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ON PSEUDOBOUNDED AND PREMEAGER PARATOPOLOGICAL GROUPS

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Let G be a paratopological group. Following F. Lin and S. Lin, we say that the group G is pseudobounded, if for any neighborhood U of the identity of G , there exists a natural number n such that $U^n = G$. The group G is ω -pseudobounded, if for any neighborhood U of the identity of G , the group G is a union of sets U^n , where n is a natural number. The group G is premeager, if $G \neq N^n$ for any nowhere dense subset N of G and any positive integer n . In this paper we investigate relations between the above classes of groups and answer some questions posed by F. Lin, S. Lin, and Sánchez.

A *topologized group* (G, τ) is a group G endowed with a topology τ . A *left topological group* is a topologized group such that each left shift $G \rightarrow G, x \mapsto gx, g \in G$, is continuous. A *semitopological group* G is a topologized group such that the multiplication map $G \times G \rightarrow G, (x, y) \mapsto xy$, is separately continuous. Moreover, if the multiplication is continuous then G is called a *paratopological group*. A paratopological group with the continuous inversion map $G \rightarrow G, x \mapsto x^{-1}$, is called a *topological group*. A classical example of a paratopological group failing to be a topological group is the *Sorgenfrey line* \mathbb{S} , that is the group \mathbb{R} endowed with the topology generated by the base consisting of all half-intervals $[a, b), a < b$.

Whereas an investigation of topological groups already is one of fundamental branches of topological algebra (see, for instance, [6, 17] and [1]), other topologized groups are not so well-investigated and have more variable structure.

Basic properties of semitopological or paratopological groups are described in book [1] by Arhangel'skii and Tkachenko, in author's PhD thesis [21] and papers [19, 20]. New Tkachenko's survey [23] presents recent advances in this area. Let ω be the set of finite ordinals and $\mathbb{N} = \omega \setminus \{0\}$.

A subset A of a left topological group G is

- *left* (resp. *right*) *precompact*, if for any neighborhood U of the identity of G there exists a finite subset F of G such that $FU \supseteq A$ (resp. $UF \supseteq A$);

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- *precompact*, if A is both left and right precompact;
- *left ω -precompact*, if for any neighborhood U of the identity of G there exists a countable set $F \subseteq G$ such that $FU \supseteq A$;
- *pseudobounded*, if for any neighborhood U of the identity of G there exists $n \in \mathbb{N}$ such that $U^n = G$;
- *ω -pseudobounded*, if for any neighborhood U of the identity of G , $G = \bigcup_{n \in \mathbb{N}} U^n$.

Proposition 2.1 from [20] implies that a paratopological group is left precompact iff it is right precompact, so iff it is precompact. Moreover, precompact Hausdorff topological groups are exactly subgroups of compact Hausdorff groups [24]. A left topological group is called *locally left* (resp. *ω -precompact*) if it has a left (resp. *ω -precompact*) neighborhood of the identity.

The notion of ω -pseudobounded paratopological groups was introduced by F. Lin and S. Lin in their paper [13], generalizing a notion of Azar [2] provided for topological groups. In [13] and also in a subsequent paper [14] with Sánchez they investigated basic properties of these groups and asked some related questions. In this paper we answer some of them.

Similarly to the proof of Theorems 3 and 6 from [13] we can show that if H is a normal subgroup of a topologized group G then G is (ω -)pseudobounded provided both groups H and G/H are (ω -)pseudobounded.

In Problem 2.27 from [14] is asked whether every pseudobounded (para)topological group is (ω -)precompact. The following example provides its negative solution.

Example 1. Consider the compact topological group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$ endowed with the operation of multiplication of complex numbers. Let G be \mathbb{T}^ω endowed with the topology, generated by the sup-metric $d(x, y) = \sup_{i \in \omega} |x(i) - y(i)|$ for $x, y \in \mathbb{T}^\omega$. It is easy to check that G is pseudobounded but not locally ω -precompact.

On the other hand, the following proposition provides an affirmative solution to a special case of the above problem. Recall that a neighborhood U of an identity of a paratopological group G is *invariant*, if $U = g^{-1}Ug$ for each $g \in G$. A paratopological group G is a *SIN-group*, if it has a base at the identity consisting of invariant neighborhoods.

Proposition 1. *Each pseudobounded locally precompact paratopological SIN-group G is precompact.*

Proof. Let U be any left precompact neighborhood of the identity e of G . Since the group G is pseudobounded, there exists a natural number n such that $G = U^n$. Since G is a SIN-group, there exists an invariant neighborhood V of e such that $V^n \subseteq U$. Since the set U is precompact, there exists a finite subset F of G such that $FV \supseteq U$. Then $G = U^n \subseteq (FV)^n = F^nV^n \subseteq F^nU$. □

Example 2. Each real or complex linear topological space is ω -pseudobounded. Since a first countable topological group (in particular, a normed space) is ω -precompact iff it is separable, each nonseparable real or complex normed topological space is ω -pseudobounded but not ω -precompact. For instance, so is the Banach space $\ell_\infty(X)$ of bounded real-valued functions on an infinite set X , endowed with the supremum norm.

A left topological group G is *2-pseudocompact* if $\bigcap_{n \in \omega} \overline{U_n^{-1}} \neq \emptyset$ for each nonincreasing sequence $(U_n)_{n \in \omega}$ of nonempty open subsets of G . By [14, Proposition 2.9], each ω -pseudobounded 2-pseudocompact paratopological group is pseudobounded. Since each 2-pseudocompact left topological group is feebly compact and Baire (by Proposition 3.13 and Lemma 3.7 from [4]), the following proposition generalizes this result.

Proposition 2. *Each ω -pseudobounded feebly compact Baire left topological group G is pseudobounded.*

Proof. Let U be any neighborhood of the identity of G . Since the group G is ω -pseudobounded, $G = \bigcup_{n \in \mathbb{N}} U^{-n}$. Since the space G is Baire, there exists $n \in \mathbb{N}$ such that $\overline{U^{-n}}$ contains a nonempty open set V . Then $U^{-n-1} = U^{-n}U^{-1} \supseteq \overline{U^{-n}} \supseteq V$. Pick any point $y \in V$. Since $G = \bigcup_{n \in \mathbb{N}} U^n$, there exists $m \in \mathbb{N}$ such that $y \in U^{m-1}$. Then

$$W := y^{-1}V \subseteq U^{-m+1}U^{-n-1} = U^{-m-n}.$$

Suppose for a contradiction that $\overline{G} \neq U^k$ for every $k \in \mathbb{N}$. Taking into account that $\overline{U^{-k}} \subseteq U^{-k}U^{-1}$, we conclude that $\overline{U^{-k}} \neq G$ for every $k \in \mathbb{N}$. Since the group G is feebly compact, there exists a point $x \in \bigcap_{k \in \mathbb{N}} G \setminus \overline{U^{-k}}$. Since the group G is ω -pseudobounded, there exists $l \in \mathbb{N}$ such that $x \in W^l \subseteq U^{-l(m+n)}$. But then W^l is a neighborhood of x , disjoint from $G \setminus \overline{U^{-l(m+n)}}$, a contradiction. \square

The following proposition answers Question 6 from [13].

Proposition 3. *Let G be a pseudobounded left topological group and d be any left-invariant quasi-pseudometric generating the topology of G . Then d is bounded on G .*

Proof. Since G is pseudobounded, for the neighborhood $U = \{x \in G : d(e, x) < 1\}$ of the identity e in G , there exists a number $n \in \mathbb{N}$ such that $G = U^n$. Now let y, z be any elements of G . There exist elements $x_1, \dots, x_n \in G$ such that $y^{-1}z = x_1 \cdots x_n$. Then

$$\begin{aligned} d(y, z) &= d(e, y^{-1}z) = d(e, x_1 \cdots x_n) \leq \\ &\leq d(e, x_1) + d(x_1, x_1x_2) + \cdots + d(x_1 \cdots x_{n-1}, x_1 \cdots x_n) = \\ &= d(e, x_1) + d(e, x_2) + \cdots + d(e, x_n) < n. \end{aligned}$$

\square

Following F. Lin and S. Lin [13], we call a left topological group G *premeager*, if $G \neq N^n$ for any nowhere dense subset N of G and any $n \in \mathbb{N}$.

A *Lusin space* is an uncountable crowded T_1 space containing no uncountable nowhere dense subsets. A space X is *crowded* if every nonempty open set in X is infinite. Clearly, each Lusin left topological group is premeager. A trivial example of a Lusin space is any uncountable space X endowed with the T_1 -topology $\{\emptyset\} \cup \{X \setminus A : A \text{ is finite}\}$. On the other hand, the existence of a Hausdorff Lusin space is independent of the axioms of ZFC: Lusin [16] showed that such a space exists under Continuum Hypothesis, and Kunen [12]

showed that there are no Hausdorff Lusin spaces under Martin's Axiom and the negation of Continuum Hypothesis.

In order to construct a nonpremeager paratopological group we introduce the following definition. A subset of a space X is *meager*, if it is contained in a countable union of nowhere dense subsets of X . By \mathcal{M}_X we denote the family of all meager subsets of X . Let

- $\text{cov}(\mathcal{M}_X) = \min\{|\mathcal{A}|: \mathcal{A} \subseteq \mathcal{M}_X \ (\bigcup \mathcal{A} = X)\}$ and
- $\text{cof}(\mathcal{M}_X) = \min\{|\mathcal{A}|: \mathcal{A} \subseteq \mathcal{M}_X \ \forall B \in \mathcal{M}_X \ \exists A \in \mathcal{A} \ (B \subseteq A)\}$.

The family \mathcal{A} in the latter definition is called a *cofinal* in \mathcal{M}_X . The family $\mathcal{M}_{\mathbb{R}}$ will be denoted by \mathcal{M} . A space is called *Polish*, if it is homeomorphic to a separable complete metric space. By Theorem 15.6 in [11], for any crowded Polish space X we have $\text{cof}(\mathcal{M}_X) = \text{cof}(\mathcal{M})$ and $\text{cov}(\mathcal{M}_X) = \text{cov}(\mathcal{M})$.

A left topological group G *meagerly divisible* if for every meager subset M of G and every nonzero integer number n , the set $\{x \in G: x^n \in M\}$ is meager.

Proposition 4. *A paratopological group G is meagerly divisible provided for each positive integer n , the power map $p_n: G \rightarrow G, x \mapsto x^n$, is open.*

Proof. Since G is a topological group, for every $n \in \mathbb{N}$ the openness of the map $p_n: G \rightarrow G$ implies the openness of the map $p_{-n}: G \rightarrow G, p_{-n}: x \mapsto x^{-n}$. Let N be any closed nowhere dense subset of G and n be any nonzero integer. Since the map p_n is continuous, the preimage $p_n^{-1}(N)$ is closed. If $p_n^{-1}(N)$ contains a nonempty open subset U of G then $p_n(U)$ is a nonempty open subset of N , that is impossible.

Let M be any meager subset of G . There exist a countable family \mathcal{A} of nowhere dense closed subsets of G such that $M \subset \bigcup \mathcal{A}$. Then $p_n^{-1}(M) \subseteq \bigcup \{p_n^{-1}(N): N \in \mathcal{A}\}$ and each set $p_n^{-1}(N)$ is nowhere dense. □

Corollary 1. *The paratopological groups $\mathbb{R}, \mathbb{S}, \mathbb{R}/\mathbb{Z}$, and \mathbb{S}/\mathbb{Z} are meagerly divisible.*

Proof. The topological group \mathbb{R} and \mathbb{R}/\mathbb{Z} are meagerly divisible by Proposition 4. The Sorgenfrey line \mathbb{S} (resp. \mathbb{S}/\mathbb{Z}) has a common π -base with the topological group \mathbb{R} (resp. \mathbb{R}/\mathbb{Z}), so $\mathcal{M}_{\mathbb{S}}$ (resp. $\mathcal{M}_{\mathbb{S}/\mathbb{Z}}$) equals \mathcal{M} and the group \mathbb{S} (resp. \mathbb{S}/\mathbb{Z}) is meagerly divisible. □

We recall that a space is *analytic* if it is a continuous image of a Polish space. The Open Mapping Principle (see, for instance, Corollary 3.10 in [3]) states that any continuous surjective homomorphism from an analytic topological group to a Polish topological group is open.

Proposition 5. *Each divisible Abelian Polish topological group G is meagerly divisible.*

Proof. Let n be any nonzero integer and $p_n: G \rightarrow G, x \mapsto nx$, be the power map. We claim the the map p_n is open. Indeed, since G is Abelian, p_n is a homomorphism. Since G is divisible, p_n is surjective. By the Open Mapping Principle, p_n is open. By Proposition 4, G is meagerly divisible. □

Proposition 6. *If an Abelian Polish topological group G is meagerly divisible, then for each nonzero integer n , the power map $p_n: G \rightarrow G$, $x \mapsto nx$, is open.*

Proof. Let n be any nonzero integer. Since p_n is a homomorphism, it suffices to show that for each neighborhood U of the identity, $p_n(U)$ is a neighborhood of the identity. Pick an open neighborhood V of the identity such that $V - V \subseteq U$. Since the group G is Polish, the set V is nonmeager. The group G is meagerly divisible, the set $p_n(V)$ is nonmeager. Since the map p_n is continuous, the space $p_n(V)$ is analytic. Piccard-Pettis' Theorem implies (see, for instance, Corollary 3.9 in [3]) that $p_n(V) - p_n(V)$ is a neighborhood of the identity of G . It remains to note that $p_n(U) \supseteq p_n(V) - p_n(V)$. \square

Proposition 7. *Let H be a meagerly divisible Abelian semitopological group such that $\text{cov}(\mathcal{M}_H) = \text{cof}(\mathcal{M}_H) = \kappa > \omega$. Then H contains a free group G of size κ such that every meager subset $M \subseteq G$ of H has cardinality less than κ and hence does not generate G . If H is T_1 (and $\kappa = \omega_1$), then G is crowded (and Lusin).*

Proof. Let $\{S_\alpha: \alpha < \kappa\}$ be a cofinal family in \mathcal{M}_H such that $S_\alpha \ni e$ for each α . Since the group H is meagerly divisible, it contains a nonperiodic element x_0 . For each nonzero integer n let $p_n: H \rightarrow H$, $x \mapsto nx$, be the power map. Since $\kappa = \text{cov}(\mathcal{M}_H)$ and H is meagerly divisible, we can inductively choose for each nonzero ordinal $\beta < \kappa$ a point

$$x_\beta \in H \setminus \bigcup \{p_n^{-1}(S_\alpha + G_\beta): n \in \mathbb{Z} \setminus \{0\}, \alpha < \beta\},$$

where $G_\gamma = \langle \{x_\alpha: \alpha < \gamma\} \rangle$ for each $\gamma \leq \kappa$. Put $G = G_\kappa$.

We claim that G is a free Abelian group over the set $\{x_\alpha: \alpha < \kappa\}$. Indeed, suppose for a contradiction that there exist a finite subset F of κ and a map $f: F \rightarrow \mathbb{Z} \setminus \{0\}$ such that $\sum_{\alpha \in F} f(\alpha)x_\alpha = 0$. Put $\gamma = \max F$. Then $x_\gamma \in p_{f(\gamma)}^{-1}(G_\gamma)$, a contradiction.

Let $M \subseteq G$ be a meager subset of the space H . Then M is contained in the set S_β for some ordinal $\beta < \kappa$. For every $x \in M \subseteq G$, there exist a finite subset F of κ and a function $f: F \rightarrow \mathbb{Z} \setminus \{0\}$ such that $x = \sum_{\alpha \in F} f(\alpha)x_\alpha$. Put $\gamma = \max F$. Then $x_\gamma \in p_{f(\gamma)}^{-1}(S_\beta + G_\gamma)$ and hence $\gamma \leq \beta$, which implies that $x \in G_{\beta+1}$. Thus $M \subseteq S_\beta \cap G \subseteq G_{\beta+1}$ and $|M| \leq |G_{\beta+1}| < \kappa$.

If H is a T_1 space, then H is not discrete (otherwise $\mathcal{M}_H = \emptyset$) and $\text{cof}(\mathcal{M}_H) = 1$ and the group G is crowded (otherwise G is discrete, meager in H and hence $|G| < \kappa = |G|$). \square

Example 3. The assumption $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \kappa > \omega$ is consistent (for instance, $MA_{\text{countable}}$ implies $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \mathfrak{c}$). Since, by Corollary 1, the topological group \mathbb{R} is meagerly divisible, by Proposition 7, it contains a premeager crowded group G of size κ .

Corollary 2. *The existence of a Lusin Hausdorff (para)topological group is independent on ZFC.*

Proof. By Corollary 1, the topological group \mathbb{R} is meagerly divisible. By Proposition 7, under the consistent assumption $\text{cof}(\mathcal{M}) = \omega_1$, \mathbb{R} contains a Lusin group G of size ω_1 .

On the other hand, by [12], under Martin's Axiom and the negation of Continuum Hypothesis, there are no Hausdorff Lusin spaces. \square

It turns out that it is independent on ZFC whether each pseudobounded Lusin Hausdorff paratopological group is a topological group, see Question 4 from [13]. Indeed, under Martin's Axiom and the negation of Continuum Hypothesis, each Lusin paratopological group has an isolated point and so it is a topological group. On the other hand, under $\text{cof}(\mathcal{M}) = \omega_1$ we have the following example (which is also a counterexample for Questions 1 and 3 from [13]).

Example 4. Assume $\text{cof}(\mathcal{M}) = \omega_1$. Since, by Corollary 1, the paratopological group \mathbb{S}/\mathbb{Z} is meagerly divisible, by Proposition 7, it contains a Lusin free group G of size ω_1 . It follows that G is nondiscrete and so not a topological group.

We claim that that the group G is pseudobounded. Indeed, let U be any open neighborhood of the identity of the group G . Let τ be a topology on G inherited from \mathbb{R}/\mathbb{Z} . Since the group \mathbb{R}/\mathbb{Z} is compact, the group (G, τ) is precompact, see [24]. Since the set U has nonempty interior in (G, τ) , there exists a finite subset F of G such that $G = F + U$. Since F is finite, it suffices to show that F is contained in a subsemigroup S of G , generated by U . We claim that $S = G$. Indeed, following [4, Section 5.1], consider a (not necessarily Hausdorff) paratopological group G_S whose topology consists of the sets $A + S$ where $A \subseteq G$ is any subset. Since the topology of the group G_S is weaker than the topology of G and G is precompact, the group G_S is precompact too, and so by Proposition 5.8 from [4], S is a subgroup of G . Since G is nondiscrete, U contains cosets $a + \mathbb{Z}$ for arbitrarily small positive numbers a , so S is dense in G . Since U is open in G , S is open in G . So S is an open dense subgroup of G , that implies $S = G$.

Question 1. Whether there exists under ZFC a nondiscrete premeager (regular) Hausdorff (para)topological group?

The class of nonpremeager topological groups is rather wide. For instance, according to exercises from [18, Section 13], a topological group G contains a closed nowhere dense (and left Haar null in all cases below but the first) subset N such that $NN^{-1} = G$ in the following cases:

- G is nondiscrete Polish Abelian;
- G is nondiscrete locally compact T_1 ;
- G is metrizable and contains a closed connected Lie subgroup;
- G is a T_1 real linear topological space, which is metrizable or locally convex;
- G is the group of homeomorphisms of the Hilbert cube, endowed with the compact-open topology.

Moreover, according to [18, Question 13.6] it is not known even whether there exists under ZFC a T_1 topological group that cannot be generated by its meager subset.

A special case of a nowhere dense subset of a T_1 crowded space is a discrete subset. There is a known problem when a T_1 (para)topological group G can be (topologically) generated by its discrete subset S such that $S \cup \{e\}$ is closed. Namely, such sets for topological groups were considered by Hofmann and Morris in [10] and by Tkachenko in [22]. Fundamental

results were obtained by Comfort et al. in [5] and Dikranjan et al. in [7] and in [8]. In [15] Lin et al. extended this research to paratopological groups.

A sample result is Theorem 2.2 from [5] stating that each countable Hausdorff topological group is generated by a closed discrete set. Protasov and Banakh in Theorem 13.3 of [18] generalized this and Guran's [9] results to Hausdorff left topological groups.

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