UDK 517.535.1+517.537.72

M. M. Sheremeta

PSEUDOSTARLIKE AND PSEUDOCONVEX SOLUTIONS OF A DIFFERENTIAL EQUATION WITH EXPONENTIAL COEFFICIENTS

M. M. Sheremeta. *Pseudostarlike and pseudoconvex solutions of a differential equation with exponential coefficients*, Mat. Stud. **56** (2021), 39–47.

Dirichlet series $F(s)=e^s+\sum_{k=1}^{\infty} f_k e^{s\lambda_k}$ with the exponents $1<\lambda_k\uparrow+\infty$ and the abscissa of absolute convergence $\sigma_a[F]\ge 0$ is said to be pseudostarlike of order $\alpha\in[0,\,1)$ and type $\beta\in(0,\,1]$ if

$$\left|\frac{F'(s)}{F(s)}-1\right|<\beta\left|\frac{F'(s)}{F(s)}-(2\alpha-1)\right| \text{ for all } s\in\Pi_0=\{s\colon\ \mathrm{Re}\, s<0\}.$$

Similarly, the function F is said to be pseudoconvex of order $\alpha\in[0,\,1)$ and type $\beta\in(0,\,1]$ if

$$\left|\frac{F''(s)}{F'(s)}-1\right|<\beta\left|\frac{F''(s)}{F'(s)}-(2\alpha-1)\right| \text{ for all } s\in\Pi_0.$$

Some conditions are found on the parameters $b_0,\ b_1,\ c_0,\ c_1,\ c_2$ and the coefficients a_n, under which the differential equation

$$\frac{d^2w}{ds^2}+(b_0e^s+b_1)\frac{dw}{ds}+(c_0e^{2s}+c_1e^s+c_2)w=\sum_{n=1}^{\infty}a_ne^{ns}$$

has an entire solution which is pseudostarlike or pseudoconvex of order $\alpha\in[0,\,1)$ and type $\beta\in(0,\,1]$. It is proved that by some conditions for such solution the asymptotic equality holds

$$\ln\max\{|F(\sigma+it)|\colon t\in\mathbb{R}\}=\frac{1+o(1)}{2}\left(|b_0|+\sqrt{|b_0|^2+4|c_0|}\right) \text{ as } \sigma\to+\infty.$$

1. Introduction and auxiliary results. An analytic univalent in $\mathbb{D}=\{z\colon\ |z|<1\}$ function $f(z)=\sum_{n=0}^{\infty} f_nz^n$ is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [1, p. 203] that the condition $\mathrm{Re}\,\{1+zf''(z)/f'(z)\}>0$ $(z\in\mathbb{D})$ is necessary and sufficient for the convexity of f. By W. Kaplan [2] the function f is said to be close-to-convex in $\mathbb{D}$ (see also [1, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,(f'(z)/\Phi'(z))>0$ $(z\in\mathbb{D})$. Close-to-convex function f has a characteristic property that the complement G of the domain $f(\mathbb{D})$ can be filled with rays L which go from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0)\ne 0$. Hence, it follows that the function f is close-to-convex in $\mathbb{D}$ if and only if the function $g(z)=z+\sum_{n=2}^{\infty} g_nz^n$ is close-to-convex in $\mathbb{D}$, where $g_n=f_n/f_1$. Such function g is said to be starlike if $f(\mathbb{D})$ is a starlike domain. It is well known [1, p. 203] that the condition $\mathrm{Re}\,\{zf'(z)/f(z)\}>0$ $(z\in\mathbb{D})$ is necessary and sufficient for the starlikeness of g.

S.M. Shah [3] indicated conditions on real parameters $\beta_0,\ \beta_1,\ \gamma_0,\ \gamma_1,\ \gamma_2$ of the differential equation $z^2w''+(\beta_0z^2+\beta_1z)w'+(\gamma_0z^2+\gamma_1z+\gamma_2)w=0$, under which there exists an entire transcendental solution f such that f and all its derivatives are close-to-convex in $\mathbb{D}$. The

2010 *Mathematics Subject Classification:* 30B10, 30B50, 30C45, 35B08.
Keywords: linear differential equation; Dirichlet series; pseudostarlikeness; pseudoconvexity.
doi:10.30970/ms.56.1.39-47

investigations are continued in the papers [4–9]. In [10, 11] it is studied the closeness-to-convexity of the second order non-homogeneous linear differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = A(z),$$

where $A(z) = \sum_{n=1}^{\infty} a_n z^n$ and radius of convergence of the last power series is $R[A] \geq 1$. Substituting $z = e^s$ we obtain the differential equation

$$\frac{d^2w}{ds^2} + (b_0 e^s + b_1)\frac{dw}{ds} + (c_0 e^{2s} + c_1 e^s + c_2)w = \sum_{n=1}^{\infty} a_n e^{ns}, \tag{1}$$

where $b_0 = \beta_0$, $b_1 = \beta_1 - 1$, $c_j = \gamma_j$ and Dirichlet series $\sum_{n=1}^{\infty} a_n e^{ns}$ is absolutely convergent in a half-plane $\{s\colon \mathrm{Re} s < a\}$ with $a \geq 0$.

Now, let $\Lambda = (\lambda_k)$ be an increasing to $+\infty$ sequence of positive numbers ($\lambda_1 > 1$) and $SD(\Lambda, 0)$ be a class of Dirichlet series

$$F(s) = e^s + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}, \quad f_k \neq 0, \quad s = \sigma + it, \tag{2}$$

with the exponents Λ and the abscissa of absolute convergence $\sigma_a[F] = 0$. It is known [12] (see also [13, p. 135] that each function $F \in SD(\Lambda, 0)$ is non-univalent in $\Pi_0 = \{s\colon \mathrm{Re}\, s < 0\}$, but there exist conformal in Π_0 functions (2), and if $\sum\limits_{k=1}^{\infty} \lambda_k |f_k| \leq 1$ then function (2) is conformal in Π_0. A conformal function (2) in Π_0 is said to be pseudostarlike if $\mathrm{Re}\{F'(s)/F(s)\} > 0$ for $s \in \Pi_0$. In [12] (see also [13, p. 139]) it is proved that if $\sum\limits_{k=1}^{\infty} \lambda_k |f_k| \leq 1$ then function (2) is pseudostarlike.

A conformal function (2) in Π_0 is said to be *pseudostarlike of order* α if

$$\mathrm{Re}\{F'(s)/F(s)\} > \alpha \in [0,\, 1), \quad s \in \Pi_0. \tag{3}$$

Since the inequality $|w - 1| < |w - (2\alpha - 1)|$ holds if and only if $\mathrm{Re}\, w > \alpha$, function (3) is pseudostarlike of the order α if and only if

$$\left|\frac{F'(s)}{F(s)} - 1\right| < \left|\frac{F'(s)}{F(s)} - (2\alpha - 1)\right| \text{ for } s \in \Pi_0.$$

Therefore, as in [14] the conformal function (2) in Π_0 is called pseudostarlike of order $\alpha \in [0,\, 1)$ and type $\beta \in (0,\, 1]$ if

$$\left|\frac{F'(s)}{F(s)} - 1\right| < \beta \left|\frac{F'(s)}{F(s)} - (2\alpha - 1)\right|, \quad s \in \Pi_0. \tag{4}$$

Lemma 1 ([14]). *If*

$$\sum_{k=1}^{\infty} \{(1+\beta)\lambda_k - \beta(2\alpha - 1) - 1\}|f_k| \leq 2\beta(1 - \alpha) \tag{5}$$

then (2) *is pseudostarlike of order* α *and type* β.

Similarly, a conformal function (2) in Π_0 is said to be *pseudoconvex* if
$$\mathrm{Re}\{F''(s)/F'(s)\} > 0 \text{ for } s \in \Pi_0.$$
In [12] and [13, p. 139] it is proved that if
$$\sum_{k=1}^{\infty} \lambda_k^2 |f_k| \le 1$$
then function (2) is pseudoconvex. Here we call the function (2) pseudoconvex of the order $\alpha \in [0, 1)$ if $\mathrm{Re}\{F''(s)/F'(s)\} > \alpha$, and pseudoconvex of order α and type $\beta \in (0, 1]$ if [14]
$$\left|\frac{F''(s)}{F'(s)} - 1\right| < \beta \left|\frac{F''(s)}{F'(s)} - (2\alpha - 1)\right|, \quad s \in \Pi_0.$$
Since $F''(s)/F'(s) = G'(s)/G(s)$, where $G(s) = e^s + \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\}$ and $g_k = \lambda_k f_k$, the function F is pseudoconvex of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$ if and only if the function G is pseudostarlike of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$. Therefore, from Lemma 1 one can easily obtain the corresponding result for pseudoconvex functions.

Lemma 2 ([14]). *If*
$$\sum_{k=1}^{\infty} \lambda_k\{(1+\beta)\lambda_k - \beta(2\alpha - 1) - 1\}|f_k| \le 2\beta(1-\alpha)$$
then (2) *is pseudoconvex of order* α *and type* β.

Here we investigate the conditions under which equation (1) has solutions that are pseudostarlike or pseudoconvex of order α and type β. We remark that if $b_0 = b_1 = 0$ then such a problem is solved in [12-13] for the case of $\alpha = 0$, $\beta = 1$.

2. Recurrent formulas. Suppose that Dirichlet series (2) satisfies (1). Then
$$(1 + b_1 + c_2)e^s + (b_0 + c_1)e^{2s} + +c_0 e^{3s} + \sum_{k=1}^{\infty}(\lambda_k^2 + b_1\lambda_k + c_2) f_k \exp\{s\lambda_k\} +$$
$$+ \sum_{k=1}^{\infty}(b_0\lambda_k + c_1) f_k \exp\{s(\lambda_k + 1)\} + \sum_{k=1}^{\infty} c_0 f_k \exp\{s(\lambda_k + 2)\} = a_1 e^s + \sum_{n=2}^{\infty} a_n e^{ns}. \quad (6)$$
Since $\lambda_1 > 1$, hence as $s \to -\infty$ we have $(1+b_1+c_2)e^s = (1+o(1))a_1 e^s$, i. e. $1+b_1+c_2 = a_1$. Therefore, (6) implies
$$(b_0 + c_1)e^{2s} + c_0 e^{3s} + (\lambda_1^2 + b_1\lambda_1 + c_2) f_1 \exp\{s\lambda_1\} + \sum_{k=2}^{\infty}(\lambda_k^2 + b_1\lambda_k + c_2) f_k \exp\{s\lambda_k\} +$$
$$+ \sum_{k=1}^{\infty}(b_0\lambda_k + c_1) f_k \exp\{s(\lambda_k + 1)\} + \sum_{k=1}^{\infty} c_0 f_k \exp\{s(\lambda_k + 2)\} = a_2 e^{2s} + \sum_{n=3}^{\infty} a_n e^{ns}. \quad (7)$$
Since $\lambda_1 + 1 > 1$ and $\lambda_2 > \lambda_1$, hence as $s \to -\infty$ we have
$$(\lambda_1^2 + b_1\lambda_1 + c_2) f_1 \exp\{s\lambda_1\} + o(e^{s\lambda_1}) = (a_2 - b_0 - c_1)e^{2s} + o(e^{2s}).$$

Therefore, if $\lambda_1^2+b_1\lambda_1+c_2\neq 0$ then $\lambda_1=2$ and

$$f_1=\frac{a_2-b_0-c_1}{4+2b_1+c_2}. \tag{8}$$

Therefore, (7) implies

$$c_0e^{3s}+(\lambda_2^2+b_1\lambda_2+c_2)f_2\exp\{s\lambda_2\}+\sum_{k=3}^{\infty}(\lambda_k^2+b_1\lambda_k+c_2)f_k\exp\{s\lambda_k\}+$$
$$+(2b_0+c_1)f_1\exp\{3s\}+\sum_{k=2}^{\infty}(b_0\lambda_k+c_1)f_k\exp\{s(\lambda_k+1)\}+$$
$$+\sum_{k=1}^{\infty}c_0f_k\exp\{s(\lambda_k+2)\}=a_3e^{3s}+\sum_{n=4}^{\infty}a_ne^{ns}. \tag{9}$$

Hence, it follows that

$$(\lambda_2^2+b_1\lambda_2+c_2)f_2\exp\{s\lambda_2\}+o(e^{s\lambda_2})=(a_3-c_0-(2b_0+c_1)f_1)e^{3s}+o(e^{3s}),$$

as $s\to-\infty$ and if $\lambda_2^2+b_1\lambda_2+c_2\neq 0$ then $\lambda_2=3$ and

$$f_2=\frac{a_3-c_0}{9+3b_1+c_2}-\frac{2b_0+c_1}{9+3b_1+c_2}f_1. \tag{10}$$

Therefore, (9) implies

$$(\lambda_3^2+b_1\lambda_3+c_2)f_3\exp\{s\lambda_3\}+\sum_{k=4}^{\infty}(\lambda_k^2+b_1\lambda_k+c_2)f_k\exp\{s\lambda_k\}+$$
$$+(3b_0+c_1)f_2e^{4s}+\sum_{k=3}^{\infty}(b_0\lambda_k+c_1)f_k\exp\{s(\lambda_k+1)\}+$$
$$+c_0f_1e^{4s}+\sum_{k=2}^{\infty}c_0f_k\exp\{s(\lambda_k+2)\}=a_4e^{4s}+\sum_{n=5}^{\infty}a_ne^{ns}. \tag{11}$$

Hence, it follows that

$$(\lambda_3^2+b_1\lambda_3+c_2)f_3\exp\{s\lambda_3\}+o(e^{s\lambda_3})=(a_4-c_0f_1-(3b_0+c_1)f_2)e^{3s}+o(e^{3s}),$$

as $s\to-\infty$ and if $\lambda_3^2+b_1\lambda_3+c_2\neq 0$ then $\lambda_3=4$ and

$$f_3=\frac{a_4}{16+4b_1+c_2}-\frac{3b_0+c_1}{16+4b_1+c_2}f_2-\frac{c_0}{16+4b_1+c_2}f_1. \tag{12}$$

Therefore, (11) implies

$$(\lambda_4^2+b_1\lambda_4+c_2)f_4\exp\{s\lambda_4\}+\sum_{k=5}^{\infty}(\lambda_k^2+b_1\lambda_k+c_2)f_k\exp\{s\lambda_k\}+$$
$$+(4b_0+c_1)f_3e^{5s}+\sum_{k=4}^{\infty}(b_0\lambda_k+c_1)f_k\exp\{s(\lambda_k+1)\}+$$

$$+c_0 f_2 e^{5s} + \sum_{k=3}^{\infty} c_0 f_k \exp\{s(\lambda_k+2)\} = a_5 e^{5s} + \sum_{n=6}^{\infty} a_n e^{ns},$$

whence as above it follows that if $\lambda_4^2 + b_1\lambda_4 + c_2 \neq 0$ then $\lambda_4 = 5$ and

$$f_4 = \frac{a_5}{25+5b_1+c_2} - \frac{4b_0+c_1}{25+5b_1+c_2} f_3 - \frac{c_0}{25+5b_1+c_2} f_2.$$

Continuing this process, we will come to the formulas $\lambda_k = k+1$ and

$$f_k = \frac{a_{k+1}}{(k+1)^2+(k+1)b_1+c_2} -$$
$$-\frac{kb_0+c_1}{(k+1)^2+(k+1)b_1+c_2} f_{k-1} - \frac{c_0}{(k+1)^2+(k+1)b_1+c_2} f_{k-2} \qquad (13)$$

for $k \geq 3$, provided $\lambda_k^2 + b_1\lambda_k + c_2 \neq 0$.

Thus, the following statement is correct.

Lemma 3. *If $1 + b_1 + c_2 = a_1$ and $k^2 + kb_1 + c_2 \neq 0$ for all $k \geq 2$ then differential equation* (1) *has the solution*

$$F(s) = e^s + \sum_{k=1}^{\infty} f_k \exp\{s(k+1)\}, \qquad (14)$$

where the coefficients f_1 and f_2 are defined by formulas (8) *and* (10), *and for $k \geq 3$ recurrent formula* (13) *is true.*

2. Pseudostarlikeness. At the first, we remark that for function (14) condition (5) has the form

$$\sum_{k=1}^{\infty} B_k |f_k| \leq 2\beta(1-\alpha), \quad B_k = (1+\beta)k + 2\beta(1-\alpha). \qquad (15)$$

We put $A_k = (k+1)^2 + (k+1)b_1 + c_2$ and suppose that $b_1 \geq 0$ and $c_2 \geq 0$. Then $A_k > 0$ for all $k \geq 1$ and from (8), (10) and (13) we get $f_1 = \frac{a_2-b_0-c_1}{A_1}$, $f_2 = \frac{a_3-c_0}{A_2} - \frac{2b_0+c_1}{A_2} f_1$ and $f_k = \frac{a_{k+1}}{A_k} - \frac{kb_0+c_1}{A_k} f_{k-1} - \frac{c_0}{A_k} f_{k-2}$ for $k \geq 3$. Therefore,

$$\sum_{k=1}^{\infty} B_k|f_k| \leq B_1|f_1| + B_2|f_2| + \sum_{k=1}^{\infty} B_k|f_k| \leq$$
$$\leq B_1|f_1| + B_2|f_2| + \sum_{k=3}^{\infty} B_k \frac{|a_{k+1}|}{A_k} + \sum_{k=3}^{\infty} B_k \frac{k|b_0|+|c_1|}{A_k}|f_{k-1}| + \sum_{k=3}^{\infty} B_k \frac{|c_0|}{A_k}|f_{k-2}| =$$
$$= B_1|f_1| + B_2|f_2| + \sum_{k=3}^{\infty} B_k \frac{|a_{k+1}|}{A_k} + \sum_{k=2}^{\infty} B_{k+1} \frac{(k+1)|b_0|+|c_1|}{A_{k+1}}|f_k| + \sum_{k=1}^{\infty} B_{k+2} \frac{|c_0|}{A_{k+2}}|f_k| =$$
$$= B_1|f_1| + B_2|f_2| + \sum_{k=3}^{\infty} B_k \frac{|a_{k+1}|}{A_k} - B_2 \frac{|2|b_0|+c_1|}{A_2}|f_1| +$$
$$+ \sum_{k=1}^{\infty} \left(B_{k+1} \frac{(k+1)|b_0|+|c_1|}{A_{k+1}} + B_{k+2} \frac{|c_0|}{A_{k+2}} \right) |f_k|.$$

Since $B_2|f_2| - B_2\frac{2|b_0|+|c_1|}{A_2}|f_1| \le B_2\frac{|a_3-c_0|}{A_2}$, hence we obtain

$$\sum_{k=1}^{\infty} B_k|f_k| \le B_1\frac{|a_2-b_0-c_1|}{A_1} + B_2\frac{|a_3-c_0|}{A_2} + \sum_{k=3}^{\infty} B_k\frac{|a_{k+1}|}{A_k} + \\ + \sum_{k=1}^{\infty}\left(\frac{B_{k+1}}{B_k}\frac{(k+1)|b_0|+|c_1|}{A_{k+1}} + \frac{B_{k+2}}{B_k}\frac{|c_0|}{A_{k+2}}\right)B_k|f_k|. \tag{16}$$

If we put

$$\eta = \frac{2(1+\beta)+2\beta(1-\alpha)}{1+\beta+2\beta(1-\alpha)}\frac{3|b_0|+|c_1|+2|c_0|}{9+3b_1}$$

and

$$Q = ‘\frac{1+\beta+2\beta(1-\alpha)}{4+2b_1+c_2}(|a_2-b_0-c_1|+2|a_3-c_0|) + \sum_{k=3}^{\infty}\frac{(1+\beta)k+2\beta(1-\alpha)}{(k+1)^2+(k+1)b_1+c_2}|a_{k+1}|$$

then using (16) we can prove the following theorem.

Theorem 1. *Let $b_1 \ge 0$, $c_2 \ge 0$ and $a_1 = 1+b_1+c_2$. If*

$$Q \le 2\beta(1-\eta)(1-\alpha) \tag{17}$$

then differential equation (1) *has solution* (14) *which is pseudostarlike in Π_0 of order α and type β.*

Proof. The conditions $b_1 \ge 0$ and $c_2 \ge 0$ imply $k^2+kb_1+c_2 \ne 0$ for all $k \ge 2$ and, thus, the conditions of Lemma 3 are valid.

Since $\frac{B_{k+1}}{B_k} = 1 + \frac{1+\beta}{(1+\beta)k+2\beta(1-\alpha)} \downarrow 1$ as $k \to \infty$, we have $\frac{B_{k+1}}{B_k} \le \frac{B_2}{B_1}$ and $\frac{B_{k+2}}{B_k} \le \frac{B_3}{B_1}$ for all $k \ge 1$. Also, for $k \ge 1$

$$\frac{(k+1)|b_0|+|c_1|}{A_{k+1}} = \frac{(k+1)|b_0|+|c_1|}{(k+2)^2+(k+2)b_1+c_2} \le \frac{(k+1)|b_0|+|c_1|}{(k+2)^2+(k+2)b_1} \le \\ \le \frac{(k+2)|b_0|}{(k+2)^2+(k+2)b_1} + \frac{|c_1|}{(k+2)^2+(k+2)b_1} \le \frac{|b_0|}{3+b_1} + \frac{|c_1|}{9+3b_1} = \frac{3|b_0|+|c_1|}{9+3b_1}$$

and

$$\frac{|c_0|}{A_{k+2}} = \frac{|c_0|}{(k+3)^2+(k+3)b_1+c_2} \le \frac{|c_0|}{(k+3)^2+(k+3)b_1} \le \frac{|c_0|}{16+4b_1}.$$

Thus,

$$\frac{B_{k+1}}{B_k}\frac{(k+1)|b_0|+|c_1|}{A_{k+1}} + \frac{B_{k+2}}{B_k}\frac{|c_0|}{A_{k+2}} \le \\ \le \frac{2(1+\beta)+2\beta(1-\alpha)}{1+\beta+2\beta(1-\alpha)}\frac{3|b_0|+|c_1|}{9+3b_1} + \frac{3(1+\beta)+2\beta(1-\alpha)}{1+\beta+2\beta(1-\alpha)}\frac{|c_0|}{16+4b_1}$$

and, since $\frac{3(1+\beta)+2\beta(1-\alpha)}{16+4b_1} \le 2\frac{2(1+\beta)+2\beta(1-\alpha)}{9+3b_1}$, we have

$$\frac{B_{k+1}}{B_k}\frac{(k+1)|b_0|+|c_1|}{A_{k+1}} + \frac{B_{k+2}}{B_k}\frac{|c_0|}{A_{k+2}} \le \eta. \tag{18}$$

Since $\frac{B_2}{A_2} = \frac{2(1+\beta)++2\beta(1-\alpha)}{9+3b_1+c_2} \le 2\frac{B_1}{A_1} = 2\frac{1+\beta+2\beta(1-\alpha)}{4+2b_1+c_2}$, we have

$$B_1\frac{|a_2-b_0-c_1|}{A_1} + B_2\frac{|a_3-c_0|}{A_2} + \sum_{k=3}^{\infty} B_k\frac{|a_{k+1}|}{A_k} =$$

$$= (1+\beta+2\beta(1-\alpha))\frac{|a_2-b_0-c_1|}{4+2b_1+c_2} + (2(1+\beta)+2\beta(1-\alpha))\frac{|a_3-c_0|}{9+3b_1+c_2} +$$

$$+\sum_{k=3}^{\infty}((1+\beta)k+2\beta(1-\alpha))\frac{|a_{k+1}|}{(k+1)^2+(k+1)b_1+c_2} \le Q. \quad (19)$$

From (16), (18) and (19) we get $\sum_{k=1}^{\infty} B_k|f_k| \le Q + \sum_{k=1}^{\infty} \eta B_k|f_k|$. Since condition (17) implies $\eta < 1$, it follows that $(1-\eta)\sum_{k=1}^{\infty} B_k|f_k| \le Q \le 2\beta(1-\eta)(1-\alpha)$ and, thus, condition (15) holds. By Lemma 1 function (14) is pseudoconvex in Π_0 of order α and type β. □

3. Pseudoconvexity. By Lemma 2 function (14) is pseudoconvex of order α and type β if $\sum_{k=1}^{\infty} kB_k|f_k| \le 2\beta(1-\alpha)$. As above instead (16) we have

$$\sum_{k=1}^{\infty} kB_k|f_k| \le B_1\frac{|a_2-b_0-c_1|}{A_1} + 2B_2\frac{|a_3-c_0|}{A_2} + \sum_{k=3}^{\infty} kB_k\frac{|a_{k+1}|}{A_k} +$$

$$+\sum_{k=1}^{\infty}\left(\frac{(k+1)B_{k+1}}{kB_k}\frac{(k+1)|b_0|+|c_1|}{A_{k+1}} + \frac{(k+2)B_{k+2}}{kB_k}\frac{|c_0|}{A_{k+2}}\right)kB_k|f_k|. \quad (20)$$

As above, we have $\frac{(k+1)B_{k+1}}{kB_k} \le \frac{2B_2}{B_1}$ and $\frac{(k+2)B_{k+2}}{kB_k} \le \frac{3B_3}{B_1}$ for all $k \ge 1$ and, therefore,

$$\frac{(k+1)B_{k+1}}{kB_k}\frac{(k+1)|b_0|+|c_1|}{A_{k+1}} + \frac{(k+2)B_{k+2}}{kB_k}\frac{|c_0|}{A_{k+2}} \le$$

$$\le \frac{4(1+\beta)+4\beta(1-\alpha)}{1+\beta+2\beta(1-\alpha)}\frac{3|b_0|+|c_1|}{9+3b_1} + \frac{9(1+\beta)+6\beta(1-\alpha)}{1+\beta+2\beta(1-\alpha)}\frac{|c_0|}{16+4b_1} \le \eta^*, \quad (21)$$

where

$$\eta^* = \frac{4(1+\beta)+4\beta(1-\alpha)}{1+\beta+2\beta(1-\alpha)}\frac{3|b_0|+|c_1|+2|c_0|}{9+3b_1}.$$

Finally, since $\frac{B_2}{A_2} \le 2\frac{B_1}{A_1}$, as above we get

$$B_1\frac{|a_2-b_0-c_1|}{A_1} + 2B_2\frac{|a_3-c_0|}{A_2} + \sum_{k=3}^{\infty} kB_k\frac{|a_{k+1}|}{A_k} \le Q^*, \quad (22)$$

where

$$Q^* = \frac{1+\beta+2\beta(1-\alpha)}{4+2b_1+c_2}(|a_2-b_0-c_1|+4|a_3-c_0|) + \sum_{k=3}^{\infty}\frac{(1+\beta)k^2+2\beta(1-\alpha)k}{(k+1)^2+(k+1)b_1+c_2}|a_{k+1}|.$$

Using (20), (21) and (22) easy to prove the following theorem.

Theorem 2. *Let $b_1 \ge 0$, $c_2 \ge 0$ and $a_1 = 1+b_1+c_2$. If $Q^* \le 2\beta(1-\eta^*)(1-\alpha)$ then differential equation* (1) *has solution* (14) *pseudoconvex in Π_0 of order α and type β.*

4. Growth. If $a_1(z)$ and $a_0(z) \not\equiv 0$ are entire functions then every solution of the homogeneous linear differential equation $w'' + a_1(z)w' + a_0(z)w = 0$ is entire, and in order to study its growth there is preferably used the value distribution theory of memorphic functions or results of the Wiman-Valiron method (see, for example, [15, p. 114–144]). For Dirichlet series such results are obtained in [16–19].

Using formula (13) here we considerably simpler will obtain information about the growth of entire solution of differential equation (1). Suppose that $b_1 \ge 0$, $c_2 \ge 0$, $b_0 \le 0$, $c_0 \le 0$, $a_2 - b_0 - c_1 > 0$ and $a_{k+1} = 0$ for all $k \ge k_0 \ge 3$. Then $f_k > 0$ for all $k \ge 1$ and (13) implies for $k \ge k_0$

$$f_k = \frac{k|b_0| + |c_1|}{A_k} f_{k-1} + \frac{|c_0|}{A_k} f_{k-2} = \frac{(1+o(1))|b_0|}{k} f_{k-1} + \frac{(1+o(1))|c_0|}{k(k-1)} f_{k-2}$$

as $k_0 \le k \to \infty$, whence

$$\frac{kf_k}{f_{k-1}} = (1+o(1))|b_0| + (1+o(1))|c_0| \frac{f_{k-2}}{(k-1)f_{k-1}}, \quad k \to \infty. \tag{23}$$

We put $A = \varlimsup\limits_{k\to\infty} \frac{kf_k}{f_{k-1}}$ and $a = \varliminf\limits_{k\to\infty} \frac{kf_k}{f_{k-1}}$. Then from (23) we get $A = |b_0| + |c_0|/a$ and $a = |b_0| + |c_0|/A$, i. e. $A^2 - |b_0|A - |c_0| = 0$ and $a^2 - |b_0|a - |c_0| = 0$. Thus, $A \ge 0$ and $a \ge 0$ are the roots of the quadratic equation $x^2 - |b_0|x - |c_0| = 0$ and, therefore, $A = a = \frac{1}{2}\left(|b_0| + \sqrt{|b_0|^2 + 4|c_0|}\right)$. Hence, it follows that $f_k = \frac{(1+o(1))af_{k-1}}{k}$ as $k \to \infty$. Therefore, for every $\varepsilon \in (0, a)$ and $k \ge k_1 = k_1(\varepsilon)$

$$\frac{(a-\varepsilon)^k}{k!} \le f_k \le \frac{(a-\varepsilon)^k}{k!}. \tag{24}$$

For entire Dirichlet series (2) we put $M(\sigma, F) = \sup\{|F(\sigma + it)|\colon t \in \mathbb{R}\}$. The values $\varrho_R = \varlimsup\limits_{\sigma\to+\infty} \frac{\ln\ln M(\sigma,F)}{\sigma}$, $\lambda_R = \varliminf\limits_{\sigma\to+\infty} \frac{\ln\ln M(\sigma,F)}{\sigma}$, $T_R = \varlimsup\limits_{\sigma\to+\infty} e^{-\varrho_R\sigma} \ln M(\sigma, F)$ and $t_R = \varliminf\limits_{\sigma\to+\infty} e^{-\varrho_R\sigma} \ln M(\sigma, F)$ are called R-order, lower R-order, R-type and lower R-type accordingly. Since $f_k > 0$ for all $k \ge 1$, we have $M(\sigma, F) = F(\sigma) = O(e^{k_1\sigma}) + e^{\sigma} \sum_{k=k_1}^{\infty} f_k e^{k\sigma}$ as $\sigma \to +\infty$. From (24) it follows that

$$O(e^{k_1\sigma}) + \exp\{(a-\varepsilon)e^{\sigma}\} = \sum_{k=k_1}^{\infty} \frac{(a-\varepsilon)^k}{k!} e^{k\sigma} \le \sum_{k=k_1}^{\infty} f_k e^{k\sigma} \le$$

$$\le \sum_{k=k_1}^{\infty} \frac{(a+\varepsilon)^k}{k!} e^{k\sigma} = O(e^{k_1\sigma}) + \exp\{((a+\varepsilon)e^{\sigma}\}, \quad \sigma \to +\infty,$$

whence $(a-\varepsilon)e^{\sigma} + o(1) \le \ln M(\sigma, F) \le (a+\varepsilon)e^{\sigma} + o(1)$ as $\sigma \to +\infty$, and in view of the arbitrariness of ε we get $\varrho_R = \lambda_R = 1$ and $T_R = t_R = a$. Thus, the following statement is proved.

Proposition 1. *Let* $b_1 \ge 0$, $c_2 \ge 0$, $b_0 \le 0$, $c_0 \le 0$, $a_2 - b_0 - c_1 > 0$, $a_1 = 1 + b_1 + c_2$ *and* $a_{k+1} = 0$ *for all* $k \ge k_0 \ge 3$. *Then differential equation* (1) *has entire solution* (14) *such that* $\ln M(\sigma, F) = \frac{1+o(1)}{2}\left(|b_0| + \sqrt{|b_0|^2 + 4|c_0|}\right)$ *as* $\sigma \to +\infty$. *If* $Q \le 2\beta(1-\eta)(1-\alpha)$ *then function* (14) *is pseudostarlike in* Π_0 *of order* α *and type* β. *If* $Q^* \le 2\beta(1-\eta^*)(1-\alpha)$ *then function* (14) *is pseudoconvex in* Π_0 *of order* α *and type* β.

REFERENCES

1. Goluzin G.M., *Geometric theory of functions of a complex variable*, M.: Nauka, 1966 (in Russian). Engl. transl. Providence: AMS, 1969.
2. Kaplan W., *Close-to-convex schlicht functions*, Michigan Math. J., **1** (1952), №2, 169–185.
3. Shah S.M., *Univalence of a function f and its successive derivatives when f satisfies a differential equation, II*, J. Math. anal. and appl., **142** (1989), 422–430.
4. Sheremeta Z.M., *Close-to-convexity of entire solutions of a differential equation*, Matem. Metody ta Fiz.-Mekh. Polya, **42** (1999), №3, 31–35 (in Ukrainian).
5. Sheremeta Z.M., *The properties of entire solutions of one differential equation*, Diff. Equ., **36** (2000), №8, 1155–1161. doi: 10.1007/BF02754183
6. Sheremeta Z.M., *On entire solutions of a differential equation*, Mat. Stud., **14** (2000), №1, 54–58.
7. Sheremeta Z.M., *On the close-to-convexity of entire solutions of a differential equation,* Visnyk Lviv Univ. Series Mech. Math., **58** (2000), 98–106 (in Ukrainian).
8. Sheremeta Z.M., Sheremeta M.N., *Closeness to convexity for entire solutions of a differential equation*, Diff. Equ., **38** (2002), №4, 496-501. doi: 10.1023/A:1016355531151
9. Sheremeta Z.M., Sheremeta M.M., *Convexity of entire solutions of a differential equations*, Matem. Metody ta Fiz.-Mekh. Polya, **47** (2004), №2, 186–191 (in Ukrainian).
10. Trukhan Yu., Sheremeta M., *Closeness-to-convexity of solutions of a second order nonhomogeneous differential equation*, Visnyk Lviv Univ. Series Mech. Math., **88**, (2019), 98–106. http://dx.doi.org/10.30970/vmm.2019.88.098-106 (in Ukrainian)
11. Sheremeta M.M., Trukhan Yu.S. *Properties of analytic solutions of a differential equation*, Mat. Stud., **52** (2019), №2, 138–143. doi: 10.30970/ms.52.2.138-143
12. Holovata O.M., Mulyava O.M., Sheremeta M.M., *Pseudostarlike, pseudoconvex, and close-to-pseudoconvex Dirichlet series satisfying differential equations with exponential coefficients,* J. Math. Sci. (United States), **249** (2020), №3, 369–388. doi: 10.1007/s10958-020-04948-1
13. Sheremeta M.M., Geometric properties of analytic solutions of differential equations. Lviv: Publisher I.E. Chyzhykov, 2019.
14. Sheremeta M.M., *Pseudostarlike and pseudoconvex Dirichlet series of the order α and the type β*, Mat. Stud., **54** (2020), №1, 23–31. doi: https://doi.org/10.30970/ms.54.1.23-31
15. Wittich H., Neuere Untersuchungen uber eindeutige analytische Functionen, Springer-Verlag, Berlin, 1955.
16. Sheremeta M.N., *Asymptotic properties of entire functions defined by Dirichlet series and of their derivatives,* Ukr. Math. J., **31** (1979), №6, 558–564. doi: 10.1007/BF01092538
17. Sheremeta M.N., *On the derivative of an entire Dirichlet series,* Mathematics of the USSR - Sbornik, **65** (1990), №1, 133–145. doi: 10.1070/SM1990v065n01ABEH002076
18. Salo T.M., Skaskiv O.B., Stasyuk Ya.Z., *On a central exponent of entire Dirichlet series*, Mat. Stud., **19** (2003), №1, 61–72.
19. Skaskiv O.B., Stasyuk Ya.Z., *On the Wiman theorem for absolutely convergent Dirichlet series*, Mat. Stud., **20** (2003), №2, 133–142.

Ivan Franko National University of Lviv, Ukraine
m.m.sheremeta@gmail.com

Received 30.03.2021
Revised 15.08.2021