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OPTIMAL RECOVERY OF OPERATOR SEQUENCES

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In this paper we solve two problems of optimal recovery based on information given with an error. The first one is the problem of optimal recovery of the class

$$W_q^T = \{(t_1h_1, t_2h_2, \ldots) : \|h\|_{\ell_q} \le 1\},$$

where $1 \le q < \infty$ and $t_1 \ge t_2 \ge \ldots \ge 0$ are given, in the space ℓ_q. Information available about a sequence $x \in W_q^T$ is provided either (i) by an element $y \in \mathbb{R}^n$, $n \in \mathbb{N}$, whose distance to the first n coordinates $(x_1, \ldots, x_n)$ of x in the space ℓ_r^n, $0 < r \le \infty$, does not exceed given $\varepsilon \ge 0$, or (ii) by a sequence $y \in \ell_\infty$ whose distance to x in the space ℓ_r does not exceed ε. We show that the optimal method of recovery in this problem is either operator Φ_m^* with some $m \in \mathbb{Z}_+$ ($m \le n$ in case $y \in \ell_r^n$), where

$$\Phi_m^*(y) = \left\{ y_1 \left(1 - \frac{t_{m+1}^q}{t_1^q}\right), \ldots, y_m\left(1 - \frac{t_{m+1}^q}{t_m^q}\right), 0, \ldots \right\}, \quad y \in \mathbb{R}^n \text{ or } y \in \ell_\infty,$$

or convex combination $(1-\lambda)\Phi_{m+1}^* + \lambda\Phi_m^*$.

The second one is the problem of optimal recovery of the scalar product operator acting on the Cartesian product $W_{p,q}^{T,S}$ of classes W_p^T and W_q^S, where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $s_1 \ge s_2 \ge \ldots \ge 0$ are given. Information available about elements $x \in W_p^T$ and $y \in W_q^S$ is provided by elements $z, w \in \mathbb{R}^n$ such that the distance between vectors $(x_1y_1, x_2y_2, \ldots, x_ny_n)$ and $(z_1w_1, \ldots, z_nw_n)$ in the space ℓ_r^n does not exceed ε. We show that the optimal method of recovery is delivered either by operator Ψ_m^* with some $m \in \{0, 1, \ldots, n\}$, where

$$\Psi_m^* = \sum_{k=1}^m z_kw_k\left(1 - \frac{t_{m+1}s_{m+1}}{t_ks_k}\right), \quad z, w \in \mathbb{R}^n,$$

or by convex combination $(1-\lambda)\Psi_{m+1}^* + \lambda\Psi_m^*$.

As an application of our results we consider the problem of optimal recovery of classes in Hilbert spaces by the Fourier coefficients of its elements known with an error measured in the space ℓ_p with $p > 2$.

1. Introduction. Let X, Z be complex linear spaces, Y be a complex normed space, $A\colon X \to Y$ be an operator, in general non-linear, with domain $\mathcal{D}(A)$, $W \subset \mathcal{D}(A)$ be some class of elements. Denote by $\mathfrak{B}(Z)$ the set of non-empty subsets of Z, and let $I\colon \overline{\operatorname{span} W} \to \mathfrak{B}(Z)$ be a given mapping called *information.* When saying that information about element $x \in W$ is available we mean that some element $z \in I(x)$ is known. An arbitrary mapping $\Phi\colon Z \to Y$ is called a *method of recovery* of the operator A. Define the *error of method of recovery* Φ of the operator A on the set W given information I:

$$\mathcal{E}(A, W, I, \Phi) = \sup_{x\in W} \sup_{z\in I(x)} \|Ax - \Phi(z)\|_Y. \tag{1}$$

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The quantity

$$\mathcal{E}(A,W,I) = \inf_{\Phi\colon Z\to Y} \mathcal{E}(A,W,I,\Phi) \tag{2}$$

is called the *error of optimal recovery* of the operator A on elements of class W given information I. The method Φ^* delivering inf in (2) (if any exists) is called *optimal.*

The problem of recovery of linear operators in Hilbert spaces based on exact information was studied in [14]. In the case when information mapping I has the form $Ix = i(x) + B$, where i is a linear operator and B is a ball of some radius defining information error, recovery problem (2) was considered in [12] (see also [15]–[16]). Alternative approach to the study of optimal recovery problems based on standard principles of convex optimization was proposed in [10]. In [12] it was shown that among optimal methods of recovery there exists a linear one, and in [10] explicit representations for optimal methods of recovery were found in cases when the error of information is measured with respect to the uniform metric. For a thorough overview of optimal recovery and related problems we refer the reader to books [17, 16] and survey [1].

Remark that results of the present work supplement and generalize results of paper [10] on optimal recovery of functions and its derivatives and paper [7].

2. Elementary lower estimate. Let us present a trivial yet effective lower estimate for the error of optimal recovery (2). Denote by θ_Z the null element of the space Z and let I be some information mapping.

Lemma 1. *Let $\theta_Z \in I(W)$. Then*

$$\mathcal{E}(A,W,I) \ge \frac{1}{2} \sup_{\substack{x,y\in W:\\ \theta_Z\in Ix\cap Iy}} \|Ax - Ay\|_Y.$$

Proof. Indeed, for every method of recovery $\Phi\colon Z \to Y$,

$$\mathcal{E}(A,W,I,\Phi) \ge \sup_{\substack{x\in W:\\ \theta_Z\in Ix}} \|Ax - \Phi(\theta_Z)\|_Y \ge \frac{1}{2}\Bigg(\sup_{\substack{x\in W:\\ \theta_Z\in Ix}} \|Ax - \Phi(\theta_Z)\|_Y +$$

$$+ \sup_{\substack{y\in W:\\ \theta_Z\in Iy}} \|Ay - \Phi(\theta_Z)\|_Y \Bigg) \ge \frac{1}{2} \sup_{\substack{x,y\in W:\\ \theta_Z\in Ix\cap Iy}} \|Ax - Ay\|_Y.$$

Taking inf over methods Φ we finish the proof. □

From Lemma 1 we easily derive the following consequences.

Corollary 1. *Let A be an odd operator, $\tilde{x} \in W$ be such that $-\tilde{x} \in W$ and $\theta_Z \in I(\tilde{x}) \cap I(-\tilde{x})$. Then*

$$\mathcal{E}(A,W,I) \ge \|A\tilde{x}\|_X.$$

Corollary 2. *Let $Y = \mathbb{C}$, R be a (complex) normed space, $X = R \times R^*$, $W_1 \subset R$ and $W_2 \subset R^*$ be given classes. Also, let A be the scalar product of elements in $R \times R^*$, i.e. $A(x,y) = \langle y, x\rangle$, $x \in R$ and $y \in R^*$. Assume that there exist $\tilde{x}_1 \in W_1$ and $\tilde{x}_2 \in W_2$ such that either*

$$-\tilde{x}_1 \in W_1 \quad \text{and} \quad \theta_Z \in I(\tilde{x}_1, \tilde{x}_2) \cap I(-\tilde{x}_1, \tilde{x}_2)$$

or

$$-\tilde{x}_2 \in W_2 \quad \text{and} \quad \theta_Z \in I(\tilde{x}_1, \tilde{x}_2) \cap I(\tilde{x}_1, -\tilde{x}_2).$$

Then

$$\mathcal{E}(A, W_1 \times W_2, I) \ge |\langle \tilde{x}_2, \tilde{x}_1\rangle|.$$

Remark that similar and related lower estimates were established in many papers (see, *e.g.*, [10, 5]).

3. Optimal recovery of sequences. Let us present notations used in the rest of the paper. Let $1 \le p, q \le \infty$, ℓ_q be the standard space of sequences $x = \{x_k\}_{k=1}^{\infty}$, complex-valued in general, with corresponding norm $\|x\|_q$, and ℓ_q^n, $n \in \mathbb{N}$, be the spaces of finite sequences. Denote by θ the null element of ℓ_q and by θ^n the null element of ℓ_q^n.

For a given non-increasing sequence $t = \{t_k\}_{k=1}^{\infty}$ of non-negative numbers, consider a bounded operator $T\colon \ell_q \to \ell_q$ defined as follows

$$Th := \{t_k h_k\}_{k=1}^{\infty}, \qquad h \in \ell_q,$$

and the class

$$W_q^T := \{x = Th : \ h \in \ell_q, \ \|h\|_q \le 1\}.$$

In this section we will study the problem of optimal recovery of identity operator $A = \mathrm{id}_X$ on the class W_q^T, also called the problem of optimal recovery of class W_q^T, when the information mapping I is given in one of the following forms:

1. $Ix = I_{\bar{\varepsilon}}^n x = (x_1, \ldots, x_n) + B[\varepsilon_1] \times B[\varepsilon_n]$, where $n \in \mathbb{N}$, $\varepsilon_1, \ldots, \varepsilon_n \ge 0$ and $B[\varepsilon_j] = [-\varepsilon_j, \varepsilon_j]$;
2. $Ix = I_{\varepsilon,p}^n x = (x_1, \ldots, x_n) + B\left[\varepsilon, \ell_p^n\right]$, where $n \in \mathbb{N}$, $\varepsilon \ge 0$ and $B\left[\varepsilon, \ell_p^n\right]$ is the ball of radius ε in the space ℓ_p^n centered at θ^n;
3. $Ix = I_{\varepsilon,p} x = x + B[\varepsilon, \ell_p]$, where $\varepsilon \ge 0$ and $B[\varepsilon, \ell_p]$ is the ball of radius ε in the space ℓ_p centered at θ.

To simplify further notations, we set

$$\mathcal{E}(W, I) := \mathcal{E}(\mathrm{id}_X, W, I), \qquad \mathcal{E}(W, I, \Phi) := \mathcal{E}(\mathrm{id}_X, W, I, \Phi),$$

and, for $m \in \mathbb{N}$ and $q < \infty$, introduce the method of recovery $\Phi_m^*\colon \ell_p \to \ell_q$:

$$\Phi_m^*(a) = \left\{a_1\Big(1 - \frac{t_{m+1}^q}{t_1^q}\Big), \ldots, a_m\Big(1 - \frac{t_{m+1}^q}{t_m^q}\Big), 0, \ldots\right\}, \quad a \in \ell_p,$$

that would be optimal in many situations. Also, we set $\Phi_0^*(a) := \theta$, $a \in \ell_p$.

In what follows we define $\sum_{k=1}^{0} a_k := 0$ for numeric a_k's. In addition, for simplicity we assume that $t_k > 0$ for every $k \in \mathbb{N}$. Results in this paper remain true in the case when t_k can attain zero value with the substitution of $1/t_k$ with $+\infty$ and t_s/t_k, $s \ge k$ with 1.

3.1. Information mapping $I_{\bar{\varepsilon}}^n(x) = (x_1, \ldots, x_n) + B[\varepsilon_1] \times \ldots \times B[\varepsilon_n]$.

Theorem 1. *Let $n \in \mathbb{N}$, $1 \le q < \infty$ and $\varepsilon_1, \ldots, \varepsilon_n \ge 0$. If $1 - \sum_{k=1}^{n} \frac{\varepsilon_k^q}{t_k^q} \ge 0$, we set $m = n$. Otherwise we choose $m \in \mathbb{Z}_+$, $m \le n$, to be such that $1 - \sum_{k=1}^{m} \frac{\varepsilon_k^q}{t_k^q} \ge 0$ and $1 - \sum_{k=1}^{m+1} \frac{\varepsilon_k^q}{t_k^q} < 0$. Then*

$$\mathcal{E}\left(W_q^T, I_{\bar{\varepsilon}}^n\right) = \mathcal{E}\left(W_q^T, I_{\bar{\varepsilon}}^n, \Phi_m^*\right) = \Big(t_{m+1}^q + \sum_{k=1}^{m} \Big(1 - \frac{t_{m+1}^q}{t_k^q}\Big)\varepsilon_k^q\Big)^{1/q}.$$

Proof. Using convexity inequality, relations $|x_k - a_k| \le \varepsilon_k$, $k = 1, \dots, n$, and monotony of the sequence t, we obtain that, for $x = Th \in W_q^T$ and $a \in I_{\bar\varepsilon}^n(x)$,

$$\|x - \Phi_m^*(a)\|_q^q = \sum_{k=1}^m \left| x_k - a_k \left(1 - \frac{t_{m+1}^q}{t_k^q}\right)\right|^q + \sum_{k=m+1}^\infty |x_k|^q =$$

$$= \sum_{k=1}^m \left| \Big(1 - \frac{t_{m+1}^q}{t_k^q}\Big)(x_k - a_k) + \frac{t_{m+1}^q}{t_k^q} x_k \right|^q + \sum_{k=m+1}^\infty t_k^q |h_k|^q \le$$

$$\le \sum_{k=1}^m \left(\Big(1 - \frac{t_{m+1}^q}{t_k^q}\Big)|x_k - a_k|^q + \frac{t_{m+1}^q}{t_k^q}|x_k|^q \right) + t_{m+1}^q \sum_{k=m+1}^\infty |h_k|^q =$$

$$= \sum_{k=1}^m \Big(1 - \frac{t_{m+1}^q}{t_k^q}\Big)|x_k - a_k|^q + \sum_{k=1}^m t_{m+1}^q |h_k|^q + t_{m+1}^q \sum_{k=m+1}^\infty |h_k|^q \le$$

$$\le \sum_{k=1}^m \Big(1 - \frac{t_{m+1}^q}{t_k^q}\Big)\varepsilon_k^q + t_{m+1}^q.$$

To obtain the lower estimate, we choose

$$u_k := \frac{\varepsilon_k}{t_k}, \quad k = 1, \dots, m, \quad \text{and} \quad u_{m+1} := \Big(1 - \sum_{k=1}^m \frac{\varepsilon_k^q}{t_k^q}\Big)^{1/q},$$

and consider $h^* = (u_1, \dots, u_{m+1}, \dots) \in l_q$. It is clear that $Th^* \in W_q^T$, as $\|h^*\|_q \le 1$. Furthermore, by the choice of number m we have that $\theta \in I_{\bar\varepsilon}^n(Th^*)$. Hence, by Corollary 1,

$$\left(\mathcal{E}\left(W_q^T, I_{\bar\varepsilon}^n\right)\right)^q \ge \|Th^*\|_q^q = \sum_{k=1}^m t_k^q u_k^q + t_{m+1}^q u_{m+1}^q =$$

$$= \sum_{k=1}^m \varepsilon_k^q + t_{m+1}^q \Big(1 - \sum_{k=1}^m \frac{\varepsilon_k^q}{t_k^q}\Big) = t_{m+1}^q + \sum_{k=1}^m \varepsilon_k^q \Big(1 - \frac{t_{m+1}^q}{t_k^q}\Big),$$

which finishes the proof. □

3.2. Information mapping $I_{\varepsilon,p}^n(x) = (x_1, \dots, x_n) + B\left[\varepsilon, \ell_p^n\right]$.

We consider three cases separately: $p = \infty$, $p \le q$ and $p > q$.

3.2.1. Case $p = \infty$.

Setting $\varepsilon_1 = \ldots = \varepsilon_n = \varepsilon$, we obtain from Theorem 1 the following corollary.

Theorem 2. *Let $n \in \mathbb{N}$, $1 \le q < \infty$ and $\varepsilon \ge 0$. If $1 - \varepsilon^q \sum_{k=1}^n \frac{1}{t_k^q} \ge 0$ then we set $m = n$. Otherwise we choose $m \in \mathbb{Z}_+$, $m \le n$, to be such that $1 - \varepsilon^q \sum_{k=1}^m \frac{1}{t_k^q} \ge 0$ and $1 - \varepsilon^q \sum_{k=1}^{m+1} \frac{1}{t_k^q} < 0$. Then*

$$\mathcal{E}\left(W_q^T, I_{\varepsilon,\infty}^n\right) = \mathcal{E}\left(W_q^T, I_{\varepsilon,\infty}^n, \Phi_m^*\right) = \left(t_{m+1}^q + \varepsilon^q \sum_{k=1}^m \Big(1 - \frac{t_{m+1}^q}{t_k^q}\Big)\right)^{1/q}.$$

3.2.2. Case $0 < p \le q$.

Theorem 3. *Let $n \in \mathbb{N}$, $1 \le q < \infty$ and $0 < p \le q$. If $\varepsilon \in [0, t_1]$ then*

$$\mathcal{E}\left(W_q^T, I_{\varepsilon,p}^n\right) = \mathcal{E}\left(W_q^T, I_{\varepsilon,p}^n, \Phi_n^*\right) = \left(t_{n+1}^q + \varepsilon^q\Big(1 - \frac{t_{n+1}^q}{t_1^q}\Big)\right)^{1/q},$$

and if $\varepsilon > t_1$ then $\mathcal{E}\left(W_q^T, I_{\varepsilon,p}^n\right) = \mathcal{E}\left(W_q^T, I_{\varepsilon,p}^n, \Phi_0^*\right) = t_1$.

Proof. First, consider the case $\varepsilon \in [0, t_1]$. For $x = Th \in W_q^T$, $\|h\|_q \le 1$, and $a \in I_{\varepsilon,p}^n(x)$, we have

$$\|x - \Phi_n^*(a)\|_q^q = \sum_{k=1}^n \left| x_k - a_k\Big(1 - \frac{t_{n+1}^q}{t_k^q}\Big)\right|^q + \sum_{k=n+1}^\infty |x_k|^q =$$

$$= \sum_{k=1}^n \left| \Big(1 - \frac{t_{n+1}^q}{t_k^q}\Big)(x_k - a_k) + \frac{t_{n+1}^q}{t_k^q} x_k\right|^q + \sum_{k=n+1}^\infty t_k^q |h_k|^q \le$$

$$\le \sum_{k=1}^n \left(\Big(1 - \frac{t_{n+1}^q}{t_k^q}\Big)|x_k - a_k|^q + \frac{t_{n+1}^q}{t_k^q}|x_k|^q\right) + t_{n+1}^q \sum_{k=n+1}^\infty |h_k|^q =$$

$$= \sum_{k=1}^n \Big(1 - \frac{t_{n+1}^q}{t_k^q}\Big)\big(|x_k - a_k|^p\big)^{q/p} + t_{n+1}^q \sum_{k=1}^\infty |h_k|^q \le$$

$$\le \Big(1 - \frac{t_{n+1}^q}{t_1^q}\Big)\Big(\sum_{k=1}^n |x_k - a_k|^p\Big)^{q/p} + t_{n+1}^q \le \Big(1 - \frac{t_{n+1}^q}{t_1^q}\Big)\varepsilon^q + t_{n+1}^q.$$

Now, we establish the lower estimate for $\mathcal{E}\left(W_q^T, I_{\varepsilon,p}^n\right)$. Let u_1 and u_{n+1} be such that $t_1 u_1 = \varepsilon$ and $u_1^q + u_{n+1}^q = 1$, *i.e.* $u_1 = \varepsilon / t_1$ and $u_{n+1}^q = 1 - \varepsilon^q / t_1^q$. Set $h^* := (u_1, 0, \dots, 0, u_{n+1}, 0, \dots)$. Obviously, $\|h\|_q \le 1$ and $\theta \in I_{\varepsilon,p}^n(Th^*)$. Then by Corollary 1,

$$\left(\mathcal{E}\left(W_q^T, I_{\varepsilon,p}^n\right)\right)^q \ge \|Th^*\|_q^q = t_1^q u_1^q + t_{n+1}^q u_{n+1}^q =$$

$$= \varepsilon^q + t_{n+1}^q\Big(1 - \frac{\varepsilon^q}{t_1^q}\Big) = t_{n+1}^q + \varepsilon^q\Big(1 - \frac{t_{n+1}^q}{t_1^q}\Big).$$

Finally, consider the case $\varepsilon > t_1$. For $x = Th \in W_q^T$ and $a \in I_{\varepsilon,p}^n(x)$, we have

$$\|x - \Phi_0^*(a)\|_q^q = \|Th\|_q^q = \sum_{n=1}^\infty t_n^q |h_n|^q \le t_1^q \sum_{n=1}^\infty |h_n|^q \le t_1^q.$$

Taking $h^* := (1, 0, \dots)$, it is clear that $\theta \in I_{\varepsilon,p}^n(Th^*)$ and by Corollary 1,

$$\mathcal{E}\left(W_q^T, I_{\varepsilon,p}^n\right) \ge \|Th^*\|_q = t_1.$$

Theorem 3 is proved. □

3.2.3. Case $1 \le q < p < \infty$. This case is the most technical one. We introduce some preliminary notations. For $m = 1, \dots, n$, define

$$\delta_{j,m} := \Big(1 - \frac{t_{m+1}^q}{t_j^q}\Big)^{\frac{p}{p-q}}, \quad j = 1, \dots, m-1,$$

and set $c_1 := t_1$ and, for $m \ge 2$,

$$c_m := \Big(\sum_{j=1}^m \delta_{j,m}\Big)^{1/p} \Big(\sum_{j=1}^m \frac{\delta_{j,m}^{q/p}}{t_j^q}\Big)^{-1/q}. \tag{3}$$

The sequence $\{c_m\}_{m=1}^n$ is non-increasing. Indeed, let $\delta_{j,m}(\xi) := \left(1 - \frac{\xi t_m^q + (1-\xi)t_{m+1}^q}{t_j^q}\right)^{\frac{p}{p-q}}$ and consider the function

$$g(\xi) := \Big(\sum_{j=1}^m \delta_{j,m}(\xi)\Big)^{1/p}\Big(\sum_{j=1}^m \frac{\delta_{j,m}^{q/p}(\xi)}{t_j^q}\Big)^{-1/q}, \qquad \xi \in [0,1].$$

Differentiating g and applying the Cauchy-Swartz inequality we have

$$g'(\xi) = \frac{t_{m+1}^q - t_m^q}{p-q}\Big(\sum_{j=1}^m \delta_{j,m}(\xi)\Big)^{\frac{1}{p}-1}\Big(\sum_{j=1}^m \frac{\delta_{j,m}^{q/p}(\xi)}{t_j^q}\Big)^{-1/q-1}\times$$

$$\times\Big(\Big(\sum_{j=1}^m \frac{\delta_{j,m}^{q/p}(\xi)}{t_j^q}\Big)^2 - \Big(\sum_{j=1}^m \delta_{j,m}(\xi)\Big)\Big(\sum_{j=1}^m \frac{\delta_{j,m}^{2q/p-1}(\xi)}{t_j^{2q}}\Big)\Big) \geq 0.$$

Hence, $c_{m+1} = g(0) \leq g(1) = c_m$.

For convenience, for $\lambda \in [0,1]$ denote $t_{m,\lambda}^q := (1-\lambda)t_{m+1}^q + \lambda t_m^q$.

Theorem 4. *Let $n \in \mathbb{N}$ and $1 \leq q < p < \infty$.*
1. *If $\varepsilon \leq c_n$ then*

$$\mathcal{E}(W_q^T, I_{\varepsilon,p}^n) = \mathcal{E}(W_q^T, I_{\varepsilon,p}^n, \Phi_n^*) = \left(t_{n+1}^q + \varepsilon^q\Big(\sum_{j=1}^n \Big(1 - \frac{t_{n+1}^q}{t_j^q}\Big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\right)^{1/q}.$$

2. *If $\varepsilon \in (c_n, c_1]$ then there exist $m \in \{1,\dots,n-1\}$ such that $\varepsilon \in (c_{m+1}, c_m]$ and $\lambda = \lambda(\varepsilon) \in [0,1)$ such that*

$$\varepsilon = \Big(\sum_{j=1}^m \Big(1 - \frac{t_{m,\lambda}^q}{t_j^q}\Big)^{\frac{p}{p-q}}\Big)^{\frac{1}{p}}\left(\sum_{j=1}^m \frac{\Big(1 - \frac{t_{m,\lambda}^q}{t_j^q}\Big)^{\frac{q}{p-q}}}{t_j^q}\right)^{-1/q}. \tag{4}$$

Then

$$\mathcal{E}(W_q^T, I_{\varepsilon,p}^n) = \mathcal{E}(W_q^T, I_{\varepsilon,p}^n, \Phi_{m,\lambda}^*) = \left(t_{m,\lambda}^q + \varepsilon^q \cdot \Big(\sum_{j=1}^m \Big(1 - \frac{t_{m,\lambda}^q}{t_j^q}\Big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\right)^{1/q},$$

where

$$\Phi_{m,\lambda}^*(a) = \left(a_1\Big(1 - \frac{t_{m,\lambda}^q}{t_1^q}\Big), \dots, a_m\Big(1 - \frac{t_{m,\lambda}^q}{t_m^q}\Big), 0, \dots\right),\ a \in \ell_p.$$

3. *If $\varepsilon > c_1$ then $\mathcal{E}(W_q^T, I_{\varepsilon,p}^n) = \mathcal{E}(W_q^T, I_{\varepsilon,p}^n, \Phi_0^*) = t_1$.*

Proof. Let $m \in \{0,\dots,n\}$, $\lambda \in [0,1]$ and Φ be either Φ_n^*, or Φ_0^*, or $\Phi_{m,\lambda}^*$. For $x \in W_q^T$ and $a \in I_{\varepsilon,p}^n(x)$,

$$\|x - \Phi(a)\|_q^q \leq \sum_{k=1}^m \left|\Big(1 - \frac{t_{m,\lambda}^q}{t_k^q}\Big)(x_k - a_k) + \frac{t_{m,\lambda}^q}{t_k^q}x_k\right|^q + \sum_{k=m+1}^\infty |x_k|^q \leq$$

$$\leq \sum_{k=1}^m \Big(1 - \frac{t_{m,\lambda}^q}{t_k^q}\Big)|x_k - a_k|^q + \sum_{k=1}^m t_{m,\lambda}^q |h_k|^q + \sum_{k=m+1}^\infty t_k^q |h_k|^q.$$

Using the Hölder inequality with parameters $p/(p-q)$ and p/q to estimate the first term and the inequality $t_k^q \le t_{m,\lambda}^q$, $k = m+1, m+2, \ldots$, we obtain

$$\|x - \Phi(a)\|_q^q \le \left\{ \sum_{k=1}^{m} \left(1 - \frac{t_{m,\lambda}^q}{t_k^q}\right)^{\frac{p}{p-q}} \right\}^{1-q/p} \left\{ \sum_{k=1}^{m} |x_k - a_k|^p \right\}^{q/p} + t_{m,\lambda}^q \sum_{k=1}^{\infty} h_k^q \le$$
$$\le \left\{ \sum_{k=1}^{m} \left(1 - \frac{t_{m,\lambda}^q}{t_k^q}\right)^{\frac{p}{p-q}} \right\}^{1-q/p} \varepsilon^q + t_{m,\lambda}^q,$$

which proves the estimate from above.

Now, we turn to the proof of the lower estimate. First, let $\varepsilon \le c_n$, and define

$$u_j := \frac{\varepsilon\, \delta_{j,n}^{1/p}}{t_j} \left(\sum_{j=1}^{n} \delta_{j,n} \right)^{-1/p}, \quad j = 1, \ldots, n, \qquad \text{and} \qquad u_{n+1} := \left(1 - \sum_{j=1}^{n} u_j^q \right)^{1/q}.$$

Consider $h^* := (u_1, \ldots, u_{n+1}, 0, \ldots)$. Evidently, u_{n+1} is well-defined as

$$\sum_{j=1}^{n} u_j^q = \varepsilon^q \left(\sum_{j=1}^{n} \delta_{j,n} \right)^{-q/p} \sum_{j=1}^{n} \frac{\delta_{j,n}^{q/p}}{t_j^q} = \frac{\varepsilon^q}{c_n^q} \le 1,$$

$\|h^*\|_q = 1$ and $\theta \in I_{\varepsilon,p}^n(Th^*)$ as $\sum_{j=1}^n t_j^p h_j^p = \varepsilon^p$. Hence, by Corollary 1,

$$\left(\mathcal{E}\left(W_q^T, I_{\varepsilon,p}^n\right)\right)^q \ge \|Th^*\|_q^q = \varepsilon^q \sum_{j=1}^{n} \delta_{j,n}^{q/p} \left(\sum_{j=1}^{n} \delta_{j,n} \right)^{-q/p} + t_{n+1}^q - t_{n+1}^q \sum_{j=1}^{n} \varepsilon^q \frac{\delta_{j,n}^{q/p}}{t_j^q} \left(\sum_{j=1}^{n} \delta_{j,n} \right)^{-q/p} =$$
$$= \varepsilon^q \left(\sum_{j=1}^{n} \delta_{j,n} \right)^{-q/p} \sum_{j=1}^{n} \delta_{j,n}^{q/p} \left(1 - \frac{t_{n+1}^q}{t_j^q} \right) + t_{n+1}^q = t_{n+1}^q + \varepsilon^q \cdot \left(\sum_{j=1}^{n} \left(1 - \frac{t_{n+1}^q}{t_j^q} \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}}.$$

Next, let $m \in \{1, 2, \ldots, n-1\}$ be such that $c_{m+1} < \varepsilon \le c_m$ and $\lambda = \lambda_\varepsilon \in [0, 1)$ be defined by (4). Set

$$u_j := \frac{\varepsilon \delta_{j,m}^{1/p}(\lambda)}{t_j} \left(\sum_{j=1}^{m} \delta_{j,m}(\lambda) \right)^{-1/p}, \quad j = 1, \ldots, m,$$

and consider $h^* = (u_1, \ldots, u_m, 0, \ldots)$. Clearly, $\|h\|_q = 1$ and $\theta \in I_{\varepsilon,p}^n(Th^*)$. Using Corollary 1, we obtain the desired lower estimate for $\mathcal{E}\left(W_q^T, I_{\varepsilon,p}^n\right)$.

Finally, let $\varepsilon > c_1$. Consider $h^* := (1, 0, 0, \ldots)$. Since $c_1 = t_1$, we have $\theta \in I_{\varepsilon,p}^n(Th^*)$. Hence, by Corollary 1, $\mathcal{E}\left(W_q^T, I_{\varepsilon,p}^n\right) \ge \|Th^*\|_q = t_1^q$. □

3.3. Information mapping $I(x) = I_{\varepsilon,p}(x) := x + B\left[\varepsilon, \ell_p\right]$. As a limiting case from Theorem 2, 3 and 4 we can obtain the following corollaries.

Theorem 5. *Let $1 \le q < \infty$ and $\varepsilon \ge 0$. Choose $m \in \mathbb{Z}_+$ to be such that*

$$1 - \varepsilon^q \sum_{k=1}^{m} \frac{1}{t_k^q} \ge 0 \quad \textit{and} \quad 1 - \varepsilon^q \sum_{k=1}^{m+1} \frac{1}{t_k^q} < 0.$$

Then

$$\mathcal{E}\left(W_q^T, I_{\varepsilon,\infty}\right) = \mathcal{E}\left(W_q^T, I_{\varepsilon,\infty}, \Phi_m^*\right) = \left(t_{m+1}^q + \varepsilon^q \sum_{k=1}^{m} \left(1 - \frac{t_{m+1}^q}{t_k^q} \right) \right)^{1/q}.$$

Theorem 6. *Let $1 \le q < \infty$ and $0 < p \le q$. If $0 \le \varepsilon \le t_1$ then $\mathcal{E}\left(W_q^T, I_{\varepsilon,p}\right) = \mathcal{E}\left(W_q^T, I_{\varepsilon,p}, \mathrm{id}\right) = \varepsilon$, and if $\varepsilon > t_1$ then $\mathcal{E}\left(W_q^T, I_{\varepsilon,p}\right) = \mathcal{E}\left(W_q^T, I_{\varepsilon,p}, \Phi_0^*\right) = t_1$.*

Define the sequence $\{c_n\}_{n=1}^\infty$ using formulas (3). It is not difficult to verify that $\{c_n\}_{n=1}^\infty$ is non-increasing and tend to 0 as $n \to \infty$.

Theorem 7. *Let $1 \le q < p < \infty$. If $\varepsilon \in (0, c_1]$ then there exists $m \in \mathbb{N}$ such that $\varepsilon \in (c_{m+1}, c_m]$ and $\lambda = \lambda(\varepsilon) \in [0,1)$ such that*

$$\varepsilon = \left(\sum_{j=1}^m \left(1 - \frac{t_{m,\lambda}^q}{t_j^q}\right)^{\frac{p}{p-q}}\right)^{1/p} \left(\sum_{j=1}^m \frac{\left(1 - \frac{t_{m,\lambda}^q}{t_j^q}\right)^{\frac{q}{p-q}}}{t_j^q}\right)^{-1/q}. \tag{5}$$

Then

$$\mathcal{E}\left(W_q^T, I_{\varepsilon,p}\right) = \mathcal{E}\left(W_q^T, I_{\varepsilon,p}, \Phi_{m,\lambda}^*\right) = \left(t_{m,\lambda}^q + \varepsilon^q \cdot \left(\sum_{j=1}^m \left(1 - \frac{t_{m,\lambda}^q}{t_j^q}\right)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right)^{1/q}.$$

where the method $\Phi_{m,\lambda}^*$ is defined in Theorem 3. Otherwise, if $\varepsilon > c_1$ then $\mathcal{E}\left(W_q^T, I_{\varepsilon,p}\right) = \mathcal{E}\left(W_q^T, I_{\varepsilon,p}, \Phi_0^*\right) = t_1$.

4. Recovery of scalar products. Following [3] (see also [4, 6, 7]), let us consider the problem of optimal recovery of scalar product. Let $1 \le p, q \le \infty$ and given operators $T\colon \ell_p \to \ell_p$ and $S : \ell_q \to \ell_q$ be defined as follows: for fixed non-increasing sequences $t = \{t_k\}_{k=1}^\infty$ and $s = \{s_k\}_{k=1}^\infty$,

$$Th := \{t_k h_k\}_{k=1}^\infty, \quad h \in \ell_p, \quad \text{and} \quad Sg := \{s_k g_k\}_{k=1}^\infty, \quad g \in \ell_q.$$

Consider classes of sequences

$$W_p^T := \{x = Th : \ h \in \ell_p, \ \|h\|_p \le 1\}, \quad W_q^S := \{y = Tg : \ g \in \ell_q, \ \|g\|_q \le 1\}.$$

and define the scalar product $A = \langle \cdot, \cdot \rangle : \ell_p \times \ell_q \to \mathbb{C}$ as usually:

$$\langle x, y \rangle = \sum_{k=1}^\infty x_k y_k, \qquad x \in \ell_p, \ y \in \ell_q.$$

For brevity, we denote $\langle x, y \rangle_n := \sum_{k=1}^n x_k y_k$.

In this section we will consider the problem of optimal recovery of the scalar product operator A on the class $W_{p,q}^{T,S} := W_p^T \times W_q^S$, when information mapping I is given in one of the following forms:

1. $I(x,y) = J_{\bar{\varepsilon}}^n(x,y) = \{(a,b) \in \mathbb{C}^n \times \mathbb{C}^n : \ \forall k = 1, ..., n \Rightarrow |x_k y_k - a_k b_k| \le \varepsilon_k\}$, where $n \in \mathbb{N}$ and $\varepsilon_1, \dots, \varepsilon_n \ge 0$;
2. $I(x,y) = J_{\varepsilon,r}^n(x,y) = \{(a,b) \in \mathbb{C}^n \times \mathbb{C}^n : \ \|\langle x, y \rangle_n - \langle a, b \rangle_n\|_{\ell_r^n} \le \varepsilon\}$, where $n \in \mathbb{N}$ and $1 \le r \le \infty$.

Finally, for $m \in \mathbb{N}$, we define methods of recovery $\Psi_m^*\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$:

$$\Psi_m^*(a,b) = \sum_{k=1}^{m} a_k b_k \Big(1 - \frac{t_{m+1}s_{m+1}}{t_k s_k}\Big), \qquad a, b \in \mathbb{C}^n,$$

that will be optimal in many situations and set $\Psi_0^*(a,b) := 0$.

4.1. Information mapping $J_{\bar\varepsilon}^n$.

Theorem 8. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $q = p/(p-1)$, $\varepsilon_1, \dots, \varepsilon_n \ge 0$ and $\bar\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$. If*

$$1 - \sum_{k=1}^{n} \frac{\varepsilon_k}{t_k s_k} \ge 0$$

we set $m = n$. Otherwise we choose $m \in \mathbb{Z}_+$, $m \le n$, to be such that

$$1 - \sum_{k=1}^{m} \frac{\varepsilon_k}{t_k s_k} \ge 0 \qquad \text{and} \qquad 1 - \sum_{k=1}^{m} \frac{\varepsilon_k}{t_k s_k} - \frac{\varepsilon_{m+1}}{t_{m+1}s_{m+1}} < 0.$$

Then

$$\mathcal{E}\left(A, W_{p,q}^{T,S}, J_{\bar\varepsilon}^n\right) = \mathcal{E}\left(A, W_{p,q}^{T,S}, J_{\bar\varepsilon}^n, \Psi_m^*\right) = t_{m+1}s_{m+1} + \sum_{k=1}^{m} \varepsilon_k \Big(1 - \frac{t_{m+1}s_{m+1}}{t_k s_k}\Big).$$

Proof. Using the triangle inequality, relations $|x_k y_k - a_k b_k| \le \varepsilon_k$, $k = 1, \dots, n$, and monotony of sequences t and s, we obtain that, for $(x,y) = (Th, Sg) \in W_{p,q}^{T,S}$ and $(a,b) \in J_{\bar\varepsilon}^n(x,y)$,

$$|\langle x, y\rangle - \Psi_m^*(a,b)| = \left|\sum_{k=1}^{\infty} x_k y_k - \sum_{k=1}^{m} a_k b_k \left(1 - \frac{t_{m+1}s_{m+1}}{t_k s_k}\right)\right| \le$$
$$\le \sum_{k=1}^{m} \left(1 - \frac{t_{m+1}s_{m+1}}{t_k s_k}\right) |x_k y_k - a_k b_k| + \sum_{k=1}^{m} \frac{t_{m+1}s_{m+1}}{t_k s_k} |x_k y_k| + \sum_{k=m+1}^{\infty} |x_k y_k| \le$$
$$\le \sum_{k=1}^{m} \left(1 - \frac{t_{m+1}s_{m+1}}{t_k s_k}\right) \varepsilon_k + t_{m+1}s_{m+1} \sum_{k=1}^{m} |h_k g_k| + \sum_{k=m+1}^{\infty} t_k s_k |h_k g_k| \le$$
$$\le \sum_{k=1}^{m} \left(1 - \frac{t_{m+1}s_{m+1}}{t_k s_k}\right) \varepsilon_k + t_{m+1}s_{m+1} \sum_{k=1}^{\infty} |h_k g_k| \le$$
$$\le t_{m+1}s_{m+1} + \sum_{k=1}^{m} \left(1 - \frac{t_{m+1}s_{m+1}}{t_k s_k}\right) \varepsilon_k,$$

which proves the upper estimate.

To establish the lower estimate, we set

$$u_k := \left(\frac{\varepsilon_k}{t_k s_k}\right)^{1/p}, \qquad v_k := \left(\frac{\varepsilon_k}{t_k s_k}\right)^{1/q}, \qquad k = 1, \dots, m, \tag{6}$$

$$u_{m+1} = \left(1 - \sum_{k=1}^{m} \frac{\varepsilon_k}{t_k s_k}\right)^{1/p}, \qquad v_{m+1} = \left(1 - \sum_{k=1}^{m} \frac{\varepsilon_k}{t_k s_k}\right)^{1/q}, \tag{7}$$

and consider $u^* = (u_1, \dots, u_{m+1}, 0, \dots)$ and $v^* = (v_1, \dots, v_{m+1}, 0, \dots)$. It is clear that $(Tu^*, Sv^*) \in W^{T,S}_{p,q}$ and $\theta \in J^n_{\bar\varepsilon}(Tu^*, Sv^*) \cap J^n_{\bar\varepsilon}(-Tu^*, Sv^*)$ due to the choice of number m. Hence, by Corollary 2,

$$E\left(A, W^{T,S}_{p,q}, J^n_{\bar\varepsilon}\right) \geq |\langle Tu^*, Sv^*\rangle| = \sum_{k=1}^{m} t_k s_k u_k v_k + t_{m+1}s_{m+1}\left(1 - \sum_{k=1}^{m} \frac{\varepsilon_k}{t_k s_k}\right) =$$
$$= t_{m+1}s_{m+1} + \sum_{k=1}^{m}\left(1 - \frac{t_{m+1}s_{m+1}}{t_k s_k}\right)\varepsilon_k.$$

This proves the sharpness of the upper estimate. □

4.2. Information mapping $J^n_{\varepsilon,r}$. We consider three cases separately: $r = \infty$, $0 < r \leq 1$ and $1 < r < \infty$.

4.2.1. Case $r = \infty$. Setting $\varepsilon_1 = \ldots = \varepsilon_n = \varepsilon$, we obtain the following corollary from Theorem 8.

Theorem 9. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $q = p/(p-1)$ and $\varepsilon \geq 0$. If*

$$1 - \varepsilon \sum_{k=1}^{n} \frac{1}{t_k s_k} \geq 0,$$

we set $n = m$. Otherwise we choose $m \in \mathbb{Z}_+$, $m \leq n$, to be such that

$$1 - \varepsilon \sum_{k=1}^{m} \frac{1}{t_k s_k} \geq 0 \qquad \text{and} \qquad 1 - \varepsilon \sum_{k=1}^{m+1} \frac{1}{t_k s_k} < 0.$$

Then

$$\mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,\infty}\right) = \mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,\infty}, \Psi^*_m\right) = t_{m+1}s_{m+1} + \varepsilon \sum_{k=1}^{m}\left(1 - \frac{t_{m+1}s_{m+1}}{t_k s_k}\right).$$

4.2.2. Case $0 < r \leq 1$.

Theorem 10. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $q = p/(p-1)$ and $r \in (0,1]$. If $\varepsilon \leq t_1 s_1$ then*

$$\mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,r}\right) = \mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,r}, \Psi^*_n\right) = t_{n+1}s_{n+1} + \varepsilon\left(1 - \frac{t_{n+1}s_{n+1}}{t_1 s_1}\right),$$

and if $\varepsilon > t_1 s_1$ then $\mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,r}\right) = \mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,r}, \Psi^*_0\right) = t_1 s_1$.

Proof. First, we consider the case $\varepsilon \leq t_1 s_1$. Let $(x, y) = (Th, Sg) \in W^{T,S}_{p,q}$ and $(a, b) \in J^n_{\varepsilon,r}(x, y)$. Similarly to the proof of Theorem 8 in the case $m = n$ we obtain

$$|\langle x, y\rangle - \Psi^*_n(a, b)| \leq \sum_{k=1}^{n}\left(1 - \frac{t_{n+1}s_{n+1}}{t_k s_k}\right)|x_k y_k - a_k b_k| + t_{n+1}s_{n+1}\sum_{k=1}^{\infty} h_k g_k. \qquad (8)$$

Using the Hölder inequality and inequality $\varepsilon_1^{1/r} + \ldots + \varepsilon_n^{1/r} \le (\varepsilon_1 + \ldots + \varepsilon_n)^{1/r}$, we have

$$|\langle x, y\rangle - \Psi_n^*(a,b)| \le \max_{k=\overline{1,n}} \left(1 - \frac{t_{n+1}s_{n+1}}{t_k s_k}\right) \sum_{k=1}^n |x_k y_k - a_k b_k| + t_{n+1}s_{n+1}\|h\|_p \|g\|_q \le$$
$$\le \left(1 - \frac{t_{n+1}s_{n+1}}{t_1 s_1}\right)\varepsilon + t_{n+1}s_{n+1}.$$

The upper estimate is proved.

Now, we establish the lower estimate. Let

$$u_1 = \left(\frac{\varepsilon}{s_1 t_1}\right)^{1/p}, \quad u_{n+1} = \left(1 - \frac{\varepsilon}{t_1 s_1}\right)^{1/p}, \quad v_1 = \left(\frac{\varepsilon}{s_1 t_1}\right)^{1/q}, \quad v_{n+1} = \left(1 - \frac{\varepsilon}{t_1 s_1}\right)^{1/q},$$

and consider elements $u^* = (u_1, 0, \ldots, 0, u_{n+1}, 0, \ldots)$, $v^* = (u_1, 0, \ldots, 0, u_{n+1}, 0, \ldots)$. Obviously, $(Tu^*, Sv^*) \in W_{p,q}^{T,S}$ and $(\theta, \theta) \in J_{\varepsilon,r}^n(Tu^*, Sv^*) \cap J_{\varepsilon,r}^n(-Tu^*, Sv^*)$. Then by Corollary 2,

$$\mathcal{E}\left(A, W_{p,q}^{T,S}, J_{\varepsilon,r}^n\right) \ge |\langle Tu^*, Sv^*\rangle| = t_1 s_1 \cdot \frac{\varepsilon}{t_1 s_1} + t_{n+1}s_{n+1} \cdot \left(1 - \frac{\varepsilon}{t_1 s_1}\right)$$
$$= \left(1 - \frac{t_{n+1}s_{n+1}}{t_1 s_1}\right)\varepsilon + t_{n+1}s_{n+1},$$

which finishes the proof of the desired estimate.

Next, we let $\varepsilon > t_1 s_1$. For $(x, y) = (Th, Sg) \in W_{p,q}^{T,S}$, and $(a, b) \in J_{\varepsilon,r}^n(x, y)$, we have

$$|\langle x, y\rangle - \Psi_0^*(a,b)| = |\,\langle x, y\rangle\,| \le \sum_{k=1}^\infty t_k s_k |h_k g_k| \le t_1 s_1 \|h\|_p \|g\|_q \le t_1 s_1.$$

Taking $u^* = v^* = (1, 0, ...)$, it is clear that $(\theta, \theta) \in J_{\varepsilon,r}^n(Tu^*, Sv^*) \cap J_{\varepsilon,r}^n(-Tu^*, Sv^*)$. By Corollary 2,

$$\mathcal{E}\left(A, W_{p,q}^{T,S}, J_{\varepsilon,r}^n\right) \ge |(Tu^*, Sv^*)| = t_1 s_1.$$

□

4.2.3. Case $1 < r < \infty$**.** First, we introduce some preliminary notations. For $m = 1, \ldots, n$, we define

$$\tau_{j,m} := \left(1 - \frac{t_{m+1}s_{m+1}}{t_j s_j}\right)^{\frac{1}{r-1}}, \quad j = 1, \ldots, m-1,$$

and set $d_1 := t_1 s_1$ and, for $m \ge 2$,

$$d_m := \left(\sum_{j=1}^m \tau_{j,m}^r\right)^{1/r} \left(\sum_{j=1}^m \frac{\tau_{j,m}}{t_j s_j}\right)^{-1}.$$

The sequence $\{d_m\}_{m=1}^n$ is non-increasing, which can be verified using the arguments similar to those applied to prove monotony of sequence $\{c_m\}_{m=1}^n$ in subsection 3.2.3. In addition, for convenience, for $\lambda \in [0, 1]$, we denote

$$t_{m,\lambda} := (1-\lambda)t_{m+1} + \lambda t_m \quad \text{and} \quad s_{m,\lambda} := (1-\lambda)s_{m+1} + \lambda s_m.$$

Theorem 11. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $q = p/(p-1)$ and $1 < r < \infty$.*

1. *If $\varepsilon \le d_{n+1}$ then*

$$\mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,r}\right) = \mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,r}, \Psi^*_n\right) = t_{n+1}s_{n+1} + \varepsilon \cdot \left(\sum_{j=1}^{n}\left(1 - \frac{t_{n+1}s_{n+1}}{t_js_j}\right)^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}.$$

2. *If $\varepsilon \in (d_n, d_1]$ then there exists $m \in \{1, \dots, n-1\}$ such that $\varepsilon \in (d_{m+1}, d_m]$ and $\lambda = \lambda(\varepsilon) \in [0,1)$ such that*

$$\varepsilon = \left(\sum_{j=1}^{m}\left(1 - \frac{t_{m,\lambda}s_{m,\lambda}}{t_js_j}\right)^{\frac{r}{r-1}}\right)^{1/r}\left(\sum_{j=1}^{m}\frac{\left(1 - \frac{t_{m,\lambda}s_{m,\lambda}}{t_js_j}\right)^{\frac{1}{r-1}}}{t_js_j}\right)^{-1}. \tag{9}$$

Then

$$\mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,r}\right) = \mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,r}, \Psi^*_{m,\lambda}\right) = t_{m,\lambda}s_{m,\lambda} + \varepsilon\left(\sum_{j=1}^{m}\left(1 - \frac{t_{m,\lambda}s_{m,\lambda}}{t_js_j}\right)^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}},$$

where

$$\Psi^*_{m,\lambda}(a,b) = \sum_{j=1}^{m} a_jb_j\left(1 - \frac{t_{m,\lambda}s_{m,\lambda}}{t_js_j}\right), \qquad a, b \in \ell_r.$$

3. *If $\varepsilon > d_1$ then $\mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,r}\right) = \mathcal{E}\left(A, W^{T,S}_{p,q}, J^n_{\varepsilon,r}, \Psi^*_0\right) = t_1s_1$.*

Proof. Let $m \in \{0, \dots, n\}$, $\lambda \in [0,1]$ and Ψ be either Ψ^*_n or Ψ^*_0, or $\Psi^*_{m,\lambda}$. Using the Hölder inequality with parameters r and $\frac{r}{r-1}$, for $(x,y) = (Th, Sg) \in W^{T,S}_{p,q}$ and $(a,b) \in J^n_{\varepsilon,r}(x,y)$, we have

$$\begin{aligned}|\langle x, y\rangle - \Psi(a,b)| &\le \sum_{j=1}^{m}\left(1 - \frac{t_{m,\lambda}s_{m,\lambda}}{t_js_j}\right)|x_jy_j - a_jb_j| + t_{m,\lambda}s_{m,\lambda}\sum_{j=1}^{m}\frac{x_jy_j}{t_js_j} + \sum_{j=m+1}^{\infty}x_jy_j \le \\ &\le t_{m,\lambda}s_{m,\lambda} + \varepsilon\left(\sum_{j=1}^{m}\left(1 - \frac{t_{m,\lambda}s_{m,\lambda}}{t_js_j}\right)^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}},\end{aligned}$$

which proves the upper estimate.

Now, we turn to the proof of the lower estimate. We let $\varepsilon \le d_n$, and, for $j = 1, \dots, n$, set

$$u_j = \left(\frac{\varepsilon\tau_{j,n}}{t_js_j}\right)^{1/p}\left(\sum_{k=1}^{n}\tau^r_{k,n}\right)^{-\frac{1}{rp}}, \qquad v_j = \left(\frac{\varepsilon\tau_{j,n}}{t_js_j}\right)^{1/q}\left(\sum_{k=1}^{n}\tau^r_{k,n}\right)^{-\frac{1}{rq}},$$

$u_{n+1} := (1 - u_1^p - \dots - u_n^p)^{1/p}$, and $v_{n+1} := (1 - v_1^q - \dots - v_n^q)^{1/q}$. In addition, we define $u^* = (u_1, \dots, u_n, u_{n+1}, 0, \dots)$ and $v^* := (v_1, \dots, v_n, v_{n+1}, 0, \dots)$. By the choice of ε, numbers u_{n+1} and v_{n+1} are well defined and, hence, $(Tu^*, Sv^*) \in W^{T,S}_{p,q}$. Also,

$$\sum_{j=1}^{n}\left|Tu^*_j \cdot Sv^*_j\right|^r = \sum_{j=1}^{n}|t_js_ju_jv_j|^r = \varepsilon^r,$$

yielding that $(\theta,\theta)\in J^n_{\varepsilon,r}(Tu^*,Sv^*)\cap J^n_{\varepsilon,r}(-Tu^*,Sv^*)$. By Corollary 2,

$$\mathcal{E}\left(A,W^{T,S}_{p,q},J^n_{\varepsilon,r}\right)\geq|(Tu^*,Sv^*)|=$$
$$=\varepsilon\sum_{j=1}^n\tau_{j,n}\left(\sum_{j=1}^n\tau_{j,n}^r\right)^{-1/r}+t_{n+1}s_{n+1}-\varepsilon\sum_{j=1}^n\frac{t_{n+1}s_{n+1}\tau_{j,n}}{t_js_j}\left(\sum_{j=1}^n\tau_{j,n}^r\right)^{-1/r}=$$
$$=t_{n+1}s_{n+1}+\varepsilon\left(\sum_{j=1}^n\left(1-\frac{t_{n+1}s_{n+1}}{t_js_j}\right)^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}},$$

which proves the desired lower estimate.

Next, let $m\in\{1,\dots,n-1\}$ be such that $d_{m+1}<\varepsilon\leq d_m$ and $\lambda=\lambda_\varepsilon\in[0,1)$ be defined by (9). Set

$$u_j:=\left(\frac{\varepsilon\tau_j}{t_js_j}\right)^{1/p}\left(\sum_{k=1}^m\tau_k^r\right)^{-\frac{1}{rp}}\quad\text{and}\quad v_j:=\left(\frac{\varepsilon\tau_j}{t_js_j}\right)^{1/q}\left(\sum_{k=1}^m\tau_k^r\right)^{-\frac{1}{rq}},$$

and define $u^*:=(u_1,\dots,u_m,0,\dots)$ and $v^*:=(v_1,\dots,v_m,0,\dots)$. It is not difficult to verify that $(\theta,\theta)\in J^n_{\varepsilon,r}(Tu^*,Sv^*)\cap J^n_{\varepsilon,r}(-Tu^*,Sv^*)$. Using Corollary 2 we obtain the desired estimate for $\mathcal{E}\left(A,W^{T,S}_{p,q},J^n_{\varepsilon,r}\right)$.

Finally, let $\varepsilon>t_1s_1$. Consider $u^*=v^*=(1,0,\dots)$. Since $d_1=t_1s_1$, we have $(\theta,\theta)\in J^n_{\varepsilon,r}(Tu^*,Sv^*)\cap J^n_{\varepsilon,r}(-Tu^*,Sv^*)$. Hence, by Corollary 2, $\mathcal{E}\left(A,W^{T,S}_{p,q},J^n_{\varepsilon,r}\right)\geq|\langle Tu^*,Sv^*\rangle|=t_1s_1$. □

4.3. Applications. Let H be a complex Hilbert space with orthonormal basis $\{\varphi_n\}_{n=1}^\infty$, $\{t_k\}_{k=1}^\infty$ be a non-increasing sequence; $T\colon\ell_2\to\ell_2$ be an operator mapping sequence $x=(x_1,x_2,\dots)$ into sequence $Tx=(t_1x_1,t_2x_2,\dots)$. Consider the class

$$\mathcal{W}^T:=\left\{x=\sum_{n=1}^\infty t_nc_n\varphi_n:\ \sum_{n=1}^\infty|c_n|^2\leq 1\right\},$$

and information operator $\mathcal{I}_{p,\varepsilon}\colon H\to\ell_p$, with $2<p\leq\infty$, mapping an element $x=\sum_{n=1}^\infty x_n\varphi_n$ into the set $\mathcal{I}_{p,\varepsilon}x=(x_1,x_2,\dots)+B[\varepsilon,\ell_p]\in\ell_p$. Due to isomorphism between ℓ_2 and H, under notations of Section 3 we have

$$\mathcal{E}\left(\mathcal{W}^T,\mathcal{I}_{\varepsilon,p}\right)=\mathcal{E}\left(W_2^T,I^\infty_{\varepsilon,p}\right).\tag{10}$$

Moreover, methods of recovery $F^*_{m,\lambda}:=\mathfrak{A}\circ\Phi^*_{m,\lambda}$ are optimal, where $\mathfrak{A}\colon\ell_2\to H$ is the natural isomorphism between ℓ_2 and H: $\mathfrak{A}(x_1,x_2,\dots)=\sum_{n=1}^\infty x_n\varphi_n$. Remark that $F_{m,\lambda}$ are triangular methods of recovery that play an important role in the theory of ill-posed problems (see, *e.g.* [11, Theorem 2.1] and references therein).

Consider an important case when $t_n=n^{-\mu}$, $n\in\mathbb{N}$, with some fixed $\mu>0$. It corresponds *e.g.*, to the space $H=L_2(\mathbb{T})$ of square integrable functions defined on a period and the class $\mathcal{W}^T=W_2^\mu(\mathbb{T})$ of functions having L_2-bounded Weyl derivative of order μ. Using equality (10) and Theorems 7 and 5, we obtain

$$\lim_{\varepsilon\to0^+}\varepsilon^{-\lambda}\mathcal{E}\left(\mathcal{W}^T,\mathcal{I}_{\varepsilon,p}\right)=\left(\frac{\alpha+\beta}{\beta}\right)^{1/2}\left(\frac{\beta^{1/2}}{\alpha^{1/p}}\right)^\lambda,\qquad\lambda=\frac{\mu}{\mu+1/2-1/p},$$

where

$$\alpha = \frac{1}{2\mu} B\left(\frac{2p-2}{p}, \frac{1}{2\mu}\right), \qquad \beta = \frac{1}{2\mu} B\left(\frac{2p-2}{p}, 2+\frac{1}{2\mu}\right),$$

and $B(\alpha,\beta)$ is the Euler beta function. Indeed, in case $2<p<\infty$, by selecting $n=n_\varepsilon\in\mathbb{N}$ and $\lambda\in[0,1)$ such that equation (5) is satisfied, we can easily verify that

$$\lim_{\varepsilon\to 0^+} n_\varepsilon^{\mu/\lambda} c_{n_\varepsilon} = \lim_{\varepsilon\to 0^+} n_\varepsilon^{\mu/\lambda} c_{n_\varepsilon+1} = \alpha^{1/p}\beta^{-1/2} \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \varepsilon^{-\mu/\lambda} n_\varepsilon^{-\mu} = \left(\frac{\beta^{1/2}}{\alpha^{1/p}}\right)^{\mu/\lambda}.$$

Similar arguments are applicable for $p=\infty$, in which case $1/p$ should be replaced with 0.

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