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CHARACTERIZATIONS OF UPPER AND LOWER $(\alpha, \beta, \theta, \delta, \mathcal{I})$-CONTINUOUS MULTIFUNCTIONS

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Let (X, τ) and (Y, σ) be topological spaces in which no separation axioms are assumed, unless explicitly stated and if $\mathcal{I}$ is an ideal on X. Given a multifunction $F\colon (X, \tau) \to (Y, \sigma)$, α, β operators on (X, τ), θ, δ operators on (Y, σ) and $\mathcal{I}$ a proper ideal on X. We introduce and study upper and lower $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous multifunctions. A multifunction $F\colon (X, \tau) \to (Y, \sigma)$ is said to be: 1) upper-$(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous if $\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V))) \in \mathcal{I}$ for each open subset V of Y; 2) lower-$(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous if $\alpha(F^-(\delta(V))) \setminus \beta(F^-(\theta(V))) \in \mathcal{I}$ for each open subset V of Y; 3) $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous if it is upper- and lower-$(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous. In particular, the following statements are proved in the article (Theorem 2): Let α, β be operators on (X, τ) and θ, θ^*, δ operators on (Y, σ):

1. The multifunction $F\colon (X, \tau) \to (Y, \sigma)$ is upper $(\alpha, \beta, \theta \cap \theta^*, \delta, \mathcal{I})$-continuous if and only if it is both upper $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous and upper $(\alpha, \beta, \theta^*, \delta, \mathcal{I})$-continuous.
2. The multifunction $F\colon (X, \tau) \to (Y, \sigma)$ is lower $(\alpha, \beta, \theta \cap \theta^*, \delta, \mathcal{I})$-continuous if and only if it is both lower $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous and lower $(\alpha, \beta, \theta^*, \delta, \mathcal{I})$-continuous, provided that $\beta(A \cap B) = \beta(A) \cap \beta(B)$ for any subset A, B of X.

1. Introduction. It is well known today, that the notion of multifunction is playing a very important role in general topology, upper and lower continuity have been extensively studied on multifunctions $F\colon (X, \tau) \to (Y, \sigma)$. Currently using the notion of topological ideal introduced by Kuratowski [24], different types of upper and lower continuity in multifunction $F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ have been studied and characterized [1], [14], [15], [34], [37], [33], [30], [31], [10], [28], [13]. Kasahara in [21], introduced the concept of an operator on a topology τ on a set X as a map $\alpha\colon \tau \to P(X)$ such that $U \subseteq \alpha(U)$ for all $U \in \tau$. Vielma and Rosas [36], modified the above definition by considering the operator α to be defined on $P(X)$; they are called operators on (X, τ). By a multifunction $F\colon X \to Y$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F\colon X \to Y$, the upper and lower inverse of any subset A of Y denoted by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X\colon F(x) \subseteq A\}$ and $F^-(A) = \{x \in X\colon F(x) \cap A \neq \varnothing\}$. In particular, $F^+(y) = \{x \in X\colon y \in F(x)\}$ for each point $y \in Y$. Jankovic and Hamlett [18], introduced the notion of $\mathcal{I}$-open set in a topological space (X, τ) with an ideal $\mathcal{I}$ on X. In this direction, we introduce the concept of upper and lower $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous multifunctions, where α, β are operators on (X, τ), θ, δ are operators on (Y, σ) and $\mathcal{I}$ a proper ideal on X. Selecting the operators $(\alpha, \beta, \theta, \delta)$ and $\mathcal{I}$ in an adequate form we can obtain many well known forms of continuity in multifunctions.

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Finally we obtain some decomposition forms for upper and lower $(\alpha,\beta,\theta,\delta,\mathcal{I})$-continuous multifunctions.

2. Preliminaries. Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed, unless explicitly stated and if $\mathcal{I}$ is an ideal on X, $(X,\tau,\mathcal{I})$ mean an ideal topological space. For a subset A of (X,τ), $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset A is said to be regular open [35] (resp. semiopen [19], preopen [25]) if $A=\mathrm{int}(\mathrm{cl}(A))$ (resp.$A\subseteq \mathrm{cl}(\mathrm{int}(A)), A\subseteq \mathrm{int}(\mathrm{cl}(A))$). The complement of a regular open (resp. semiopen, semi-preopen) set is called regular closed (resp. semiclosed) set. A subset S of $(X,\tau,\mathcal{I})$ is an $\mathcal{I}$-open[18], if $S\subseteq \mathrm{int}(S^*)$. The complement of an $\mathcal{I}$-open set is called $\mathcal{I}$-closed set. The $\mathcal{I}$-closure and the $\mathcal{I}$-interior, can be defined in the same way as $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively, will be denoted by $\mathcal{I}\mathrm{cl}(A)$ and $\mathcal{I}\mathrm{int}(A)$, respectively. The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed, regular open, regular closed, semiopen, semi closed, preopen) subsets of a $(X,\tau,\mathcal{I})$, is denoted by $IO(X)$(resp. $IC(X), RO(X), RC(X), SO(X), SC(X), PO(X)$). We set $IO(X,x)=\{A\colon A\in IO(X) \text{ and } x\in A\}$. It is well known that in an ideal topological space $(X,\tau,\mathcal{I})$, $\alpha\colon P(X)\to P(X)$ (resp. $\beta\colon P(X)\to P(X)$) defined as $\alpha(U)=\mathcal{I}\mathrm{cl}(U)$ (resp. $\beta(U)=\mathcal{I}\mathrm{int}(U)$) are operators on $(X,\tau,\mathcal{I})$.

3. General form of continuity on multifunctions. We introduce the notion of continuity for a multifunction F in a natural form. In the following definition, let (X,τ), (Y,σ), α,β be operators on (X,τ), θ,δ operators on (Y,σ) and $\mathcal{I}$ a proper ideal on X.

Definition 1. A multifunction $F\colon (X,\tau)\to(Y,\sigma)$ is said to be:

1. *upper*-$(\alpha,\beta,\theta,\delta,\mathcal{I})$-*continuous* if for each open subset V of Y,
$$\alpha(F^+(\delta(V)))\setminus\beta(F^+(\theta(V)))\in\mathcal{I};$$
2. *lower*-$(\alpha,\beta,\theta,\delta,\mathcal{I})$-*continuous* if for each open subset V of Y,
$$\alpha(F^-(\delta(V)))\setminus\beta(F^-(\theta(V)))\in\mathcal{I};$$
3. $(\alpha,\beta,\theta,\delta,\mathcal{I})$-*continuous* if it is upper- and lower-$(\alpha,\beta,\theta,\delta,\mathcal{I})$-continuous.

We can see in a natural form that Definition 1 generalize many forms of weak upper (resp. lower)-continuous multifunctions as we can see:

1. The notion of upper (resp. lower)-continuous multifunction [1], when we choose $\alpha=$ identity operator, $\beta=$ interior operator, $\delta=$ identity operator, $\theta=$ identity operator, and $\mathcal{I}=\{\varnothing\}$.
2. The notion of upper (resp. lower)-semi continuous multifunction [9], when we choose $\alpha=$ identity operator, $\beta=$ closure interior operator, $\delta=$ identity operator, $\theta=$ identity operator, and $\mathcal{I}=\{\varnothing\}$.
3. The notion of upper (resp. lower)-pre continuous multifunction [29], when we choose $\alpha=$ identity operator, $\beta=$ interior closure operator, $\delta=$ identity operator, $\theta=$ identity operator, and $\mathcal{I}=\{\varnothing\}$.
4. The notion of upper (resp. lower) weakly continuous multifunction [34], when we choose $\alpha=$ identity operator, $\beta=$ interior operator, $\delta=$ identity operator, $\theta=$ closure operator, and $\mathcal{I}=\{\varnothing\}$. That means $F^+(V)\setminus\mathrm{int}(F^+(\mathrm{cl}(V)))\in\{\varnothing\}$, (resp. $F^-(V)\setminus\mathrm{int}(F^-(\mathrm{cl}(V)))\in\{\varnothing\}$).
5. The notion of upper (resp. lower) weakly $\mathcal{I}$-continuous multifunction [3], when we choose $\alpha=$ identity operator, $\beta=\mathcal{I}$-interior operator, $\delta=$ identity operator, $\theta=$ closure operator, and $\mathcal{I}=\{\varnothing\}$. That means $F^+(V)\setminus \mathrm{Iint}(F^+(\mathrm{cl}(V)))\in\{\varnothing\}$ (resp. $F^-(V)\setminus \mathrm{Iint}(F^-(\mathrm{cl}(V)))\in\{\varnothing\}$).

6. The notion of upper (resp. lower)-almost continuous multifunction [28], $F^+(V)$ (resp. $F^-(V)$) is preopen subset in X for any open subset V in Y. When we choose $\alpha =$ identity operator, $\beta =$ interior closure operator, $\delta =$ identity operator, $\theta =$ identity operator, and $\mathcal{I} = \{\varnothing\}$.

7. The notion of upper(resp. lower) almost continuous multifunction [7], when we choose $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ identity operator, $\theta =$ interior closure operator, and $\mathcal{I} = \{\varnothing\}$. That means $F^+(V) \setminus \operatorname{int}(F^+(intcl(V))) \in \{\varnothing\}$, (resp. $F^-(V) \setminus (\operatorname{int} F^-(\operatorname{intcl}(V))) \in \{\varnothing\}$).

8. The notion of upper(resp. lower) almost $\mathcal{I}$-continuous multifunction [7], when we choose $\alpha =$ identity operator, $\beta = \mathcal{I}$-interior operator, $\delta =$ identity operator, $\theta =$ interior closure operator, and $\mathcal{I} = \{\varnothing\}$. That means $F^+(V) \setminus I\operatorname{int}(F^+(\operatorname{intcl}(V))) \in \{\varnothing\}$ (resp. $F^-(V) \setminus (\operatorname{Iint} F^-(\operatorname{intcl}(V))) \in \{\varnothing\}$).

9. The notion of upper (resp. lower)-ω-continuous multifunction [37], (briefly u-w-c (l-w-c)), when we choose $\alpha =$ identity operator, $\beta = \omega$-interior operator, $\delta =$ identity operator, $\theta =$ identity operator, and $I = \{\varnothing\}$.

10. The notion of upper (resp. lower)-$\mathcal{I}$-continuous multifunction [1], when we choose $\alpha =$ identity operator, $\beta = \mathcal{I}$-interior operator, $\delta =$ identity operator, $\theta =$ identity operator, and $\mathcal{I} = \{\varnothing\}$.

11. The notion of upper (resp. lower)-continuous multifunction [1], when we choose $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ identity operator, $\theta =$ identity operator, and $\mathcal{I} = \{\varnothing\}$.

12. The notion of upper (resp. lower)-completely continuous multifunction [23], when we choose $\alpha =$ identity operator, $\beta =$ interior closure operator, $\delta =$ identity operator, $\theta =$ identity operator, and $\mathcal{I} = \{\varnothing\}$.

13. The notion of upper (resp. lower)-perfectly continuous multifunction [22], when we choose $\alpha =$ closure operator, $\beta =$ interior operator, $\delta =$ identity operator, $\theta =$ identity operator, and $\mathcal{I} = \{\varnothing\}$.

It is well known that there exist some definitions of upper and lower continuous multifunction that need some additional condition on a subcollection of open subsets of Y. For example see [14], [15], [20], [11], [10], [5], [32]. We now introduce the following definition.

Definition 2. A multifunction $F\colon (X, \tau) \to (Y, \sigma)$ is said to be:

1. *upper-P-continuous* if for each open subset V of Y, satisfying property P, $F^+(V)$ is an open subset in X.
2. *lower-P-continuous* if for each open subset V of Y, satisfying property P, $F^-(V)$ is an open subset in X.
3. *P-continuous* if it is upper-P-continuous and lower-P-continuous.

Let (Y, σ) be a topological space and define $\theta_P\colon P(Y) \to P(Y)$ by $\theta_P(A) = A$ if A is an open and satisfies property P and $\theta_P(A) = Y$ otherwise.
Note that θ_P is an operator on Y.

Theorem 1. *A multifunction $F\colon (X, \tau) \to (Y, \sigma)$ is upper (resp. lower) P-continuous if and only if it is upper (resp. lower) $(\alpha, \beta, \theta_P, \delta, \mathcal{I})$-continuous, where $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ identity operator and $\mathcal{I} = \{\varnothing\}$.*

Proof. Suppose that F is upper P-continuous and let V be an open subset of Y. Consider two cases:

Case 1. V satisfies property P, follows $\theta_P(V) = V$ and then $F^+(V)$ is an open subset in X and satisfies $F^+(V) \subseteq \mathrm{int}(F^+(\theta_P(V))) = \mathrm{int}(F^+(V))$. So F is upper $(\mathrm{id}, \mathrm{int}, \theta_P, \mathrm{id}, \{\varnothing\})$-continuous.
Case 2. V does not satisfy property P, follows $\theta_P(V) = Y$ and $F^+(V) \subseteq \mathrm{int}(F^+(\theta_P(V))) = \mathrm{int}(F^+(Y)) = X$ and then F is upper $(\mathrm{id}, \mathrm{int}, \theta_P, \mathrm{id}, \{\varnothing\})$-continuous.
Conversely, suppose that F is upper $(\mathrm{id}, \mathrm{int}, \theta_P, \mathrm{id}, \{\varnothing\})$-continuous multifunction, then for any open subset V in Y, $F^+(V) \subseteq \mathrm{int}(F^+(\theta_P(V)))$. If we take V satisfying property p, $\theta_P(V) = V$, and then $F^+(V) \subseteq \mathrm{int}(F^+(V))$. In consequence, $F^+(V) = \mathrm{int}(F^+(V))$ and therefore, $F^+(V)$ is an open subset in X. $\square$

Observe that Theorem 1, generalize many forms of weak upper (resp. lower)-continuous multifunction when the open subset V in Y, satisfies a particular condition as we can see:

1. E. Ekici in [14], defined upper (resp. lower) nearly continuous multifuncion F, $F^+(V)$ (resp. $F^-(V)$) is an open subset in X for any open subset V in Y, having N-closed complement.
2. E. Ekici in [15], defined upper (resp. lower) almost nearly continuous multifunction F, $F^+(V)$ (resp. $F^-(V)$) is an open subset in X for any regular open subset V in Y, having N-closed in Y.
3. A. Kambir and I. L. Reilly in [20], defined upper (resp. lower) almost l-continuous multifuncion F, $F^+(V)$ (resp. $F^-(V)$) is an open subset in X for any regular open subset V in Y, having lindelöof complement in Y.
4. C. Carpintero, E. Rosas and J. Moreno in [11], defined more on upper and lower almost nearly $\mathcal{I}$-continuous multifunction F, $F^+(V)$ (resp. $F^-(V)$) is an open subset in X for any open subset V in Y, having N-closed complement in Y.
5. C. Carpintero, J. Pacheco, N. Rajesh, E. Rosas and S. Saranyasri in [10], defined upper (resp. lower) nearly ω-continuous multifunction F, $F^+(V)$ (resp. $F^-(V)$) is an ω-open subset in X for any open subset V in Y, having N-closed complement.
6. C. Arivazhagi and N. Rajesh, in [5], defined upper (resp. lower) nearly $\mathcal{I}$-continuous multifunction F, $F^+(V)$ (resp. $F^-(V)$) is an $\mathcal{I}$-open subset in X for any open subset V in Y, having N-closed complement.
7. C. E. Rosas, C. Carpintero and J. Moreno in [32], defined upper and lower nearly $(\mathcal{I}, \mathcal{J})$-continuous multifunction F, $F^+(V)$ (resp. $F^-(V)$) is an $\mathcal{I}$-open subset in X for any $\mathcal{J}$-open subset V in Y, having N-closed complement in Y.

Now there is a natural question. It is possible to used in some form the Definition 1, in order to obtain some equivalences for another forms of continuity in multifunctions and the answer is yes, as we write down in each particular case:

1. Upper (resp. lower) $(\mathcal{I}, \mathcal{J})$-continuous multifunctions defined in [32]. $F^+(V)$ (resp. $F^-(V)$) is an $\mathcal{I}$-open subset of X for each $\mathcal{J}$-open subset V of Y.
Choose α = identity operator, $\beta = \mathcal{I}$-interior operator, δ = identity operator, θ = closure operator, and $\mathcal{I} = \{\varnothing\}$.
2. Upper (resp. lower) weakly $(\mathcal{I}, \mathcal{I})$-continuous multifunctions defined in [31]. $F^+(V) \subseteq \mathrm{Iint}(F^+(\mathrm{Jcl}((V)))$ (resp. $F^-(V) \subseteq \mathrm{Iint}(F^-(\mathrm{Jcl}((V))))$.
Choose α = identity operator, $\beta = \mathcal{I}$-interior operator, δ = identity operator, $\theta = \mathcal{J}$-closure operator, and $\mathcal{I} = \{\varnothing\}$.
3. Upper (resp. lower) almost weakly-continuous multifunctions defined in [28]. $F^+(V) \subseteq \mathrm{int}(\mathrm{cl}(F^+(\mathrm{cl}((V)))))$ (resp. $F^-(V) \subseteq \mathrm{int}(\mathrm{cl}(F^-(\mathrm{cl}((V))))))$.

Choose α = identity operator, β = interior operator, δ = identity operator, θ = closure operator, and $\mathcal{I} = \{\varnothing\}$.

4. Upper (resp. lower) contra-continuous multifunctions defined in [17]. $F^+(V)$ (resp. $F^-(V)$) is open for each closed set V in Y.
Choose α = identity operator, β = interior operator, δ = identity operator, θ = closure operator, and $\mathcal{I} = \{\varnothing\}$.

5. Upper (resp. lower) contra $(\mathcal{I}, \mathcal{J})$-continuous multifunctions defined in [31]. $F^+(V)$ (resp. $F^-(V)$) is $\mathcal{I}$-open for each $\mathcal{J}$-regular closed set V in Y.
Choose α = identity operator, $\beta = \mathcal{I}$-interior operator, δ = identity operator, $\theta = \mathcal{I}$-closure $\mathcal{I}$-interior operator, and $\mathcal{I} = \{\varnothing\}$.

6. Upper (resp. lower) almost contra-continuous multifunctions defined in [16]. $F^+(V)$ $(F^-(V))$ is open for each regular closed set V in Y.
Choose α = identity operator, β = interior operator, δ = identity operator, θ = closure interior operator, and $\mathcal{I} = \{\varnothing\}$.

7. Upper (resp. lower) almost contra $\mathcal{I}$-continuous multifunctions defined in [6]. F^+ (resp. $F^-(V)$) is open for each regular closed set V in Y.
Choose α = identity operator, $\beta = \mathcal{I}$-interior operator, δ = identity operator, θ = closure interior operator, and $\mathcal{I} = \{\varnothing\}$.

8. Upper (resp. lower) almost contra $(\mathcal{I}, \mathcal{J})$-continuous multifunctions defined in [33]. $F^+(V)$ (resp. $F^-(V)$) is $\mathcal{I}$-open for each $\mathcal{J}$-regular closed set V in Y.
Choose α = identity operator, $\beta = \mathcal{I}$-interior operator, δ = identity operator, $\theta = \mathcal{J}$-closure $\mathcal{J}$-interior operator, and $\mathcal{I} = \{\varnothing\}$.

9. Upper (resp. lower) contra ω-continuous multifunctions defined in [12] $F^+(V)$ (resp. $F^-(V)$) is ω-open for each closed set V in Y.
Choose α = identity operator, β = ω-interior operator, δ = identity operator, θ = closure operator, and $\mathcal{I} = \{\varnothing\}$.

10. Upper (resp. lower) contra $\mathcal{I}$-continuous multifunctions defined in [2] $F^+(V)$ (resp. $F^-(V)$) is $\mathcal{I}$-open for each closed set V in Y.
Choose α = identity operator, $\beta = \mathcal{I}$-interior operator, δ = identity operator, θ = closure operator, and $\mathcal{I} = \{\varnothing\}$.

11. Upper (resp. lower) slightly continuous multifunctions defined in [27] $F^+(V)$ (resp. $F^-(V)$) is open for each clopen set V in Y.
Choose α =identity operator, β = interior operator, δ = identity operator, θ = interior closure operator, and $\mathcal{I} = \{\varnothing\}$.

12. Upper (resp. lower) slightly $\mathcal{I}$-continuous multifunctions defined in [8] $F^+(V)$ (resp. $F^-(V)$) is $\mathcal{I}$-open for each clopen set V in Y.
Choose α =identity operator, $\beta = \mathcal{I}$-interior operator, δ = identity operator, θ = interior closure operator, and $\mathcal{I} = \{\varnothing\}$.

13. Upper (resp. lower) faintly continuous multifunctions defined in [1] $F^+(V)$ (resp. $F^-(V)$) is open for each θ open set V in Y.
Choose α = identity operator, β = interior operator, δ = identity operator, $\theta = \theta$-interior operator, and $\mathcal{I} = \{\varnothing\}$.

14. Upper (resp. lower) faintly $\mathcal{I}$-continuous multifunctions defined in [4] $F^+(V)$ (resp. $F^-(V)$) is $\mathcal{I}$-open for each θ open set V in Y.
Choose α = identity operator, $\beta = \mathcal{I}$-interior operator, δ = identity operator, $\theta = \theta$-interior operator, and $\mathcal{I} = \{\varnothing\}$.

15. Upper (resp. lower) faintly ω-continuous multifunctions defined in [13] $F^+(V)$ (resp.

$F^-(V)$) is ω-open for each θ open set V in Y.
Choose α = identity operator, $\beta = \omega$-interior operator, δ = identity operator, $\theta = \theta$-interior operator, and $\mathcal{I} = \{\varnothing\}$.

Remark 1. In the same form as above, if X, Y are nonempty sets, m_X (resp. m_Y) is a minimal structure on X (resp. Y) and $F\colon (X, m_X) \to (Y, m_Y)$ is a multifunction. If F is upper (resp. lower) M-continuous multifunction [26]; we can obtain it choosing in an adequate form the operators $\alpha, \beta, \delta, \theta$ and the ideal $\mathcal{I}$.

Definition 3 ([36])**.** Let α and α^* be operators on (X, τ). We write $\alpha \subseteq \alpha^*$ if for all subset V of X, $\alpha(V) \subseteq \alpha^*(V)$.

Definition 4 ([36])**.** If α and α^* are operators on (X, τ), the intersection operator is defined as $\alpha \cap \alpha^*(V) = \alpha(V) \cap \alpha^*(V)$ for any subset V of X. The operators α and α^* are said to be *mutually dual* if $\alpha \cap \alpha^*$ is the identity operator.

Theorem 2. *Let (X, τ) and (Y, σ) be two topological spaces and $\mathcal{I}$ a proper ideal on X. Let α, β be operators on (X, τ) and θ, θ^*, δ operators on (Y, σ):*

1. The multifunction $F\colon (X, \tau) \to (Y, \sigma)$ is upper $(\alpha, \beta, \theta \cap \theta^*, \delta, \mathcal{I})$-continuous if and only if it is both upper $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous and upper $(\alpha, \beta, \theta^*, \delta, \mathcal{I})$-continuous.

2. The multifunction $F\colon (X, \tau) \to (Y, \sigma)$ is lower $(\alpha, \beta, \theta \cap \theta^*, \delta, \mathcal{I})$-continuous if and only if it is both lower $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous and lower $(\alpha, \beta, \theta^*, \delta, \mathcal{I})$-continuous, provided that for any subset A, B of X $\beta(A \cap B) = \beta(A) \cap \beta(B)$.

Proof. We only proof the case for upper $(\alpha, \beta, \theta \cap \theta^*, \delta, \mathcal{I})$-continuous. The proof for lower $(\alpha, \beta, \theta \cap \theta^*, \delta, \mathcal{I})$-continuous is similar.

If F is both $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous and $(\alpha, \beta, \theta^*, \delta, \mathcal{I})$-continuous, then for any open subset V of Y, $\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V))) \in \mathcal{I}$ and $\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta^*(V))) \in \mathcal{I}$. Then using the property of the ideal,

$$[\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V)))] \cup [\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta^*(V)))] \in \mathcal{I} = \alpha(F^+(\delta(V)))\setminus \\ \setminus[\beta(F^+(\theta(V))) \cap \beta(F^+(\theta^*(V)))] = \alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta \cap \theta^*(V))).$$

Follows that F is upper $(\alpha, \beta, \theta\cap\theta^*, \delta, \mathcal{I})$-continuous. Conversely, F is upper $(\alpha, \beta, \theta\cap\theta^*, \delta, \mathcal{I})$-continuous, then $\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta \cap \theta^*(V))) \in \mathcal{I}$. It follows that

$$[\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V)))] \cup [\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta^*(V))) \in \mathcal{I}].$$

Now using the hereditary property of the ideal, we obtain that $\alpha(F^+(\delta(V)))\setminus\beta(F^+(\theta(V))) \in \mathcal{I}$ and $\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta^*(V))) \in \mathcal{I}$. Hence, F is both $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous and $(\alpha, \beta, \theta^*, \delta, \mathcal{I})$-continuous. □

Theorem 3. *Let (X, τ) and (Y, σ) be two topological spaces and $\mathcal{I}$ a proper ideal on X. Let $\alpha, \alpha^*, \beta, \beta^*$ be operators on (X, τ) and θ, θ^*, δ operators on (Y, σ):*

1. If β is a monotone operator, $\theta \subseteq \theta^*$ and $F\colon (X, \tau) \to (Y, \sigma)$ is upper (resp. lower) $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous multifunction then F is upper (resp. lower) $(\alpha, \beta, \theta^*, \delta, \mathcal{I})$-continuous.

2. If $\alpha^* \subseteq \alpha$ and $F\colon (X, \tau) \to (Y, \sigma)$ is upper (resp. lower) $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous multifunction then F is upper (resp. lower) $(\alpha^*, \beta, \delta, \mathcal{I})$-continuous.

3. If $\beta \subseteq \beta^*$ and $F\colon (X, \tau) \to (Y, \sigma)$ is upper (resp. lower) $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous multifunction then F is upper (resp. lower) $(\alpha, \beta^*, \delta, \mathcal{I})$-continuous.

Proof. (1) By hypothesis for any open subset V in Y, $\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V))) \in \mathcal{I}$. Since $\theta(V) \subseteq \theta^*(V)$ and β is a monotone operator we have $\beta(F^+(\theta(V))) \subseteq \beta(F^+(\theta^*(V)))$. In consequence, $\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta^*(V))) \subseteq \alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V))) \in \mathcal{I}$ and then $\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V))) \in I$. Therefore, F is upper $(\alpha, \beta, \theta^*, \delta, \mathcal{I})$-continuous.
(2) By hypothesis for any open subset V in Y, $\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V))) \in \mathcal{I}$. Since $\alpha^*(V) \subseteq \alpha(V)$ we have $\alpha^*(F^+(\delta(V))) \subseteq \alpha(F^+(\delta(V)))$. In consequence,

$$\alpha^*(F^+(\delta(V))) \setminus \beta(F^+(\theta(V))) \subseteq \alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V))) \in \mathcal{I}.$$

Therefore, F is upper $(\alpha^*, \beta, \theta^*, \delta, \mathcal{I})$-continuous.
(3) By hypothesis for any open subset V in Y, $\alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V))) \in \mathcal{I}$. Since $\beta(V) \subseteq \beta^*(V)$, we have $\beta(F^+(\theta(V))) \subseteq \beta^*(F^+(\theta(V)))$. In consequence,

$$\alpha(F^+(\delta(V))) \setminus \beta^*(F^+(\theta(V))) \subseteq \alpha(F^+(\delta(V))) \setminus \beta(F^+(\theta(V))) \in \mathcal{I}.$$

Therefore, F is upper $(\alpha, \beta^*, \theta^*, \delta, \mathcal{I})$-continuous. The cases for lower continuity are doing in a similar form. □

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