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ON ASYMORPHISMS OF FINITARY COARSE SPACES

I. V. Protasov. *On asymorphisms of finitary coarse spaces*, Mat. Stud. **56** (2021), 212–214.

We characterize finitary coarse spaces X such that every permutation of X is an asymorphism.

1. Introduction and results. Given a set X, a family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* on X if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x, x) \in X : x \in X\}$;
- if E, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z\ ((x, z) \in E,\ (z, y) \in E')\}$, $E^{-1} = \{(y, x) : (x, y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;
- $\cup\mathcal{E} = X \times X$.

A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ and $E \in \mathcal{E}$, we denote

$$E[x] = \{y \in X : (x, y) \in E\},\ E[A] = \bigcup_{a \in A} E[a]$$

and say that $E[x]$ and $E[A]$ are *balls of radius* E *around* x and A.

The pair $(X, \mathcal{E})$ is called a *coarse space* [6] or a *ballean* [4], [5].

For a coarse space $(X, \mathcal{E})$, a subset $B \subseteq X$ is called *bounded* if $B \subseteq E[x]$ for some $E \in \mathcal{E}$ and $x \in X$. The family $\mathcal{B}_{(X,\mathcal{E})}$ of all bounded subsets of $(X, \mathcal{E})$ is called the *bornology* of $(X, \mathcal{E})$. We recall that a family $\mathcal{B}$ of subsets of a set X is a bornology if $\mathcal{B}$ is closed under taking subsets and finite unions, and $\mathcal{B}$ contains all finite subsets of X.

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \longrightarrow X'$ is called

- *bornologous* if $f(B) \in \mathcal{B}_{(X',\mathcal{E}')}$ for each $B \in \mathcal{B}_{(X,\mathcal{E})}$;
- *macro-uniform* if, for each $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that, for all $x, y \in X$, $(x, y) \in E$ implies $(f(x), f(y)) \in E'$;
- *asymorphism* if f s a bijection and f, f^{-1} are macro-uniform.

We recall that a coarse space $(X, \mathcal{E})$ is *discrete* (or *thin*) if, for each $E \in \mathcal{E}$, there exists $B \in \mathcal{B}_{(X,\mathcal{E})}$ such that $E[x] = \{x\}$ for each $x \in X \setminus B$. Every bornology $\mathcal{B}$ on a set X defines the discrete coarse space $(X, \mathcal{E}_{\mathcal{B}})$ with the base $\{E_B : B \in \mathcal{B}\}$, were $E_B[x] = B$ if $x \in B$, and $E_B[x] = \{x\}$ if $x \in X \setminus B$. Every discrete coarse space $(X, \mathcal{E})$ coincides with $(X, \mathcal{E}_{\mathcal{B}})$ for $\mathcal{B} = \mathcal{B}_{(X,\mathcal{E})}$.

For different characterizations of discrete coarse spaces, see Theorem 2.2 in [1].

2010 *Mathematics Subject Classification:* 54E05, 54E35.
Keywords: bornology; coarse space; asymorphism.
doi:10.30970/ms.56.2.212-214

Theorem 1. *Every bornologous mapping $f\colon X \longrightarrow X$ of a coarse space $(X, \mathcal{E})$ is macro-uniform if and only if $(X, \mathcal{E})$ is discrete.*

A coarse space $(X, \mathcal{E})$ is called

- *locally finite* if each ball $E[x]$ is finite, equivalently, $\mathcal{B}_{(X,\mathcal{E})} = [X]^{<\omega}$;
- *finitary* if, for each $E \in \mathcal{E}$ there exists a natural number n such that $|E[x]| < n$ for each $x \in X$.

Let G be a transitive group of permutations of a set X. We denote by X_G the set X endowed with the coarse structure with the base

$$\{\{(x, gx)\colon g \in F\}\colon F \in [G]^{<\omega},\ \ id \in F\}.$$

By [2, Theorem 1], for every finitary coarse structure $(X, \mathcal{E})$, there exists a transitive group G of permutations of X such that $(X, \mathcal{E}) = X_G$. For more general results, see [3].

Let X be a set, κ be a cardinal, $\mathcal{S}_X$ denotes the group of all permutations of X, $\mathcal{S}_X^{<\kappa} = \{g \in \mathcal{S}_X\colon |\text{supp } g| < \kappa\}$, $\text{supp } g = \{x \in X\colon g(x) \neq x\}$.

Theorem 2. *Let $(X, \mathcal{E})$ be an infinite finitary coarse space. Then the following statements are equivalent:*
(i) every permutation of X is an asymorphism of $(X, \mathcal{E})$;
(ii) there exists an infinite cardinal κ, $\kappa \leq |X|^+$ such that $(X, \mathcal{E}) = X_G$ for $G = \mathcal{S}_X^{<\kappa}$.

Open problem. *Characterize locally finite coarse spaces $(X, \mathcal{E})$ such that every permutation of X is an asymorphism.*

2. Proofs.

Proof of Theorem 1. Let $(X, \mathcal{E})$ be a discrete coarse space defined by a bornology $\mathcal{B}$ and let $f\colon X \longrightarrow X$ is bornologous. We take an arbitrary $B \in \mathcal{B}$ and note that $f(E_B[x]) \subseteq E_{f(B)}[f(x)]$ for each $x \in X$, so f is macro-uniform.

On the other hand, let $(X, \mathcal{E})$ be not discrete. Then there exists $E \in \mathcal{E}$ such that, for each bounded subset B of $(X, \mathcal{E})$, one can find $x \in X \setminus B$ such that $|E[x]| > 1$. Therefore, for some ordinal λ, we can choose inductively two injective λ-sequences $(x_\alpha)_{\alpha<\lambda}$, $(y_\alpha)_{\alpha<\lambda}$ such that the set $\{x_\alpha\colon \alpha < \lambda\}$ is unbounded, $y_\alpha) \in E[x_\alpha]$, $\alpha < \lambda$ and $y_\alpha \neq x_\beta$ for all $\alpha, \beta < \lambda$.

We define a mapping $f\colon X \longrightarrow X$ by $f(y_\alpha) = y_0$ for each $\alpha < \lambda$, and $f(x) = x$ for each $x \in X \setminus \{y_\alpha\colon \alpha < \lambda\}$. Clearly, f is bornologous. Since $\{x_\alpha\colon \alpha < \lambda\}$ is unbounded, $f(x_\alpha) = x_\alpha$, $y_\alpha \in E[x_\alpha]$ and $f(y_\alpha) = y_0$ for each $\alpha < \lambda$, we conclude that f is not macro-uniform. □

Proof of Theorem 2. $(i) \Longrightarrow (ii)$. We say that a permutation g of X is *compatible* with $\mathcal{E}$ if there exists $E \in \mathcal{E}$ such that $(x, gx) \in E$ for each $x \in X$. We note that the set G of all permutations compatible with $\mathcal{E}$ is a subgroup of $\mathcal{S}_X$ and, by Theorem 1 from [2], $(X, \mathcal{E}) = X_G$.

We say that a subset Y of X is crowded if there exists $E \in \mathcal{E}$ such that $|E[y]| > 1$ for each $y \in Y$. We take the minimal cardinal κ, $\kappa \leq |X|^+$ such that, for each $\lambda < \kappa$, $(X, \mathcal{E})$ has a crowded subset of cardinality λ.

We show that $G = \mathcal{S}_X^{<\kappa}$. If $g \in G$ then $|\text{supp } g| < \kappa$ because the set supp g is crowded, so $g \in \mathcal{S}_X^{<\kappa}$ and $G \subseteq \mathcal{S}_X^{<\kappa}$.

To prove $\mathcal{S}_X^{<\kappa} \subseteq G$, we need the following auxiliary statement.

$(*)$ *Let Y be a subset of X such that $|Y| = |X|$, λ be a cardinal, $\lambda < \kappa$. Then there exists two injective λ-sequences $(x_\alpha)_{\alpha<\lambda}$, $(y_\alpha)_{\alpha<\lambda}$ in Y and $H \in \mathcal{E}$ such that $\{x_\alpha\colon \alpha < \lambda\} \cap \{y_\alpha\colon \alpha < \lambda\} = \varnothing$ and $(x_\alpha, y_\alpha) \in H$ for each $\alpha < \lambda$.*

By the choice of κ, we can choose $E \in \mathcal{E}$ and injective λ-sequences $(a_\alpha)_{\alpha<\lambda}$, $(b_\alpha)_{\alpha<\lambda}$ such that $\{a_\alpha\colon \alpha < \lambda\} \cap \{b_\alpha\colon \alpha < \lambda\} = \varnothing$ and $(a_\alpha, b_\alpha) \in E$ for each $\alpha < \lambda$. Passing to subsequences, we may suppose that $|Y \setminus \{a_\alpha,\ b_\alpha\colon \alpha < \lambda\}| = |X|$. We choose two injective λ-sequences $(x_\alpha)_{\alpha<\lambda}$, $(y_\alpha)_{\alpha<\lambda}$ in $Y \setminus \{a_\alpha,\ b_\alpha\colon \alpha < \lambda\}$ such that $\{x_\alpha\colon \alpha < \lambda\} \cap \{y_\alpha\colon \alpha < \lambda\} = \varnothing$. Then we define an involution f of X by $fa_\alpha = x_\alpha$, $fb_\alpha = y_\alpha$ and $fx = x$ for each $x \in X \setminus \{a_\alpha,\ b_\alpha,\ x_\alpha,\ y_\alpha\colon \alpha < \lambda\}$. Since f is macro-uniform, there exists $H \in \mathcal{E}$ such that $(x, y) \in E$ implies $(fx, fy) \in H$. Hence, $(x_\alpha, y_\alpha) \in H$ for each $\alpha < \lambda$.

Now let $g \in \mathcal{S}_X^{<\kappa}$, $A = \operatorname{supp} g$. We prove that g is compatible with $\mathcal{E}$, so $g \in G$. By the 3-Sets Lemma, there exists a partition A_1, A_2, A_3 of A such that $A_i \cap gA_i = \varnothing$, $i \in \{1, 2, 3\}$.

If $|X| = |X \setminus (A_1 \cup gA_1)|$ then we denote $Y = X \setminus (A_1 \cup gA_1)$, $\lambda = |A_1|$ and apply $(*)$ to choose corresponding $(x_\alpha)_{\alpha<\lambda}$, $(y_\alpha)_{\alpha<\lambda}$ and $H \in \mathcal{E}$. We enumerate $A_1 = \{a_\alpha\colon \alpha < \lambda\}$ and define an involution h of X by $hx_\alpha = a_\alpha$, $hy_\alpha = ga_\alpha$ and $hx = x$ for each $x \in X \setminus \{x_\alpha, y_\alpha, a_\alpha, ga_\alpha\colon \alpha < \lambda\}$. Since h is macro-uniform, there exists $K \in \mathcal{E}$ such that $(x, y) \in H$ implies $(hx, hy) \in K$. Hence, $(a_\alpha, ga_\alpha) \in K$ for each $\alpha < \lambda$.

If $|X \setminus (A_1 \cup gA_1)| < |X|$ then we partition $A_1 = B \cup C$, $|B| = |C| = |X|$ and, to choose K, apply above arguments for the pair B, gB and C, gC.

Repeating above construction for A_2 and A_3, we see that g is compatible with $\mathcal{E}$.

$(ii) \Longrightarrow (i)$. Let $G = \mathcal{S}_X^{<\kappa}$, $(X, \mathcal{E}) = X_G$. We take an arbitrary $h \in \mathcal{S}_X$ and show that $h\colon X_G \to X_G$ is macro-uniform.

Let F be a finite subset of $\mathcal{S}_X^{<\kappa}$, $x \in X$, $y = hx$. Then $hFh^{-1} \subset \mathcal{S}_X^{<\kappa}$ and, for $f \in F$, we have $(hx, hfx) = (y, hfh^{-1}y)$, so h is macro-uniform. □

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Received 12.10.2021
Revised 10.12.2021