Matematico di Palermo Series 2. doi: 10.1007/s12215-021-00639-1

Department of Mathematics, Aligarh Muslim University
Aligarh-202002, India
rehman100@gmail.com
nu.rehman.mm@amu.ac.in
halnoghashi@gmail.com

Received 18.12.2021
Revised 22.03.2022