УДК 519.61

I. K. Argyros, S. M. Shakhno, H. P. Yarmola

ON THE CONVERGENCE OF KURCHATOV-TYPE METHODS USING RECURRENT FUNCTIONS FOR SOLVING EQUATIONS

I. K. Argyros, S. M. Shakhno, H. P. Yarmola. *On the convergence of Kurchatov-type methods using recurrent functions for solving equations*, Mat. Stud. **58** (2022), 103–112.

We study a local and semi-local convergence of Kurchatov's method and its two-step modification for solving nonlinear equations under the classical Lipschitz conditions for the first-order divided differences. To develop a convergence analysis we use the approach of restricted convergence regions in a combination to our technique of recurrent functions. The semi-local convergence is based on the majorizing scalar sequences. Also, the results of the numerical experiment are given.

1. Introduction. Let us consider an equation

$$F(x)=0. \tag{1}$$

Here $F\colon \Omega\subseteq X\to Y$ is a nonlinear operator, X and Y are Banach spaces, Ω is an open convex subset of X. Newton's method is very used for numerical solving of equation (1) [3, 5, 6]

$$x_{k+1}=x_k-F'(x_k)^{-1}F(x_k),\ k\in\mathbb{Z}_+:=\{0,1,2,\ldots\}. \tag{2}$$

Newton's method has a quadratic convergence order. However, its disadvantage is the need of analytically specified derivatives. Therefore, methods without derivatives are used [3, 5]. Some of difference methods are not inferior to Newton's method in the rate of convergence. One of them is Kurchatov's method (method of linear interpolation) [1, 7, 8, 9, 10, 11]

$$x_{k+1}=x_k-A_k^{-1}F(x_k),\ k\in\mathbb{Z}_+, \tag{3}$$

where $A_k=[2x_k-x_{k-1},x_{k-1};F]$, $[\cdot,\cdot;F]:\Omega\times\Omega\to L(X,Y)$ denotes the first-order divided difference.

Definition 1 ([12])**.** Let F be a nonlinear operator defined on a subset Ω of a Banach space X with values in the Banach space Y and let x,y be two points of Ω. A linear operator from X to Y which is denoted by $[x,y;F]$ and satisfies the conditions:

1) for all fixed two points $x,y\in\Omega$

$$[x,y;F](x-y)=F(x)-F(y),$$

2) if there exists the Fréchet derivative $F'(x)$, then

$$[x,x;F]=F'(x),$$

is called a *divided difference* of F at the points x and y.

2010 *Mathematics Subject Classification:* 49M15, 65J15, 65H10.
Keywords: nonlinear equation; Kurchatov's method; Banach space; divided difference; local and semi-local convergence; two-step method.
doi:10.30970/ms.58.1.103-112

In order to accelerate the convergence of single-step methods, their multi-step methods are often developed. The computational complexity of such methods is slightly greater than that of one-step methods. However, the solution of the problem is obtained in the smaller number of iterations. Some of them were studied in the works [2, 4].

In this paper, we develop such a two-step modification of the Kurchatov-type method

$$\begin{aligned} y_k &= x_k - A_k^{-1} F(x_k), \\ x_{k+1} &= y_k - B_k^{-1} F(y_k), \ k \in \mathbb{Z}_+, \end{aligned} \tag{4}$$

where $B_k = [2y_k - x_k, x_k; F]$. We study a semi-local and local convergence of methods (3) and (4) under classical Lipschitz conditions. Moreover, we give a uniqueness of the solution result.

The paper is organized as follows: Section 2 deals with the convergence of scalar majorizing sequences. Sections 3 and 4 give the semi-local and the local convergence analysis of methods (3) and (4), respectively.

2. Convergence of majorizing sequence. We base the convergence of method (3) and method (4) on scalar sequence called majorizing.

Definition 2. Let $\{\bar{u}_k\}$ be a sequence in X. We say that a nondecreasing sequence $\{u_k\}$ is *majorizing* for the sequence $\{\bar{u}_k\}$ if

$$(\forall\ k \in \mathbb{Z}_+) \colon \ \|\bar{u}_{k+1} - \bar{u}_k\| \le u_{k+1} - u_k. \tag{5}$$

Notice that according to (5) the study of $\{\bar{u}_k\}$ reduces to that of $\{u_k\}$ [3].

Let $c \ge 0$, $n \ge 0$, $L_0 > 0$ and $L > 0$ be given parameters. Define a sequence $\{v_k\}$, $\{b_k\}$ by

$$\begin{aligned} &v_{-1} = 0, \ v_0 = c, \ v_1 = c + n, \\ &(\forall\ k \in \mathbb{Z}_+) \colon \ v_{k+2} = v_{k+1} + \frac{L(v_{k+1} - v_k + 2(v_k - v_{k-1}))(v_{k+1} - v_k)}{1 - 2L_0(v_{k+1} + v_k - c)} \end{aligned} \tag{6}$$

and

$$(\forall\ k \in \mathbb{Z}_+) \colon \ b_k = \frac{L(v_{k+1} - v_k + 2(v_k - v_{k-1}))}{1 - 2L_0(v_{k+1} + v_k - c)}.$$

Notice that (6) can be written as

$$v_{k+2} - v_{k+1} = b_k(v_{k+1} - v_k). \tag{7}$$

Next we present a general result for the convergence of sequence $\{v_k\}$.

Lemma 1. *Suppose that for each $k \in \mathbb{Z}_+$*

$$2L_0(v_{k+1} + v_k - c) < 1. \tag{8}$$

Then, the sequence $\{v_k\}$ is nondecreasing, bounded from above by $v^{**} = \frac{1}{2}\left(\frac{1}{L_0} + c\right)$ and as such it converges to its unique least upper bound $v^* \in [0, v^{**}]$.

Proof. The assertion of Lemma 1 follows directly from the definition by (6) of the sequence (b_k). □

Remark 1. Condition (8) can be verified only in the special cases. That is why we develop convergence criteria that can be earlier be verified. Define a sequence of polynomials $f_k(t)$ and cubic polynomial on the interval $[0, 1)$ by

$$f_k(t) = L(t^{k-1} + 2t^{k-2})n + 2L_0((1 + t + \ldots + t^k)n + (1 + t + \ldots + t^{k-1})n + c) - 1$$

and $g(t) = 2L_0t^3 + (2L_0 + L)t^2 + Lt - 2L$. We have $g(0) = -2L$ and $g(1) = 4L_0$. Then, it follows by the mean value theorem that g has zeros in $(0, 1)$. Denote by α the smallest such a zero.

Lemma 2. *Suppose*

$$0 \le b_0 \le \alpha < 1 - \frac{4L_0n}{1 - 2L_0c}, \quad 2L_0c < 1. \tag{9}$$

Then the conclusions of Lemma 1 hold for the sequence $\{v_k\}$ *with*

$$0 \le v_{k+1} - v_k \le \alpha(v_k - v_{k-1}) \le \alpha^k n \tag{10}$$

and v^{**} *replaced by* $\bar{v}^{**} = \dfrac{n}{1 - \alpha} + c.$

Proof. We shall show

$$b_k \le \alpha. \tag{11}$$

Estimate (11) holds for $k = 0$ by (9). Then, by equality (7), we have

$$0 \le v_2 - v_1 \le \alpha(v_1 - v_0) \;\Rightarrow\; v_2 \le v_1 + \alpha(v_1 - v_0) \;\Rightarrow\; v_2 \le c + n + \alpha n = c + (1 + \alpha)n \;\Rightarrow$$
$$v_2 \le \frac{1 - \alpha^2}{1 - \alpha} n + c \le \frac{n}{1 - \alpha} + c = \bar{v}^{**}.$$

Suppose (10) holds for all k smaller or equal to n. We also get

$$v_{k+2} \le v_{k+1} + \alpha^{k+1}n \le v_k + \alpha^k n + \alpha^{k+1}n \le v_1 + \alpha n + \ldots + \alpha^{k+1}n \le$$
$$\le \frac{1 - \alpha^{k+2}}{1 - \alpha} n + c \le \frac{n}{1 - \alpha} + c = \bar{v}^{**}.$$

Endently, (11) holds if

$$L(\alpha^k n + 2\alpha^{k-1}n) + 2L_0\alpha((1 + \alpha + \ldots + \alpha^k)n + (1 + \alpha + \ldots + \alpha^{k-1})n + c) - \alpha \le 0, \tag{12}$$

or

$$f_k(\alpha) \le 0. \tag{13}$$

We need a relationship between two consecutive polynomials f_k. We get

$$f_{k+1}(t) = L(t^k + 2t^{k-1})n + 2L_0((1 + t + \ldots + t^{k+1})n + (1 + t + \ldots + t^k)n + c) - 1-$$
$$-L(t^{k-1} - 2t^{k-2})n - 2L_0((1 + t + \ldots + t^k)n - (1 + t + \ldots + t^{k-1})n + c) + 1 + f_k(t) =$$
$$= f_k(t) + g(t)t^{k-2}n. \tag{14}$$

In particular, by the definition of α we have

$$f_{k+1}(\alpha) = f_k(\alpha). \tag{15}$$

Define a function $f_\infty = \lim_{k\to\infty} f_k(t)$. Then, we have

$$f_\infty(t) = 2L_0\left(\frac{2n}{1-t} + c\right) - 1. \tag{16}$$

By (13), (15) and (16), we can show instead that

$$f_\infty(\alpha) \le 0 \tag{17}$$

which is true by the right hand side double condition in (9). The induction for (10) and (11) is completed. The rest is proved as in the proof of Lemma 1. $\square$

Remark 2. Clearly, conditions (9) implies (8) but non necessarily vice versa.

Next, we similarly study majorizing sequences for methods (4). Define the sequences $\{t_k\}$, $\{s_k\}$ by

$$\begin{aligned} &t_{-1} = 0,\ t_0 = c,\ t_1 = c + n,\\ &s_k = t_k + \frac{L(t_k - s_{k-1} + 2(s_{k-1} - t_{k-1}))(t_k - s_{k-1})}{1 - 2L_0(t_k + t_{k-1} - c)},\\ &t_{k+1} = s_k + \frac{L(s_k - t_k + 2(t_k - t_{k-1}))(s_k - t_k)}{1 - 2L_0(s_k + t_k - c)}. \end{aligned} \tag{18}$$

These equalities can also be rewritten in the following form

$$s_k - t_k = \gamma_k(t_k - s_{k-1}), \quad t_{k+1} - s_k = \delta_k(s_k - t_k),$$

where

$$\gamma_k = \frac{L(t_k - s_{k-1} + 2(s_{k-1} - t_{k-1}))}{1 - 2L_0(t_k + t_{k-1} - c)}, \quad \delta_k = \frac{L(s_k - t_k + 2(t_k - t_{k-1}))}{1 - 2L_0(s_k + t_k - c)}.$$

Lemma 3. *Suppose that*

$$2L_0(s_k + t_k - c) < 1 \ for\ each\ k \in \mathbb{Z}_+. \tag{19}$$

Then, the sequences $\{t_k\}$, $\{s_k\}$ are nondecreasing, bounded from above by $t^{**} = \frac{1}{2}(\frac{1}{L_0} + c)$ and converge to their unique least upper bound $t^* \in [0, t^{**}]$.

Proof. See the proof of Lemma 1. $\square$

Remark 3. Condition (19) can also be verified in the special cases. That is why next we present the stronger convergence criteria but it is easier to verify.

Define the sequences of polynomials $f_k^{(1)}$, $f_k^{(2)}$, g_1, g_2 on the interval $[0, 1)$ by

$$f_k^{(1)}(t) = Lt^{2k}n + 2Lt^{2k-1}n + 2L_0((1 + t + \ldots + t^{2k+1})n + (1 + t + \ldots + t^{2k-1})n + c) - 1,$$

$$\begin{aligned} f_k^{(2)}(t) &= Lt^{2k+1}n + 2L(t^{2k} + 2t^{2k-1})n +\\ &+2L_0((1 + t + \ldots + t^{2k+2})n + (1 + t + \ldots + t^{2k+3})n + c) - 1,\\ g_1(t) &= 2L_0t^4 + (2L_0 + L)t^3 + 2(L_0 + L)t^2 + (2L_0 - L)t - 2L,\\ g_2(t) &= 2L_0t^6 + 4L_0t^5 + (2L_0 + L)t^4 + 2Lt^3 + Lt^2 - 2Lt - 2L. \end{aligned}$$

We get $g_1(0) = g_2(0) = -2L$ and $g_1(1) = g_1(0) = 8L_0$.

Denote by ρ_1, ρ_2 the smallest zeros of functions g_1 and g_2, respectively on the interval $(0, 1)$. We put $\rho_0 = \max\{\gamma_1, \delta_1\}$, $\delta_0 = \min\{\rho_1, \rho_2\}$, $\delta_1 = \max\{\rho_1, \rho_2\}$.

Lemma 4. *Suppose that*

$$\rho_0 \le \delta_0 \le \delta \le \delta_1 < 1 - \frac{4L_0 n}{1 - 2L_0 c}, \;\; 2L_0 c < 1. \tag{20}$$

Then the sequences $\{s_k\}$, $\{t_k\}$ *are nondecreasing, bounded from above by* $\bar{t}^{**} = \frac{n}{1-\delta} + c$ *and converge to* $t^* \in [0, \bar{t}^{**}]$, *so that*

$$0 \le t_{k+1} - s_k \le \delta(s_k - t_k) \le \delta^{2k+1} n, \tag{21}$$

$$0 \le s_k - t_k \le \delta(t_k - s_{k-1}) \le \delta^{2k} n, \tag{22}$$

and

$$t_k \le s_k \le t_{k+1}. \tag{23}$$

Proof. Inequalities (21)–(23) hold if

$$0 \le \gamma_k \le \delta, \tag{24}$$

$$0 \le \delta_k \le \delta, \tag{25}$$

and

$$0 \le t_k \le s_k \le t_{k+1}. \tag{26}$$

It follows from the definition of these sequences and (20) that (21)–(23) hold for $k = 0$.

Suppose (24)–(26) hold for $k \in \{1, 2, \ldots, n\}$. Then, using the induction hypotheses (21) and (22), we obtain in turn that

$$s_k \le t_k + \delta^{2k} n \le s_{k-1} + \delta^{2k-1} n + \delta^{2k} n \le \ldots \le t_0 + n + \delta n + \ldots + \delta^{2k} n \le$$
$$\le \frac{1 - \delta^{2k+1}}{1 - \delta} n + c \le \frac{n}{1 - \delta} + c = \bar{t}^{**},$$

and

$$t_{k+1} \le s_k + \delta^{2k+1} n \le t_k + \delta^{2k} n + \delta^{2k+1} n \le \ldots \le t_0 + n + \delta n + \ldots + \delta^{2k+1} n \le$$
$$\le \frac{1 - \delta^{2k+2}}{1 - \delta} n + c \le \frac{n}{1 - \delta} + c.$$

Therefore, the sequences $\{s_k\}$ and $\{t_k\}$ are nondecreasing.

Then, (24) holds if

$$L\delta^{2k+1} n + 2L\delta^{2k} n + 2L_0 \delta((1 + \delta + \ldots + \delta^{2k+1})n + (1 + \delta + \ldots + \delta^{2k-1})n + c) - \delta \le 0 \tag{27}$$

or

$$f_k^{(1)}(\delta) \le 0. \tag{28}$$

But we get in turn that

$$f_{k+1}^{(1)}(t) = Lt^{2k+2} n + 2Lt^{2k+1} n + 2L_0((1 + t + \ldots + t^{2k+3})n + (1 + t + \ldots + t^{2k+1})n + c) - 1$$
$$- Lt^{2k} n - 2Lt^{2k-1} n - 2L_0((1 + t + \ldots + t^{2k+1})n + (1 + t + \ldots + t^{2k-1})n + c) + 1 + f_k^{(1)}(t) =$$
$$= f_k^{(1)}(t) + g_1(t) t^{2k-1} n.$$

In particular, we get by the definition of ρ_1 that

$$f^{(1)}_{k+1}(\rho_1) = f^{(1)}_k(\rho_1). \tag{29}$$

Define a function $f^{(1)}_\infty = \lim\limits_{k\to\infty} f^{(1)}_k(t)$. Then we have by (27) that

$$f^{(1)}_\infty(t) = \frac{4L_0 n}{1-t} + 2L_0 c - 1. \tag{30}$$

It follows from (28)–(30) that we can show instead (27) that $f^{(1)}_\infty(\rho_1) \le 0$, which is true by (20).

Similarly, (25) holds if

$$L\delta^{2k+2}n+2L(\delta^{2k+1}n+\delta^{2k}n)+2L_0\delta((1+\delta+\ldots+\delta^{2k+2})n+(1+\delta+\ldots+\delta^{2k+3})n+c)-\delta \le 0 \tag{31}$$

or

$$f^{(2)}_k(\delta) \le 0. \tag{32}$$

In this case, we get

$$\begin{gathered} f^{(2)}_{k+1}(t) = Lt^{2k+3}n + 2L(t^{2k+2}n + t^{2k+1}n) + 2L_0((1+t+\ldots+t^{2k+4})n+ \\ +(1+t+\ldots+t^{2k+5})n+c) - 1 - Lt^{2k+1}n - 2L(t^{2k}n + t^{2k-1}n)- \\ -2L_0((1+t+\ldots+t^{2k+2})n + (1+t+\ldots+t^{2k+3})n+c) + 1 + f^{(1)}_{k+1}(t) = \\ = f^{(1)}_k(t) + g_2(t)t^{2k-1}n. \end{gathered}$$

In particular, we have $f^{(2)}_{k+1}(\rho_2) = f^{(2)}_k(\rho_2)$.

Define a function $f^{(2)}_\infty = \lim\limits_{k\to\infty} f^{(2)}_k(t)$. Then, we have by (31) that $f^{(2)}_\infty(t) = f^{(1)}_\infty(t)$. Hence, we can show instead (32) that $f^{(2)}_\infty(\rho_2) \le 0$, which is true by (20). The induction is completed. Therefore, sequences $\{s_k\}$, $\{t_k\}$ are nondecreasing, bounded from above by t^{**} and such they converge to $t^* \in [0, t^{**}]$. □

3. Semi-local convergence. We first study method (3) using majorizing sequence (6), Lemma 1 or Lemma 2 and conditions (H):

(H_1) There exist $x_{-1}, x_0 \in \Omega$ such that $A_0^{-1} \in L(Y, X)$, $\|x_{-1}-x_0\| \le c$ and $\|A_0^{-1}F(x_0)\| \le R$.

(H_2) $\|A_0^{-1}([w_1, w_2; F] - A_0)\| \le L_0(\|w_1 - (2x_0 - x_{-1})\| + \|w_2 - x_{-1}\|)$ for each $w_1, w_2 \in \Omega$. Set $\Omega_0 = U\Big(x_0, \frac{1}{2}\Big(\frac{1}{2L_0} + c\Big)\Big) \cap \Omega$.

(H_3) $\|A_0^{-1}([w_1, w_2; F] - [2y-x, x; F])\| \le L(\|w_1 - (2y-x)\| + \|w_2 - x\|)$ for each $w_1, w_2, x, y, 2y-x \in \Omega_0$.

(H_4) $U(x_0, 3v^*) \subset \Omega$ (or $U(x_0, 3\bar{v}^*) \subset \Omega$).

(H_5) Conditions of Lemma 1 or Lemma 2 hold.

Next, we show the semi-local convergence analysis of method (3).

Theorem 1. *Suppose the conditions (H) hold. Then the sequence $\{x_n\}$ generated by method (3) is well-defined in $U(x_0, v^*)$, remains in $U(x_0, v^*)$ for each $n \in \mathbb{Z}_+$ and converges to a solution $x^* \in U(x_0, v^*)$ of the equation $F(x) = 0$. Moreover, the following estimates hold for each $n \in \mathbb{Z}_+$*

$$\|x_k - x^*\| \le v^* - v_k.$$

Proof. Let $x_k, x_{k-1} \in U(x_0, v^*)$. Then, using conditions (H_1) and (H_2), we get

$$\begin{aligned}\|A_0^{-1}(A_{k+1} - A_0)\| &\leq L_0(\|2x_{k+1} - x_k - (2x_0 - x_{-1})\| + \|x_k - x_{-1}\|) \leq \\ \leq L_0(2\|x_{k+1} - x_0\| + 2\|x_k - x_{-1}\|) &\leq 2L_0(\|x_{k+1} - x_0\| + \|x_k - x_0\| + \|x_0 - x_{-1}\|) \leq \\ \leq 2L_0(\|x_{k+1} - x_0\| + \|x_k - x_0\| + c) &\leq 2L_0(v_{k+1} + v_k - c) < 1. \end{aligned} \tag{33}$$

It follows from (33) and the Banach lemma on invertible operator [6] that the linear operator A_{k+1} is invertible and

$$\|A_{k+1}^{-1}A_0\| \leq \frac{1}{1 - 2L_0(v_{k+1} + v_k - c)}. \tag{34}$$

Iterated x_{k+1} is also well-defined. We can also write

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - A_k(x_{k+1} - x_k) = ([x_{k+1}, x_k; F] - A_k)(x_{k+1} - x_k), \tag{35}$$

so by (H_3), we obtain

$$\begin{aligned}\|A_0^{-1}F(x_{k+1})\| &\leq L(\|x_{k+1} - (2x_k - x_{k-1})\| + \|x_k - x_{k-1}\|)\|x_{k+1} - x_k\| \leq \\ \leq L(\|x_{k+1} - x_k\| + 2\|x_k - x_{k-1}\|)\|x_{k+1} - x_k\| &\leq L(v_{k+1} - v_k + 2(v_k - v_{k-1}))(v_{k+1} - v_k). \end{aligned} \tag{36}$$

Then, we have by method (3) and (36) that

$$\begin{aligned}\|x_{k+2} - x_{k+1}\| &= \|(A_{k+1}^{-1}A_0)(A_0^{-1}F(x_{k+1}))\| \leq \|A_{k+1}^{-1}A_0\|\|A_0^{-1}F(x_{k+1})\| \leq \\ &\leq \frac{L_0(v_{k+1} - v_k + 2(v_k - v_{k-1}))(v_{k+1} - v_k)}{1 - 2L_0(v_{k+1} + v_k - c)} = v_{k+2} - v_{k+1}.\end{aligned}$$

Notice that we also have

$$\|2x_{k+1} - x_k - x_0\| \leq \|x_{k+1} - x_0\| + \|x_{k+1} - x_k\| \leq 2\|x_{k+1} - x_0\| + \|x_k - x_0\| \leq 3v^*,$$

so $2x_{k+1} - x_k \in U(x_0, 3v^*)$. It follows that the sequence $\{x_k\}$ is Cauchy (since $\{v_k\}$ is as convergent). Hence, it converges to some $x^* \in U(x_0, v^*)$. By tending $k \to \infty$ in the estimate (36), and using the continuity of F, we conclude $F(x^*) = 0$. □

Concerning the uniqueness of the solution we have:

Proposition 1. *Suppose:*

(i) $x^* \in \Omega$ *is a solution of the equation* $F(x) = 0$.

(ii) $\|A_0^{-1}([x^*, z; F] - A_0)\| \leq L_1(\|x^* - (2x_0 - x_{-1})\| + \|z - x_{-1}\|)$ *for all* $z \in \Omega$.

(iii) There exists $\bar{v}^* \geq v^*$ *such that* $L_1(v^* + \bar{v}^* + 2c) < 1$.

Set $\Omega_1 = U(x_0, \bar{v}^*) \cap \Omega$. *Then, the only solution of the equation* $F(x) = 0$ *in the region* Ω_1 *is* x^*.

Proof. Set $T = [x^*, \bar{x}; F]$ for some $\bar{x} \in \Omega_1$ with $F(\bar{x}) = 0$. Then, using (ii) and (iii), we obtain

$$\begin{aligned}\|A_0^{-1}([x^*, \bar{x}; F] - A_0)\| &\leq L_1(\|x^* - (2x_0 - x_{-1})\| + \|\bar{x} - x_{-1}\|) \leq \\ \leq L_1(\|x^* - x_0\| + \|x_0 - x_{-1}\| + \|\bar{x} - x_0\| + \|x_0 - x_{-1}\|) &\leq L_1(v^* + 2c + \bar{v}^*) < 1,\end{aligned}$$

so $x^* = \bar{x}$ follows from the invertibility of T and the identity

$$T(\bar{x} - x^*) = F(\bar{x}) - F(x^*) = 0 - 0 = 0.$$

□

We present the semi-local convergence analysis of method (4) in a similar way under the (H) conditions with v^* replaced by t^* and using Lemma 3 or Lemma 4.

Theorem 2. *Suppose the (H) conditions hold. Then, the sequences $\{x_n\}$, $\{y_n\}$ generated by method (4) is well-defined in $U(x_0, t^*)$, remain in $U(x_0, t^*)$ for each $n \in \mathbb{Z}_+$ and converge to a solution $x^* \in U(x_0, t^*)$ of the equation $F(x) = 0$. Moreover, the following estimates hold for each $n \in \mathbb{Z}_+$*

$$\|x_k - x^*\| \leq t^* - t_k.$$

Proof. We follow the proof of Theorem 1. In this case, we get

$$\|A_0^{-1}(B_{k+1} - A_0)\| \leq L_0(\|2y_{k+1} - x_{k+1} - (2x_0 - x_{-1})\| + \|x_{k+1} - x_{-1}\|) \leq$$
$$\leq L_0(2\|y_{k+1} - x_0\| + 2\|x_{k+1} - x_{-1}\|) \leq 2L_0(\|y_{k+1} - x_0\| + \|x_{k+1} - x_0\| + \|x_0 - x_{-1}\|) \leq$$
$$\leq 2L_0(s_{k+1} + t_{k+1} - c) < 1,$$

so $\|B_{k+1}^{-1}A_0\| \leq \dfrac{1}{1 - 2L_0(s_{k+1} + t_{k+1} - c)}$. We also have

$$\|A_0^{-1}([x_{k+1}, y_k; F] - B_k)\| \leq L(\|x_{k+1} - 2y_k + x_k)\| + \|y_k - x_k\|) \leq$$
$$\leq L(\|x_{k+1} - y_k\| + 2\|y_k - x_k\|) \leq L(t_{k+1} - s_k + 2(s_k - t_k)).$$

Hence, we get

$$\|y_{k+1} - x_{k+1}\| \leq \|A_{k+1}^{-1}A_0\|\|A_0^{-1}([x_{k+1}, y_k; F] - B_k)\|\|x_{k+1} - y_k\| \leq$$
$$\leq \frac{L(t_{k+1} - s_k + 2(s_k - t_k))(t_{k+1} - s_k)}{1 - 2L_0(t_{k+1} + t_k - c)} = s_{k+1} - t_{k+1}.$$

Moreover, we have $F(y_k) = F(y_k) - F(x_k) - A_k(y_k - x_k) = ([y_k, x_k; F] - A_k)(y_k - x_k)$, so

$$\|A_0^{-1}F(y_k)\| = \|A_0^{-1}([y_k, x_k; F] - A_k)(y_k - x_k)\| \leq$$
$$\leq L(\|y_k - 2x_k + x_{k-1}\| + \|x_k - x_{k-1}\|)\|y_k - x_k\| \leq$$
$$\leq L(\|y_k - x_k\| + 2\|x_k - x_{k-1}\|)\|y_k - x_k\| \leq L(s_k - t_k + 2(t_k - t_{k-1}))(s_k - t_k).$$

Hence, we get

$$\|x_{k+1} - y_k\| = \|(B_k^{-1}A_0)(A_0^{-1}F(y_k))\| \leq \frac{L(s_k - t_k + 2(t_k - t_{k-1}))(s_k - t_k)}{1 - 2L(s_k + t_k - c)} = t_{k+1} - s_k.$$

The rest follows as in the proof of Theorem 1. □

Remark 4. (a) Condition $2y - x \in \Omega$ is satisfied if $\Omega = X$. (b) The element $2y - x$ can be replaced by the more general $u \in \Omega$. But in this case, the conditions can become stronger and the Lipschitz constants become larger, leading to a less precise convergence analysis.

4. Local convergence. We first study method (3) under conditions (C):

(C_1) There exists a simple solution x^* of the equation $F(x) = 0$.

(C_2) For each $x, y, 2x - y \in \Omega$
$$\|F'(x^*)^{-1}([2y - x, x; F] - F'(x^*))\| \leq l_0(\|2y - x - x^*\| + \|x - x^*\|).$$

(C_3) For each $x, y, 2x - y \in \Omega_2 := U\Big(x^*, \frac{1}{4l_0}\Big) \cap \Omega$
$$\|F'(x^*)^{-1}([2y - x, x; F] - [y, x^*; F])\| \leq l(\|y - x\| + \|x - x^*\|).$$

(C_4) $U(x^*, 3r) \subset \Omega$, where $r = \dfrac{1}{4l_0 + 3l}$.

Theorem 3. *Under the conditions* (C) *further suppose that* $x_{-1}, x_0 \in U(x^*, r) - \{x^*\}$. *Then the sequence* $\{x_n\}$ *generated by method (3) is well-defined in* $U(x^*, r)$, *remains in* $U(x^*, r)$ *for each* $n \in \mathbb{Z}_+$ *and converges to* x^*.

Proof. We have by (C_1) and (C_2) that $\|F'(x^*)^{-1}(A_k - F'(x^*))\| \le l_0(\|x_k - x^*\| + \|x_k - x_{k-1}\| + \|x_{k-1} - x^*\|) \le 2l_0(\|x_k - x^*\| + \|x_{k-1} - x^*\|) \le 4l_0 r < 1$, so

$$\|A_k^{-1}F'(x^*)\| \le \frac{1}{1 - 2l_0(\|x_k - x^*\| + \|x_{k-1} - x^*\|)}.$$

We also get by (C_3), $\|F'(x^*)^{-1}(A_k - [x_k, x^*; F])\| \le l(\|x_k - x^*\| + 2\|x_{k-1} - x^*\|)$, so

$$\begin{aligned} \|x_{k+1} - x^*\| &= \|x_k - x^* - A_k^{-1}F(x_k)\| \le \\ &\le \|A_k^{-1}F'(x^*)\|\|F'(x^*)^{-1}(A_k - [x_k, x^*; F])(x_k - x^*)\| \le \\ &\le \|A_k^{-1}F'(x^*)\|\|F'(x^*)^{-1}(A_k - [x_k, x^*; F])\|\|x_k - x^*\| \le \\ &\le \frac{l(\|x_k - x^*\| + 2\|x_{k-1} - x^*\|)\|x_k - x^*\|}{1 - 2l_0(\|x_k - x^*\| + \|x_{k-1} - x^*\|)} < \|x_k - x^*\| < r, \end{aligned}$$

so $x_{k+1} \in U(x^*, r)$ and $\lim_{k\to\infty} x_k = x^*$. □

We also have a uniqueness of the solution result.

Proposition 2. *Suppose:*

(i) $x^* \in \Omega$ *is a simple solution of the equation* $F(x) = 0$.

(ii) For each $z \in \Omega$ $\|F'(x^*)^{-1}([x^*, z; F] - F'(x^*))\| \le l_1\|z - x^*\|$.

Set $\Omega_3 = U(x^*, \frac{1}{l_1}) \cap \Omega$. *Then, the only solution of the equation* $F(x) = 0$ *in the region* Ω_3 *is* x^*.

Proof. Set $T = [x^*, \bar{x}; F]$ for some $\bar{x} \in \Omega_3$ with $F(\bar{x}) = 0$. Then, using (ii) and (iii), we obtain

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \le l_1\|\bar{x} - x^*\| < 1,$$

so $x^* = \bar{x}$ by the invertibility of T and the identity $T(\bar{x} - x^*) = F(\bar{x}) - F(x^*) = 0 - 0 = 0$. □

We can also show the local convergence analysis of method (4) under conditions (H).

Theorem 4. *Under the conditions* (C) *further suppose that* $x_{-1}, x_0 \in U(x^*, r) - \{x^*\}$. *Then,*

$$\lim_{n\to\infty} x_k = x^*.$$

Proof. As in Theorem 3, we get

$$\|y_k - x^*\| \le \frac{l(\|x_k - x^*\| + 2\|x_{k-1} - x^*\|)\|x_k - x^*\|}{1 - 2l_0(\|x_k - x^*\| + \|x_{k-1} - x^*\|)} < \|x_k - x^*\| < r.$$

Moreover, by estimates $\|F'(x^*)^{-1}(B_k - F'(x^*))\| \le l_0(\|2y_k - x_k - x^*\| + \|x_k - x^*\|) \le 2l_0(\|y_k - x^*\| + \|x_k - x^*\|) \le 4l_0 r < 1$, so $\|B_k^{-1}F'(x^*)\| \le \dfrac{1}{1 - 2l_0(\|y_k - x^*\| + \|x_k - x^*\|)}$, and $F'(x^*)^{-1}(B_k - [y_k, x^*; F])\| \le l(\|y_k - x^*\| + 2\|x_k - x^*\|)$, we get

$$\begin{aligned} \|x_{k+1} - x^*\| &= \|y_k - x^* - B_k^{-1}F(y_k)\| \le \|(B_k^{-1}F'(x^*))(F'(x^*)^{-1}(B_k - [y_k, x^*; F]))\times \\ &\times(y_k - x^*)\| \le \|B_k^{-1}F'(x^*)\|\|F'(x^*)^{-1}(B_k - [y_k, x^*; F])\|\|y_k - x^*\| \le \\ &\le \frac{l(\|y_k - x^*\| + 2\|x_k - x^*\|)\|y_k - x^*\|}{1 - 2l_0(\|y_k - x^*\| + \|x_k - x^*\|)} < \|y_k - x^*\| < r, \end{aligned}$$

so $y_k \in U(x^*, r)$ and $\lim_{k\to\infty} y_k = \lim_{k\to\infty} x_k = x^*$. □

REFERENCES

1. I.K. Argyros, *A Kantorovich-type analysis for a fast iterative method for solving nonlinear equations*, J. Math. Anal. Appl., **332** (2007), 97–108.
2. I.K. Argyros, S. George, *On a two-step Kurchatov-type method in Banach space*, Mediterr. J. Math., **16** (2019), Article number: 21. https://doi.org/10.1007/s00009-018-1285-7
3. I.K. Argyros, A.A. Magreñán, Iterative Methods and Their Dynamics with Applications: A Contemporary Study, CRC Press, Boca Raton Florida, USA, 2017.
4. I.K. Argyros, S. Shakhno, *Extended two-step-Kurchatov method for solving Banach space valued nondifferentiable equations*, Int. J. Appl. Comput. Math., **6** (2020), Article number: 32. https://doi.org/10.1007/s40819-020-0784-y
5. J.E. Dennis, Jr. Schnabel, B. Robert, Numerical methods for unconstrained optimization and nonlinear equations, Englewood Cliffs, Prentice-Hall, 1983.
6. L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.
7. V.A. Kurchatov, *On a method of linear interpolation for the solution of functional equations*, Dokl. Akad. Nauk. SSSR, **198** (1971), №3, 524–526 (in Russian); translation in Soviet. Math. Dokl. **12** (1971), 835–838.
8. S.M. Shakhno, *Kurchatov method of linear interpolation under generalized Lipschitz conditions for divided differences of first and second order*, Visnyk Lviv Univ. Ser. Mech. Math., **77** (2012), 235–242. (in Ukrainian)
9. S.M. Shakhno, *On a Kurchatov's method of linear interpolation for solving nonlinear equations*, PAMM: Proceedings in Applied Mathematics and Mechanics, **4** (2004), №1, 650–651.
10. S.M. Shakhno, *On the difference method with quadratic convergence for solving nonlinear operator equations*, Mat. Stud., **26** (2006), №1, 105–110. (in Ukrainian)
11. S.M. Shakhno, *Nonlinear majorants for investigation of methods of linear interpolation for the solution of nonlinear equations*, In: ECCOMAS 2004 - European congress on computational methods in applied sciences and engineering; P. Neittaanmäki,T. Rossi, K. Majava, and O. Pironneau (eds.) O. Nevanlinna and R. Rannacher (assoc. eds.) Jyväskylä, 24-28 July 2004.
12. S. Ul'm, *On generalized divided differences I, II*, Izv. Akad. Nauk. ESSR, Ser. Fiz.-Mat., **16** (1967), 13–26, 146–156. (in Russian)

Department of Mathematical Sciences,
Cameron University, Lawton, USA
iargyros@cameron.edu

Department of Theory of Optimal Processes,
Ivan Franko National University of Lviv, Lviv, Ukraine
stepan.shakhno@lnu.edu.ua

Department of Computational Mathematics,
Ivan Franko National University of Lviv, Lviv, Ukraine
halyna.yarmola@lnu.edu.ua

Received 02.12.2021
Revised 25.10.2022