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UNIVERSALLY PRESTARLIKE FUNCTIONS ASSOCIATED WITH SHELL LIKE DOMAIN

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In this paper, we introduce universally prestarlike generalized functions of order ϑ with $\vartheta \leq 1$ associated with shell like domain, and we get coefficient bounds and the second Hankel determinant $|a_2a_4 - a_3^2|$ for such functions.

1. Introduction. Let $\mathcal{A}(\Delta)$ denote the class of all analytic functions in a domain Δ. Suppose Δ contains the origin and $\mathcal{A}_0(\Delta)$ stands for the set of all functions $f \in \mathcal{A}(\Delta)$ with $f(0) = 1$ and also let

$$\mathcal{A}_1(\Delta) = \{zf : f \in \mathcal{A}_0(\Delta)\}.$$

If $\Delta = \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc, we write $\mathcal{A} \equiv \mathcal{A}(\mathbb{U}), \mathcal{A}_0 \equiv \mathcal{A}_0(\mathbb{U})$ and $\mathcal{A}_1 \equiv \mathcal{A}_1(\mathbb{U})$. Let the Hadamard Product (or convolution) of two functions

$$f(z) = \sum_{t=0}^{\infty} a_t z^t \quad \text{and} \quad g(z) = \sum_{t=0}^{\infty} b_t z^t, z \in \mathbb{U}$$

in $\mathcal{A}_0(\Delta)$ is defined as

$$(f * g)(z) = \sum_{t=0}^{\infty} a_t b_t z^t.$$

A function $f \in \mathcal{A}_1$ is called a starlike function of order ϑ $(0 \leq \vartheta < 1)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \vartheta, \quad (z \in \mathbb{U})$$

and the class of such functions is denoted by $\mathcal{S}_\vartheta$.

Due to S. Ruscheweyh [15], for $f \in \mathcal{A}_1$, let us denote by $\mathcal{R}_\vartheta$ the class of all prestarlike functions of order ϑ $(\vartheta \leq 1)$ in $\mathbb{U}$ satisfying the criteria

$$\begin{cases} h_{2-2\vartheta} * f \in \mathcal{S}_\vartheta, & \vartheta < 1, \\ \operatorname{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{2}, & \vartheta = 1, \ z \in \mathbb{U}, \end{cases}$$

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where

$$h_{2-2\vartheta}(z) = \frac{z}{(1-z)^{2-2\vartheta}} = z + \sum_{t=2}^{\infty} \mathcal{C}(\vartheta, t) z^t$$

is a well-known extremal function in $\mathcal{S}_\vartheta$ and

$$\mathcal{C}(\vartheta, t) = \frac{\prod_{k=2}^{t}(k-2\vartheta)}{(t-1)!} \quad (t \in \mathbb{N} \setminus \{1\}, \mathbb{N} := \{1, 2, 3, \cdots\}).$$

Notice that $\mathcal{C}(\vartheta, t)$ is a decreasing function of ϑ with

$$\lim_{t\to\infty} \mathcal{C}(\vartheta, t) = \begin{cases} \infty & \text{if } \vartheta < \frac{1}{2}, \\ 1 & \text{if } \vartheta = \frac{1}{2}, \\ 0 & \text{if } \vartheta > \frac{1}{2}. \end{cases}$$

While studying with prestarlike functions and convolutions, the following notation is turned out to be useful

$$(D^t f)(z) = (h_t * f)(z), \quad h_t(z) = \frac{z}{(1-z)^t},$$

where $t \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ and therefore we have $D^{t+1} f = \frac{z}{t!}\left(z^{t-1} f\right)^{(t)}$, for $t \in \mathbb{N}_0$. Using this operator we find that a function $f \in \mathcal{A}_1$ is prestarlike of order $\vartheta \leq 1$ if and only if

$$\frac{D^{3-2\vartheta} f}{D^{2-2\vartheta} f} \in \mathcal{P}$$

where $\mathcal{P} = \{g \in \mathcal{A}_0 : \operatorname{Re}(g(z)) > \frac{1}{2}, z \in \mathbb{U}\}$ or, equivalently, by the Herglotz formula

$$g \in \mathcal{P} \Leftrightarrow g(z) = \int_0^{2\pi} \frac{d\mu(\tau)}{1 - e^{-i\tau} z},$$

where μ is a probability measure on $[0, 2\pi]$.

The notion of prestarlike functions of order ϑ has recently been extended from the unit disc $\mathbb{U}$ to other discs and half-planes containing the origin (see [12, 13, 14]). Define one such a disc $\Delta_{\gamma,\rho}$ by

$$\Delta_{\gamma,\rho} = \{\varpi_{\gamma,\rho}(z) : z \in \mathbb{U}\},$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ and $\rho \in [0, 1]$ are two unique parameters and $\varpi_{\gamma,\rho}(z) = \frac{\gamma z}{1-\rho z}$. Note that $1 \notin \Delta_{\gamma,\rho}$ if and only if $|\gamma + \rho| \leq 1$. For $\vartheta \leq 1$, and for some admissible pair (γ, ρ), we define

$$\mathcal{R}_\vartheta(\Delta_{\gamma,\rho}) = \{f \in \mathcal{A}_1(\Delta_{\gamma,\rho}) : \frac{1}{\gamma} f(\varpi_{\gamma,\rho}(z)) \in \mathcal{R}_\vartheta\},$$

where $\mathcal{A}_1(\Delta_{\gamma,\rho}) = \{zf : f \in \mathcal{A}_0(\Delta_{\gamma,\rho}) \text{ with } f(0) = 1\}$. A function f in $\mathcal{R}_\vartheta(\Delta_{\gamma,\rho})$ is called a prestarlike function of order ϑ in $\Delta_{\gamma,\rho}$ (see [13]).

Definition 1 ([14]). Let $\vartheta \leq 1$ and $\Lambda = \mathbb{C}\backslash[1, \infty)$. A function $f \in \mathcal{A}_1(\Lambda)$ is named *universally prestarlike* of order ϑ in Λ if and only if f is prestarlike of order α in all sets $\varpi_{\gamma,\rho}$ with $|\gamma + \rho| \leq 1$. Denote the set of all universally prestarlike functions in Λ by $\mathcal{R}^u_\vartheta$.

Definition 2 ([11])**.** Let $\mathcal{S}^*(\varphi)$ denote the class of analytic functions f in the unit disc $\mathbb{U}$ normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition that

$$\frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2} =: \varphi(z), \quad z \in \mathbb{U}, \tag{1}$$

where the branch of the square root is chosen to be $\varphi(0) = 1$.

It may be noted from (1) in Definition 2 that the set $\varphi(\mathbb{U})$ lies in the right half-plane and it is not a starlike domain with respect to the origin, see Fig. 1 (below).

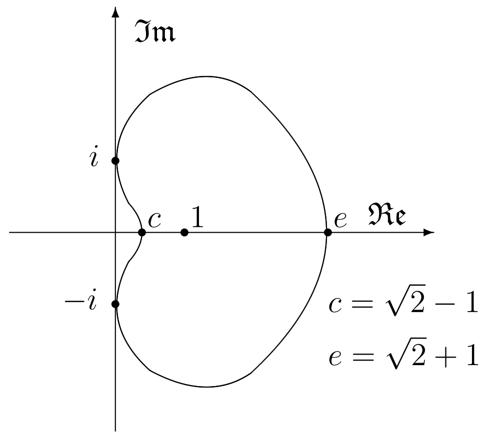

Fig. 1. The boundary of the set $\varphi(\mathbb{U})$.

Recently, Raina and Sokół[11] have studied and obtained some coefficient inequalities for the class $\mathcal{S}^*(\phi)$ and these results are further improved by Sokółand Thomas [20], for the class $\mathcal{C}(\varphi)$ in view of the Alexander result between the class $f \in \mathcal{C}(\varphi) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\varphi)$, further the Fekete-Szegő inequality for functions in $\mathcal{S}^*(\varphi)$ were also obtained.

For $\vartheta \leq 1$ and a function $f \in \mathcal{A}_1(\Lambda)$, we let $\mathcal{R}^u_\vartheta(\varphi)$ be the generalized class of universally prestarlike functions satisfying the condition

$$\frac{D^{3-2\vartheta} f}{D^{2-2\vartheta} f} \prec z + \sqrt{1+z^2} = \varphi(z),$$

where $\prec$ denotes the subordination and φ is an analytic function given by (1).

Recall that the Hankel determinants $H_q(t)$ $(t = 1, 2, 3, ...; q = 1, 2, ..)$ of the functions $f(z) = \sum_{t=1}^{\infty} a_t z^t$, $a_1 = 1$ are defined by

$$H_q(t) = \begin{vmatrix} a_t & a_{t+1} & \cdots & a_{t+q-1} \\ a_{t+1} & a_{t+2} & \cdots & a_{t+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t+q-1} & a_{t+q} & \cdots & a_{t+2q-2} \end{vmatrix}.$$

In particular, $H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1 a_3 - a_2^2$ and $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2$. For more details on the Hankel determinants, one may refer to the papers [5, 6, 7, 8, 10, 19].

Though there has been an increasing interest to study the functional $H_2(1)$ (that is $a_1a_3 - a_2^2$) for certain classes of universally prestarlike functions (see [16, 17, 18]) and in particular,the Fekete and Szegő estimates of $|a_3 - \mu a_2^2|$ (see [3]), the study of the functional $H_2(2)$ (that is, $a_2a_4 - a_3^2$) for universally prestarlike functions is in few papers [1] and not yet known for the functions related to certain conic domains. The main purpose of this paper is to obtain the upper bounds of the Hankel determinant $|a_2a_4 - a_3^2|$ for the functions $f \in \mathcal{R}_{\vartheta}^{u}(\varphi)$ related with shell-shaped regions.

2. Preliminary results. To prove our main results, we state the following lemmas.

Lemma 1 ([2], p. 41). *Let* **P** *be the class of all analytic functions* p *of the form*

$$p(z) = 1 + \sum_{t=1}^{\infty} p_t z^t \tag{2}$$

satisfying $\operatorname{Re}(p(z)) > 0 \ (z \in \mathbb{U})$ *and* $p(0) = 1$. *Then* $|p_t| \leq 2$ $(t = 1, 2, 3, \ldots)$. *This inequality is sharp for each* t. *In particular, the equality holds for all* t *and for the function*

$$p(z) = \frac{1+z}{1-z} = 1 + \sum_{t=1}^{\infty} 2z^t.$$

Lemma 2 ([9]). *If* $p_1(z) = 1 + c_1z + c_2z^2 + \cdots$ *is a function with positive real part in* $\mathbb{U}$, *then*

$$|\, c_t \,| \leq 2 \ \text{ for all } \ t \geq 1 \qquad \text{and} \qquad |c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.$$

The class of all such functions with positive real part is denoted by **P**.

Lemma 3 ([9]). *If* $p_1(z) = 1 + c_1z + c_2z^2 + \cdots$ *is a function with positive real part in* $\mathbb{U}$, *and* v *is a complex number, then*

$$|c_2 - vc_1^2| \leq 2\max(1, |2v - 1|).$$

The result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 4 ([7]). *If the function* $p \in \mathbf{P}$ *is given by (2), then*

$$2p_2 = p_1^2 + x(4 - p_1^2),$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some x, z *with* $|x| \leq 1, |z| \leq 1$ *and* $p_1 \in [0, 2]$.

Lemma 5 ([4]). *The power series for a function* p *given in (2) converges in* $\mathbb{U}$ *to a function in* **P** *if and only if the Toeplitz determinants*

$$D_t = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_t \\ p_{-1} & 2 & p_1 & \cdots & p_{t-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{-t} & p_{-t+1} & p_{-t+2} & \cdots & 2 \end{vmatrix}, \quad t = 1, 2, 3, \ldots$$

and $p_{-k} = \overline{p_k}$*, are all non-negative. They are strictly positive except for*

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{i\tau_k z}),\ \ \rho_k > 0,\ \ \tau_k \text{ real}$$

and $\tau_k \neq \tau_j$ *for* $k \neq j$*; in this case* $D_t > 0$ *for* $t < m-1$ *and* $D_t = 0$ *for* $t \geq m$*.*

This necessary and sufficient condition is due to Carathéodory and Toeplitz and can be found in [4].

3. The coefficient bounds for $f \in \mathcal{R}_{\vartheta}^{u}(\varphi)$**.** In this section, we obtain the coefficient bounds for $f \in \mathcal{R}_{\vartheta}^{u}(\varphi)$. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(\tau)}{1-\tau z},$$

where $a_k = \int_0^1 \tau^k d\mu(\tau)$, and $\mu(\tau)$ is a probability measure on $[0,1]$.

Theorem 1. *Let* $f \in \mathcal{R}_{\vartheta}^{u}(\varphi)$ *be given by* $f(z) = \sum_{t=0}^{\infty} a_t z^t$, ($a_0 = 0$ and $a_1 = 1$) *and suppose* φ *is defined by* (1)*. Then,*

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{(3-2\vartheta)} \max\left\{1, \left|2\vartheta - \frac{5}{2}\right|\right\}.$$

Further,

$$|a_3 - \varrho a_2^2| \leq \frac{1}{(3-2\vartheta)} \max\left\{1, \left|2\vartheta - \frac{5}{2} + (3-2\vartheta)\varrho\right|\right\}.$$

Proof. Since $f \in \mathcal{R}_{\vartheta}^{u}(\varphi)$, there exists a Schwarz function ϖ, analytic in $\mathbb{U}$ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ in $\mathbb{U}$ such that

$$\frac{D^{3-2\vartheta} f(z)}{D^{2-2\vartheta} f(z)} = \varphi(\varpi(z)). \tag{3}$$

Define the function ψ by

$$\psi(z) = \frac{1+\varpi(z)}{1-\varpi(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots.$$

Since ϖ is the Schwarz function, we see that $\mathrm{Re}(\psi(z)) \geq 0$ and $\psi(0) = 1$, and therefore $\psi \in \mathbf{P}$. It follows that

$$\varpi(z) = \frac{\psi(z)-1}{\psi(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) z^3 + \cdots\right]. \tag{4}$$

In view of the equations (4), we have

$$\varphi(\varpi(z)) = \varphi\left(\frac{\psi(z)-1}{\psi(z)+1}\right) = \sqrt{1 + \left(\frac{\psi(z)-1}{\psi(z)+1}\right)^2} + \frac{\psi(z)-1}{\psi(z)+1} =$$

$$= 1 + \frac{p_1}{2} z + \left(\frac{p_2}{2} - \frac{p_1^2}{8}\right) z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{4}\right) z^3 + ... \tag{5}$$

Now by (5),

$$b_1 = \frac{p_1}{2}, \quad b_2 = \frac{p_2}{2} - \frac{p_1^2}{8}, \quad b_3 = \frac{p_3}{2} - \frac{p_1 p_2}{4}. \tag{6}$$

On other hand, in view of (3) and (5), we have

$$1 + \sum_{t=1}^{\infty} b_t z^t = \frac{D^{3-2\vartheta} f(z)}{D^{2-2\vartheta} f(z)} = \frac{z + \sum_{t=2}^{\infty} \mathcal{C}_2(\vartheta, t) a_t z^t}{z + \sum_{t=2}^{\infty} \mathcal{C}_1(\vartheta, t) a_t z^t}, \tag{7}$$

where

$$\mathcal{C}_1(\vartheta, t) = \frac{\prod_{k=2}^{t} (k - 2\vartheta)}{(t-1)!}, \quad \mathcal{C}_2(\vartheta, t) = \frac{\prod_{k=2}^{t} (k + 1 - 2\vartheta)}{(t-1)!}.$$

Equating the coefficients of z, z^2 and z^3 in (7), we obtain

$$b_1 = [\mathcal{C}_2(\vartheta, 2) - \mathcal{C}_1(\vartheta, 2)] a_2, \tag{8}$$

$$b_2 = [\mathcal{C}_2(\vartheta, 3) - \mathcal{C}_1(\vartheta, 3)] a_3 + [\mathcal{C}_1(\vartheta, 2) a_2]^2 - [\mathcal{C}_1(\vartheta, 2) \mathcal{C}_2(\vartheta, 2)] {a_2}^2, \tag{9}$$

and

$$\begin{aligned} b_3 = {} & [\mathcal{C}_2(\vartheta, 4) - \mathcal{C}_1(\vartheta, 4)] a_4 + \\ & + [2\mathcal{C}_1(\vartheta, 2)\mathcal{C}_1(\vartheta, 3) - \mathcal{C}_2(\vartheta, 3)\mathcal{C}_1(\vartheta, 2) - \mathcal{C}_2(\vartheta, 2)\mathcal{C}_1(\vartheta, 3)] a_2 a_3 + \\ & + \mathcal{C}_2(\vartheta, 2) [\mathcal{C}_1(\vartheta, 2) a_2]^2 a_2 - [\mathcal{C}_1(\vartheta, 2) a_2]^3. \end{aligned} \tag{10}$$

Simplifying (8), (9) and (10) we have

$$a_2 = b_1, \qquad a_3 = \frac{b_2 + (2 - 2\vartheta) {b_1}^2}{(3 - 2\vartheta)}, \tag{11}$$

and

$$a_4 = \frac{2b_3}{(3 - 2\vartheta)(4 - 2\vartheta)} + \frac{3(2 - 2\vartheta) b_1 b_2}{(3 - 2\vartheta)(4 - 2\vartheta)} - \frac{(2 - 2\vartheta)^2 {b_1}^3}{(3 - 2\vartheta)(4 - 2\vartheta)}. \tag{12}$$

Using the equalities (6) in (11), it follows that

$$a_2 = \frac{p_1}{2}, \qquad a_3 = \frac{1}{(3 - 2\vartheta)} \left[\frac{1}{2} \left(p_2 - \frac{{p_1}^2}{2} \right) + \frac{{p_1}^2}{8} + (2 - 2\vartheta) \frac{{p_1}^2}{4} \right], \tag{13}$$

by taking absolute values and applying Lemma 1, we get

$$\begin{aligned} |a_2| \le \frac{1}{2}, \quad a_3 &= \frac{1}{(3 - 2\vartheta)} \left[\frac{1}{2} \left(p_2 - \frac{{p_1}^2}{2} \right) + \frac{{p_1}^2}{8} + (2 - 2\vartheta) \frac{{p_1}^2}{4} \right] = \\ &= \frac{1}{2(3 - 2\vartheta)} \left[p_2 - \frac{{p_1}^2}{2} \left(2\vartheta - \frac{3}{2} \right) \right], \\ |a_3| &= \frac{1}{2(3 - 2\vartheta)} \left| p_2 - \frac{{p_1}^2}{2} \left(2\vartheta - \frac{3}{2} \right) \right|. \end{aligned} \tag{14}$$

By applying Lemma 2 we get

$$|a_3| \le \frac{1}{(3 - 2\vartheta)} \max \left\{ 1, \left| 2\vartheta - \frac{5}{2} \right| \right\}$$

Now for $\varrho \in C$ and using (13) and (14), we have

$$a_3 - \varrho a_2^2 = \frac{1}{2(3-2\vartheta)}\left[p_2 - \frac{{p_1}^2}{2}\left(2\vartheta - \frac{3}{2}\right)\right] - \varrho\frac{p_1^2}{4} =$$

$$= \frac{1}{2(3-2\vartheta)}\left[p_2 - \frac{{p_1}^2}{2}\left(2\vartheta - \frac{3}{2} + (3-2\vartheta)\varrho\right)\right] = \frac{1}{2(3-2\vartheta)}\left|p_2 - \nu {p_1}^2\right|,$$

where $\nu = \frac{1}{2}\left(2\vartheta - \frac{3}{2} + (3-2\vartheta)\varrho\right)$. Thus, by applying Lemma 3 we get

$$|a_3 - \varrho a_2^2| \le \frac{1}{(3-2\vartheta)}\max\left\{1, \left|2\vartheta - \frac{5}{2} + (3-2\vartheta)\varrho\right|\right\}.$$

In particular, we put $\varrho = 1$ and get $|a_3 - a_2^2| \le \dfrac{1}{(3-2\vartheta)}$. □

4. The Hankel inequality for $f \in \mathcal{R}_\vartheta^u(\varphi)$. In this section, we obtain the upper bounds of the Hankel determinant $|a_2a_4 - a_3^2|$ for $f \in \mathcal{R}_\vartheta^u(\varphi)$.

Theorem 2. *Let $f \in \mathcal{R}_\vartheta^u(\varphi)$ be given by $f(z) = \sum_{t=0}^{\infty} a_t z^t$, ($a_0 = 0$ and $a_1 = 1$) and suppose φ is defined by* (1). *Then $|a_2a_4 - a_3^2| \le \mathcal{G}(0) = \frac{1}{(3-2\vartheta)^2}$.*

Proof. Since $f \in \mathcal{R}_\vartheta^u(\varphi)$, there exists a Schwarz function ϖ, analytic in $\mathbb{U}$ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ in $\mathbb{U}$ such that

$$\frac{D^{3-2\vartheta} f(z)}{D^{2-2\vartheta} f(z)} = \varphi(\varpi(z)).$$

Using the equalities (6) in (11) and (12), it follows that

$$a_2 = \frac{p_1}{2}, \quad a_3 = \frac{1}{(3-2\vartheta)}\left[\frac{1}{2}\left(p_2 - \frac{{p_1}^2}{2}\right) + \frac{{p_1}^2}{8} + (2-2\vartheta)\frac{{p_1}^2}{4}\right],$$

$$a_4 = \frac{1}{8(3-2\vartheta)(4-2\vartheta)}\left[8p_3 + 4(2-3\vartheta)p_1p_2 + (1-\vartheta)(1-4\vartheta){p_1}^3\right].$$

Thus, we establish that the estimate of the second Hankel determinant is given by

$$a_2a_4 - a_3^2 = \frac{1}{\mathcal{H}(\vartheta)}\left[\frac{-1}{2}\left\{12\vartheta^2 - 23\vartheta + 12\right\}{p_1}^4 - 8\vartheta(1-\vartheta){p_1}^2p_2 - \right.$$
$$\left. -8(2-\vartheta){p_2}^2 + 8(3-2\vartheta)p_1p_3\right], \quad (15)$$

where $\mathcal{H}(\vartheta) = 16(3-2\vartheta)^2(4-2\vartheta)$. Using Lemma 4 in (15), we have

$$|a_2a_4 - a_3^2| = \frac{1}{\mathcal{H}(\vartheta)}\left|\frac{-1}{2}\left[4\vartheta^2 - 11\vartheta + 8\right]{p_1}^4 + (2-2\vartheta)^2(4-{p_1}^2){p_1}^2x - \right.$$
$$\left. -\left\{(2-2\vartheta)p_1^2 + 4(4-2\vartheta)\right\}(4-{p_1}^2)x^2 + 4(3-2\vartheta)(4-{p_1}^2)p_1(1-|x|^2)z\right|. \quad (16)$$

Letting $|p_1| = \xi$ and in view of Lemma 1, we may assume without restriction that $\xi \in [0,2]$. Thus, applying the triangle inequality in (16) with $\delta = |x| \le 1$ and $|z| \le 1$, we obtain

$$|a_2a_4 - a_3^2| \le \frac{1}{\mathcal{H}(\vartheta)}\left[\frac{1}{2}\left|4\vartheta^2 - 11\vartheta + 8\right|\xi^4 + (2-2\vartheta)^2(4-\xi^2)\xi^2\delta + \right.$$
$$\left. +\{(2-2\vartheta)\xi^2 + 4(4-2\vartheta)\}(4-\xi^2)\delta^2 + 4(3-2\vartheta)\xi(4-\xi^2)(1-\delta^2)\right] =$$
$$= \frac{1}{\mathcal{H}(\vartheta)}\left[\frac{1}{2}\left|4\vartheta^2 - 11\vartheta + 8\right|\xi^4 + (2-2\vartheta)^2(4-\xi^2)\xi^2\delta + \right.$$
$$\left. +\{(2-2\vartheta)\xi^2 - 4(3-2\vartheta)\xi + 4(4-2\vartheta)\}(4-\xi^2)\delta^2 + 4(3-2\vartheta)\xi(4-\xi^2)\right] = \mathcal{F}(\xi,\delta).$$

Note that for $(\xi,\delta) \in [0,2) \times [0,1]$, differentiating $\mathcal{F}(\xi,\delta)$, partially with respect to δ yields

$$\frac{\partial \mathcal{F}}{\partial \delta} = \frac{1}{\mathcal{H}(\vartheta)}\left[\,(2-2\vartheta)^2(4-\xi^2)\xi^2 + 2\big\{(2-2\vartheta)\xi^2 - 4(3-2\vartheta)\xi + 8(2-\vartheta)\big\}(4-\xi^2)\delta\,\right]. \quad (17)$$

It is obvious that the coefficient term of δ in (17) is always a positive real number for all $(\xi,\delta) \in [0,2) \times [0,1]$. Hence it follows that the expression (17) is always positive for $\delta > 0$ and $\vartheta \le 1$, which implies that $\mathcal{F}(\xi,\delta)$ is an increasing function of δ. Therefore, there exists no point of maximum in the interior of the closed region $[0,2) \times [0,1]$. Moreover, for fixed $\xi \in [0,2)$, we have $\max \mathcal{F}(\xi,\delta) = \mathcal{F}(\xi,1) = \mathcal{G}(\xi)$. On simplification, we find that

$$\mathcal{F}(\xi,1) = \mathcal{G}(\xi) = \frac{1}{\mathcal{H}(\vartheta)}\left[\frac{1}{2}\left|4\vartheta^2 - 11\vartheta + 8\right|\xi^4 + (2-2\vartheta)^2(4-\xi^2)\xi^2 + \right.$$
$$\left. +\{(2-2\vartheta)\xi^2 - 4(3-2\vartheta)\xi + 4(4-2\vartheta)\}(4-\xi^2) + 4(3-2\vartheta)\xi(4-\xi^2)\right],$$
$$\mathcal{G}'(\xi) = \frac{1}{\mathcal{H}(\vartheta)}\left[\{2\left|4\vartheta^2 - 11\vartheta + 8\right| - 4(2-2\vartheta)(1-2\vartheta)\}\xi^3 + 16\{2\vartheta^2 - 4\vartheta + 1\}\xi\right]. \quad (18)$$

If $\mathcal{G}'(\xi) = 0$ then the root is $\xi = 0$. Also, we have

$$\mathcal{G}''(\xi) = \frac{1}{\mathcal{H}(\vartheta)}\left[\{6\left|4\vartheta^2 - 11\vartheta + 8\right| - 4(2-2\vartheta)(1-2\vartheta)\}\xi^2 + 16\{2\vartheta^2 - 4\vartheta + 1\}\right]$$

is negative for $\xi = 0$, which means that the function $\mathcal{G}(\xi)$ can take the maximum value at $\xi = 0$, also which is $|a_2a_4 - a_3^2| \le \mathcal{G}(0) = \dfrac{1}{(3-2\vartheta)^2}$. □

Remark 1. We note that by taking $\vartheta = 1/2$ in Theorem 2 we obtain the corresponding result $|a_2a_4 - a_3^2| \le \frac{1}{4}$.

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