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ON ASYMMORPHISMS OF FINITARY COARSE SPACES


We characterize finitary coarse spaces $X$ such that every permutation of $X$ is an asymorphism.

1. Introduction and results. Given a set $X$, a family $\mathcal{E}$ of subsets of $X \times X$ is called a coarse structure on $X$ if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x, x) : x \in X\}$;
- if $E, E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z ((x, z) \in E, \ (z, y) \in E')\}, \ E^{-1} = \{(y, x) : (x, y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;
- $\bigcup \mathcal{E} = X \times X$.

A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a base for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ and $E \in \mathcal{E}$, we denote

$$E[x] = \{y \in X : (x, y) \in E\}, \ E[A] = \bigcup_{a \in A} E[a]$$

and say that $E[x]$ and $E[A]$ are balls of radius $E$ around $x$ and $A$.

The pair $(X, \mathcal{E})$ is called a coarse space [6] or a ballean [4], [5].

For a coarse space $(X, \mathcal{E})$, a subset $B \subseteq X$ is called bounded if $B \subseteq E[x]$ for some $E \in \mathcal{E}$ and $x \in X$. The family $\mathcal{B}_{(X, \mathcal{E})}$ of all bounded subsets of $(X, \mathcal{E})$ is called the bornology of $(X, \mathcal{E})$. We recall that a family $\mathcal{B}$ of subsets of a set $X$ is a bornology if $\mathcal{B}$ is closed under taking subsets and finite unions, and $\mathcal{B}$ contains all finite subsets of $X$.

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \to X'$ is called

- bornologous if $f(B) \in \mathcal{B}_{(X', \mathcal{E}')} \text{ for each } B \in \mathcal{B}_{(X, \mathcal{E})}$;
- macro-uniform if, for each $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that, for all $x, y \in X$, $(x, y) \in E$ implies $(f(x), f(y)) \in E'$;
- asymorphism if $f$ s a bijection and $f, f^{-1}$ are macro-uniform.

We recall that a coarse space $(X, \mathcal{E})$ is discrete (or thin) if, for each $E \in \mathcal{E}$, there exists $B \in \mathcal{B}_{(X, \mathcal{E})}$ such that $E[x] = \{x\}$ for each $x \in X \setminus B$. Every bornology $\mathcal{B}$ on a set $X$ defines the discrete coarse space $(X, \mathcal{E}_B)$ with the base $\{E_B : B \in \mathcal{B}\}$, were $E_B[x] = B$ if $x \in B$, and $E_B[x] = \{x\}$ if $x \in X \setminus B$. Every discrete coarse space $(X, \mathcal{E})$ coincides with $(X, \mathcal{E}_B)$ for $B = \mathcal{B}_{(X, \mathcal{E})}$.

For different characterizations of discrete coarse spaces, see Theorem 2.2 in [1].

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Theorem 1. Every bornologous mapping \( f : X \to X \) of a coarse space \((X, \mathcal{E})\) is macro-uniform if and only if \((X, \mathcal{E})\) is discrete.

A coarse space \((X, \mathcal{E})\) is called

- **locally finite** if each ball \( E[x] \) is finite, equivalently, \( \mathcal{B}(X, \mathcal{E}) = [X]^{<\omega} \);
- **finitary** if, for each \( E \in \mathcal{E} \) there exists a natural number \( n \) such that \( |E[x]| < n \) for each \( x \in X \).

Let \( G \) be a transitive group of permutations of a set \( X \). We denote by \( X_G \) the set \( X \) endowed with the coarse structure with the base

\[
\{(x, gx) : g \in F \} : F \in [G]^{<\omega}, \ id \in F \}.
\]

By [2, Theorem 1], for every finitary coarse structure \((X, \mathcal{E})\), there exists a transitive group \( G \) of permutations of \( X \) such that \((X, \mathcal{E}) = X_G \). For more general results, see [3].

Let \( X \) be a set, \( \kappa \) be a cardinal, \( S_X \) denotes the group of all permutations of \( X \), \( S_X^{<\kappa} = \{g \in S_X : |\text{supp } g| < \kappa\} \), \( \text{supp } g = \{x \in X : g(x) \neq x\} \).

**Theorem 2.** Let \((X, \mathcal{E})\) be an infinite finitary coarse space. Then the following statements are equivalent:

(i) every permutation of \( X \) is an asymorphism of \((X, \mathcal{E})\);

(ii) there exists an infinite cardinal \( \kappa, \kappa \leq |X|^+ \) such that \((X, \mathcal{E}) = X_G \) for \( G = S_X^{<\kappa} \).

**Open problem.** Characterize locally finite coarse spaces \((X, \mathcal{E})\) such that every permutation of \( X \) is an asymorphism.

2. **Proofs.** Proof of Theorem 1. Let \((X, \mathcal{E})\) be a discrete coarse space defined by a bornology \( \mathcal{B} \) and let \( f : X \to X \) be bornologous. We take an arbitrary \( B \in \mathcal{B} \) and note that \( f(E_B[x]) \subseteq E_{f(B)}[f(x)] \) for each \( x \in X \), so \( f \) is macro-uniform.

On the other hand, let \((X, \mathcal{E})\) is not discrete. Then there exists \( E \in \mathcal{E} \) such that, for each bounded subset \( B \) of \((X, \mathcal{E})\), one can find \( x \in X \setminus B \) such that \( |E[x]| > 1 \). Therefore, for some ordinal \( \lambda \), we can choose inductively two injective \( \lambda \)-sequences \((x_\alpha)_{\alpha < \lambda}, (y_\alpha)_{\alpha < \lambda} \) such that the set \( \{x_\alpha : \alpha < \lambda\} \) is unbounded, \( y_\alpha \in E[x_\alpha], \alpha < \lambda \) and \( y_\alpha \neq x_\beta \) for all \( \alpha, \beta < \lambda \).

We define a mapping \( f : X \to X \) by \( f(y_\alpha) = y_\alpha \) for each \( \alpha < \lambda \), and \( f(x) = x \) for each \( x \in X \setminus \{y_\alpha : \alpha < \lambda\} \). Clearly, \( f \) is bornologous. Since \( \{x_\alpha : \alpha < \lambda\} \) is unbounded, \( f(x_\alpha) = x_\alpha, y_\alpha \in E[x_\alpha] \) and \( f(y_\alpha) = y_\alpha \) for each \( \alpha < \lambda \), we conclude that \( f \) is not macro-uniform. \( \Box \)

Proof of Theorem 2. (i) \( \implies \) (ii). We say that a permutation \( g \) of \( X \) is compatible with \( \mathcal{E} \) if there exists \( E \in \mathcal{E} \) such that \((x, gx) \in E \) for each \( x \in X \). We note that the set \( G \) of all permutations compatible with \( \mathcal{E} \) is a subgroup of \( S_X \) and, by Theorem 1 from [2], \((X, \mathcal{E}) = X_G \).

We say that a subset \( Y \) of \( X \) is crowded if there exists \( E \in \mathcal{E} \) such that \( |E[y]| > 1 \) for each \( y \in Y \). We take the minimal cardinal \( \kappa, \kappa \leq |X|^+ \) such that, for each \( \lambda < \kappa \), \((X, \mathcal{E})\) has a crowded subset of cardinality \( \lambda \).

We show that \( G = S_X^{<\kappa} \). If \( g \in G \) then \(|\text{supp } g| < \kappa \) because the set \( \text{supp } g \) is crowded, so \( g \in S_X^{<\kappa} \) and \( G \subseteq S_X^{<\kappa} \).

To prove \( S_X^{<\kappa} \subseteq G \), we need the following auxiliary statement.

(*) Let \( Y \) be a subset of \( X \) such that \( |Y| = |X| \), \( \lambda \) be a cardinal, \( \lambda < \kappa \). Then there exists two injective \( \lambda \)-sequences \((x_\alpha)_{\alpha < \lambda}, (y_\alpha)_{\alpha < \lambda} \) in \( Y \) and \( H \in \mathcal{E} \) such that \( \{x_\alpha : \alpha < \lambda\} \cap \{y_\alpha : \alpha < \lambda\} = \emptyset \) and \((x_\alpha, y_\alpha) \in H \) for each \( \alpha < \lambda \).
By the choice of $κ$, we can choose $E ∈ E$ and injective $λ$-sequences $(a_α)_{α < λ}, (b_α)_{α < λ}$ such that $\{a_α: α < λ\} ∩ \{b_α: α < λ\} = ∅$ and $(a_α, b_α) ∈ E$ for each $α < λ$. Passing to subsequences, we may suppose that $|Y \setminus \{a_α, b_α: α < λ\}| = |X|$. We choose two injective $λ$-sequences $(x_α)_{α < λ}, (y_α)_{α < λ}$ in $Y \setminus \{a_α, b_α: α < λ\}$ such that $\{x_α: α < λ\} ∩ \{y_α: α < λ\} = ∅$. Then we define an involution $f$ of $X$ by $fa_α = x_α, fb_α = y_α$ and $fx = x$ for each $x ∈ X \setminus \{a_α, b_α, x_α, y_α: α < λ\}$. Since $f$ is macro-uniform, there exists $H ∈ E$ such that $(x, y) ∈ E$ implies $(fx, fy) ∈ H$. Hence, $(x_α, y_α) ∈ H$ for each $α < λ$.

Now let $g ∈ S^κ_X, A = \text{supp } g$. We prove that $g$ is compatible with $E$, so $g ∈ G$. By the 3-Sets Lemma, there exists a partition $A_1, A_2, A_3$ of $A$ such that $A_i ∩ gA_i = ∅, i ∈ \{1, 2, 3\}$.

If $|X| = |X \setminus (A_1 ∪ gA_1)|$ then we denote $Y = X \setminus (A_1 ∪ gA_1), λ = |A_1|$ and apply (1) to choose corresponding $(x_α)_{α < λ}, (y_α)_{α < λ}$ and $H ∈ E$. We enumerate $A_1 = \{a_α: α < λ\}$ and define an involution $h$ of $Y$ by $hx_α = a_α, hy_α = ga_α$ and $hx = x$ for each $x ∈ X \setminus \{x_α, y_α, a_α, ga_α: α < λ\}$. Since $h$ is macro-uniform, there exists $K ∈ E$ such that $(x, y) ∈ H$ implies $(hx, hy) ∈ K$. Hence, $(a_α, ga_α) ∈ K$ for each $α < λ$.

If $|X \setminus (A_1 ∪ gA_1)| < |X|$ then we partition $A_1 = B ∪ C, |B| = |C| = |X|$ and, to choose $K$, apply above arguments for the pair $B, gB$ and $C, gC$.

Repeating above construction for $A_2$ and $A_3$, we see that $g$ is compatible with $E$.

(ii) $→$ (i). Let $G = S^κ_X, (X, E) = X_G$. We take an arbitrary $h ∈ S_X$ and show that $h: X_G → X_G$ is macro-uniform.

Let $F$ be a finite subset of $S^κ_X, x ∈ X, y = hx$. Then $hFh^{-1} ⊂ S^κ_X$ and, for $f ∈ F$, we have $(hx, hf x) = (y, hfh^{-1}y)$, so $h$ is macro-uniform. □

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