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THE REVERSE HÖLDER INEQUALITY FOR AN ELEMENTARY FUNCTION

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For a positive function f on the interval $[0,1]$, the power mean of order $p \in \mathbb{R}$ is defined by

$$\| f \|_p = \left(\int_0^1 f^p(x)\, dx \right)^{1/p} \quad (p \neq 0), \qquad \| f \|_0 = \exp\left(\int_0^1 \ln f(x)\, dx \right).$$

Assume that $0 < A < B$, $0 < \theta < 1$ and consider the step function $g_{A<B,\theta} = B \cdot \chi_{[0,\theta)} + A \cdot \chi_{[\theta,1]}$, where χ_E is the characteristic function of the set E.

Let $-\infty < p < q < +\infty$. The main result of this work consists in finding the term

$$C_{p<q,A<B} = \max_{0 \le \theta \le 1} \frac{\| g_{A<B,\theta} \|_q}{\| g_{A<B,\theta} \|_p}.$$

For fixed $p < q$, we study the behaviour of $C_{p<q,A<B}$ and $\theta_{p<q,A<B}$ with respect to $\beta = B/A \in (1, +\infty)$. The cases $p = 0$ or $q = 0$ are considered separately.

The results of this work can be used in the study of the extremal properties of classes of functions, which satisfy the inverse Hölder inequality, e.g. the Muckenhoupt and Gehring ones. For functions from the Gurov-Reshetnyak classes, a similar problem has been investigated in [4].

Introduction. The Muckenhoupt [1] and Gehring [2] conditions widely used in works on weighted spaces and conformal mappings, represent important examples of the reverse Hölder inequality. Initially, the expression “reverse Hölder inequality” has been applied to the Gehring condition. It is difficult to say who started using it in a more general sense. Nevertheless, the term becomes widely accepted in scientific community nowadays.

Various classes of functions satisfying the reverse Hölder inequality, often appear in applications and have numerous interesting properties. However, in order to determine exact parameters of these classes, mainly power functions are used. In this work, different elementary functions — viz. two-point step functions are considered and the relations between exact parameters of the corresponding classes are established.

The results of this work can be used in determining of exact relations between various classes of functions. Thus, a simple calculation of a sharp constant in the Gurov-Reshetnyak condition [3] for the same elementary function — cf. [4] allowed to find exact positive and negative summability orders for arbitrary functions from the Gurov-Reshetnyak class. Besides, the exact parameters of the classes containing the corresponding elementary functions, can be helpful in the construction of counter examples used in the study of general functions from these classes. In particular, for the Gurov-Reshetnyak class, it was done in [5, 6].

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1. Main results. Let f be a non-negative measurable function on the interval $[0,1]$ and p a real number. If $p \neq 0$, we write

$$\| f \|_p = \Big(\int_0^1 f^p(x)\, dx \Big)^{1/p}, \quad \text{and} \quad \| f \|_0 = \exp\Big(\int_0^1 \ln f(x)\, dx \Big)$$

if $p = 0$. Let us recall that by the Hölder inequality, the term $\| f \|_p$ grows along with p.

Let $0 < A < B < +\infty$, $0 \le \theta \le 1$. A function g defined on the interval $[0,1]$ is called elementary if it has the form

$$g \equiv g_{A<B,\theta} = B \cdot \chi_{[0,\theta)} + A\chi_{[\theta,1]},$$

where χ_E is the characteristic function of the set E.

Let $p < q$. It is clear that if $\theta(1-\theta) = 0$, then $\| g \|_p = \| g \|_q$, and for $0 < \theta < 1$ the inequality $\| g \|_p < \| g \|_q$ holds. The aim of this work is to determine the maximum

$$C_{p<q,A<B} := \max_{0 \le \theta \le 1} \frac{\| g_{A<B,\theta} \|_q}{\| g_{A<B,\theta} \|_p}$$

and the corresponding value of $\theta \equiv \theta_{p<q,A<B}$ where this maximum is attained. In other words, we are looking for the smallest constant C such that the function $g_{A<B,\theta}$ satisfies the reverse Hölder inequality

$$\| g_{A<B,\theta} \|_q \le C \| g_{A<B,\theta} \|_p \qquad (1)$$

for any $0 \le \theta \le 1$.

Remark 1. Let $A = 0$. If $p > 0$ is fixed, the reverse Hölder inequality (1) is valid for $0 < \theta < 1$ with the right-hand side constant $C = C(\theta)$, which tends to $+\infty$ as θ tends to 0. On the other hand, if $p < 0$, it is natural to set $0^p := +\infty$, $(+\infty)^{1/p} := 0$, so that we have $\| g \|_p = 0$ for $\theta < 1$. Besides, setting $\ln 0 := -\infty$, $\exp(-\infty) := 0$, we also obtain $\| g \|_0 = 0$. This justifies the condition $A > 0$. The other restriction $B < +\infty$ can be validated analogously.

The main result of this work is the following theorem.

Theorem 1. *Let $0 < A < B < +\infty$, $-\infty < p < q < +\infty$. If $p \cdot q \neq 0$, then*

$$C_{p<q,A<B} = \frac{|p|^{1/q}}{|q|^{1/p}} (q-p)^{1/p-1/q} \frac{|B^q - A^q|^{1/p}}{|B^p - A^p|^{1/q}} \left(A^p B^q - A^q B^p\right)^{1/q-1/p},$$

and the maximum is attained at the point $\theta_{p<q,A<B} = \frac{1}{q-p}\left(\frac{pA^p}{B^p - A^p} - \frac{qA^q}{B^q - A^q}\right)$.

In addition, if $p \cdot q = 0$, then

$$C_{0<q,A<B} = (\mathrm{e} \cdot q)^{-1/q} \exp\left(\frac{1}{q} \ln \frac{B^q - A^q}{\ln B - \ln A} - \frac{B^q \ln A - A^q \ln B}{B^q - A^q}\right)$$

and the maximum is attained at the point $\theta_{0<q,A<B} = \frac{1}{q} \cdot \frac{1}{\ln B - \ln A} - \frac{A^q}{B^q - A^q}$,

and

$$C_{p<0,A<B} = (-\mathrm{e} \cdot p)^{1/p} \exp\left(\frac{A^p \ln B - B^p \ln A}{A^p - B^p} - \frac{1}{p} \ln \frac{A^p - B^p}{\ln B - \ln A}\right)$$

and the maximum is attained at the point $\theta_{p<0,A<B} = \frac{1}{p} \cdot \frac{1}{\ln B - \ln A} - \frac{A^p}{B^p - A^p}$.

Set $\beta = B/A > 1$ and for $p \cdot q \neq 0$ we write

$$C_{p<q,A<B} = \frac{|p|^{1/q}}{|q|^{1/p}}(q-p)^{1/p-1/q}\left(\frac{|\beta^q-1|}{\beta^q-\beta^p}\right)^{1/p}\left(\frac{|\beta^p-1|}{\beta^q-\beta^p}\right)^{-1/q} =: C_{p<q}(\beta).$$

On the other hand, if $p \cdot q = 0$, then

$$C_{0<q,A<B} = (\mathrm{e}\cdot q)^{-1/q}\exp\left(\frac{\ln\beta}{\beta^q-1} - \frac{1}{q}\ln\frac{\ln\beta}{\beta^q-1}\right) =: C_{0<q}(\beta),$$

$$C_{p<0,A<B} = (-\mathrm{e}\cdot p)^{1/p}\exp\left(\frac{\ln\beta}{1-\beta^p} + \frac{1}{p}\ln\frac{\ln\beta}{1-\beta^p}\right) =: C_{p<0}(\beta).$$

Theorem 2. *The function $C_{p<q}(\beta)$ is continuous on $(1,+\infty)$ and strictly increases from 1 to $+\infty$.*

The proofs of Theorems 1 and 2 are given in Section 2.

Remark 2. The function $C_{p<q}(\beta)$ is defined for $\beta > 1$. It is easily seen that the equation

$$C_{p<q}(\beta) = C_{p<q}\left(\frac{1}{\beta}\right) \tag{2}$$

is valid for any $p < q$. Extending it by continuity, we set $C_{p<q}(1) = 1$. Then Theorem 2 yields that the function $C_{p<q}(\beta)$ is continuous on $(0,+\infty)$, strictly decreases on $(0,1]$ from $+\infty$ to 1 and strictly increases on $[1,+\infty)$ from 1 to $+\infty$. The equation (2) also means that the condition $A < B$ in the definition of $C_{p<q,A<B}$ can be removed.

This remarks leads to the following corollary.

Corollary 1. *If $p < q$, then for any $\overline{C} > 1$ the equation $C_{p<q}(\beta) = \overline{C}$ has two solutions $\overline{\beta}_{p<q}(\overline{C}) > 1$ and $\underline{\beta}_{p<q}(\overline{C}) = \left(\overline{\beta}_{p<q}(\overline{C})\right)^{-1} \in (0,1)$.*

Let us now consider the expression $\theta_{p<q}(\beta)$, cf. Theorem 1, for $\beta = B/A$. We represent it as

$$\theta_{p<q}(\beta) := \frac{1}{q-p}\left(\frac{p}{\beta^p-1} - \frac{q}{\beta^q-1}\right), \quad \text{if } \ p\cdot q \neq 0$$

and as

$$\theta_{0<q}(\beta) := \frac{1}{q}\cdot\frac{1}{\ln\beta} - \frac{1}{\beta^q-1}, \quad \theta_{p<0}(\beta) := \frac{1}{p}\cdot\frac{1}{\ln\beta} - \frac{1}{\beta^p-1}, \quad \text{if } \ p\cdot q = 0.$$

Theorem 3. *The function $\theta_{p<q}(\beta)$ is continuous on $(1,+\infty)$ and has the following properties:*

a) For $0 < p < q$ it strictly decreases from $1/2$ to 0.

b) For $p < q < 0$ it strictly increases from $1/2$ to 1.

c) If $p < 0 < q$, then: 1. For $q > -p$ it strictly decreases from $1/2$ to $p/(p-q) < 1/2$.
2. For $q < -p$ it strictly increases from $1/2$ to $p/(p-q) > 1/2$.
3. For $q = -p$ it takes constant value $1/2$ everywhere.

d) For $0 = p < q$ it strictly decreases from $1/2$ to 0.

e) For $p < q = 0$ it strictly increases from $1/2$ to 1.

The proof of Theorem 3 is given in Section 2.

Remark 3. The value $\theta_{p<q}(\beta)$ is well-defined for $\beta > 1$. It is easily seen that for any $p < q$, the equation

$$\theta_{p<q}\left(\frac{1}{\beta}\right) = 1 - \theta_{p<q}(\beta) \quad (\beta > 0,\ \beta \neq 1),$$

holds. By continuity, it is natural to set $\theta(1) = 1/2$.

2. Proofs.

Proof of Theorem 1. Assume first that $p \cdot q \neq 0$. Then $\| g \|_p = (B^p\theta + A^p(1-\theta))^{1/p}$, and, consequently, $\frac{\| g \|_q}{\| g \|_p} = \frac{(B^q\theta + A^q(1-\theta))^{1/q}}{(B^p\theta + A^p(1-\theta))^{1/p}}$. Set

$$\varphi(\theta) := \left(\frac{\| g \|_q}{\| g \|_p}\right)^q = \frac{B^q\theta + A^q(1-\theta)}{(B^p\theta + A^p(1-\theta))^{q/p}}.$$

Solving the equation $\varphi'(\theta) = 0$, we obtain

$$p\,(B^q - A^q)\,(A^p + \theta\,(B^p - A^p)) = q\,(B^p - A^p)\,(A^q + \theta\,(B^q - A^q))\,,$$

which yields that the unique inferior extremum point of the function $\varphi(\theta)$ is

$$\theta = \theta_{p<q,A<B} = \frac{1}{q-p}\left(\frac{pA^p}{B^p - A^p} - \frac{qA^q}{B^q - A^q}\right).$$

Let us note that $\varphi(0) = \varphi(1) = 1$, and if $0 < \theta < 1$, then $\varphi(\theta) > 1$ $(q > 0)$ and $\varphi(\theta) < 1$ $(q < 0)$. Therefore, in both cases $q > 0$ and $q < 0$, the value $\theta = \theta_{p<q,A<B}$ is the point of maximum for the function $\varphi^{1/q}(\theta) = \| g \|_q / \| g \|_p$. Substituting this value $\theta = \theta_{p<q,A<B}$ we obtain

$$\| g \|_p^p = \frac{|q|}{q-p} \cdot \frac{A^pB^q - A^qB^p}{|B^q - A^q|}, \quad \| g \|_q^q = \frac{|p|}{q-p} \cdot \frac{A^pB^q - A^qB^p}{|B^p - A^p|},$$

$$\frac{\| g \|_q}{\| g \|_p} = \frac{|p|^{1/q}}{|q|^{1/p}}(q-p)^{1/p-1/q} \cdot \frac{|B^q - A^q|^{1/p}}{|B^p - A^p|^{1/q}}\,(A^pB^q - A^qB^p)^{1/q-1/p}.$$

Considering the case $0 = p < q$, we have

$$\| g \|_0 = \exp\left((\ln B)\theta + (\ln A)(1-\theta)\right), \quad \frac{\| g \|_q}{\| g \|_0} = \frac{(B^q\theta + A^q(1-\theta))^{1/q}}{\exp\left((\ln B)\theta + (\ln A)(1-\theta)\right)}.$$

Set

$$\varphi(\theta) := \ln\frac{\| g \|_q}{\| g \|_0} = \frac{1}{q} \cdot \ln\left(B^q\theta + A^q(1-\theta)\right) - \left((\ln B)\theta + (\ln A)(1-\theta)\right).$$

Solving the equation $\varphi'(\theta) = 0$, we obtain $\frac{1}{q} \cdot \frac{B^q - A^q}{\ln B - \ln A} = A^q + \theta\,(B^q - A^q)$, which implies

$$\theta = \theta_{0<q,A<B} = \frac{1}{q} \cdot \frac{1}{\ln B - \ln A} - \frac{A^q}{B^q - A^q}.$$

The substitution of this value $\theta = \theta_{0<q,A<B}$ gives

$$\frac{\| g \|_q}{\| g \|_0} = \frac{1}{(\mathrm{e} \cdot q)^{1/q}} \exp\left(\frac{1}{q} \ln \frac{B^q - A^q}{\ln B - \ln A} - \frac{B^q \ln A - A^q \ln B}{B^q - A^q}\right).$$

It remains to consider the case $p < q = 0$. We have

$$\frac{\| g \|_0}{\| g \|_p} = \frac{\exp\left((\ln B)\theta + (\ln A)(1-\theta)\right)}{\left(B^p\theta + A^p(1-\theta)\right)^{1/p}}.$$

Set

$$\varphi(\theta) := \ln \frac{\| g \|_0}{\| g \|_p} = (\ln B)\theta + (\ln A)(1-\theta) - \frac{1}{p} \ln\left(B^p\theta + A^p(1-\theta)\right).$$

Solving again the equation $\varphi'(\theta) = 0$, we obtain

$$\frac{1}{p} \cdot \frac{B^p - A^p}{\ln B - \ln A} = A^p + \theta\left(B^p - A^p\right),$$

which implies

$$\theta = \theta_{p<0,A<B} = \frac{A^p}{A^p - B^p} - \frac{1}{|p|} \cdot \frac{1}{\ln B - \ln A}.$$

The substitution of this value $\theta = \theta_{p<0,A<B}$ gives

$$\frac{\| g \|_0}{\| g \|_p} = (\mathrm{e} \cdot (-p))^{1/p} \exp\left(\frac{A^p \ln B - B^p \ln A}{A^p - B^p} - \frac{1}{p} \ln \frac{A^p - B^p}{\ln B - \ln A}\right).$$

□

Proof of Theorem 2. The continuity of the function $C_{p<q}(\beta)$ is obvious.

If $p \cdot q \neq 0$, we use the equation $\left(|\beta^q - 1|\right)' = |q|\beta^{q-1}$, where $\beta \geq 1$ (for $\beta = 1$ the right derivative is used). Then for $\beta \to 1+0$, we have

$$C_{p<q}(\beta) = \frac{|p|^{1/q}}{|q|^{1/p}}(q-p)^{1/p-1/q}\left(\frac{|\beta^q - 1|}{\beta^{q-p} - 1}\right)^{1/p}\left(\frac{|\beta^p - 1|}{1 - \beta^{p-q}}\right)^{-1/q} \sim$$
$$\sim \frac{|p|^{1/q}}{|q|^{1/p}}(q-p)^{1/p-1/q}\left(\frac{|q|(\beta - 1)}{(q-p)(\beta - 1)}\right)^{1/p}\left(\frac{|p|(\beta - 1)}{-(p-q)(\beta - 1)}\right)^{-1/q} = 1.$$

Furthermore, if $\beta \to +\infty$, then

$$C_{0<p<q}(\beta) \sim \frac{\beta^{q/p}}{\beta^{(q-p)/p}} \cdot \beta^{-p/q} = \beta^{1-p/q} \to +\infty,$$
$$C_{p<0<q}(\beta) \sim \frac{\beta^{q/p}}{\beta^{(q-p)/p}} = \beta \to +\infty, \; C_{p<q<0}(\beta) \sim \frac{1}{\beta^{(q-p)/p}} = \beta^{1-q/p} \to +\infty.$$

In order to determine the limit of $C_{0<q}(\beta)$ as $\beta \to 1+0$, one can use the equation

$$\lim_{\beta \to 1+0} \frac{\ln \beta}{\beta^q - 1} = \lim_{\beta \to 1+0} \frac{\frac{1}{\beta}}{q\beta^{q-1}} = \frac{1}{q}.$$

It follows that $\lim\limits_{\beta\to1+0} C_{0<q}(\beta) = 1$. On the other hand, it is easily seen that $\lim\limits_{\beta\to+\infty} C_{0<q}(\beta) = +\infty$.

Analogously, in order to find the limit of $C_{p<0}(\beta)$ as $\beta \to 1+0$, we use the equation

$$\lim_{\beta\to1+0} \frac{\ln\beta}{1-\beta^p} = \lim_{\beta\to1+0} \frac{\frac{1}{\beta}}{-p\beta^{p-1}} = -\frac{1}{p}.$$

It follows that $\lim\limits_{\beta\to1+0} C_{p<0}(\beta) = 1$. On the other hand, it is easily seen that $\lim\limits_{\beta\to+\infty} C_{p<0}(\beta) = +\infty$.

It remains to show that the function $C_{p<q}(\beta)$ is strictly increasing on the interval $(1,+\infty)$. We consider five cases.

Case 1. Let $0<p<q$. Using the representation

$$C_{0<p<q}(\beta) = \frac{p^{1/q}}{q^{1/p}}(q-p)^{1/p-1/q}\left(1+\frac{\beta^p-1}{\beta^q-\beta^p}\right)^{1/p}\left(\frac{\beta^p-1}{\beta^q-\beta^p}\right)^{-1/q}$$

and the notation $t = t(\beta) = (\beta^q-\beta^p)\,/\,(\beta^p-1)$, we show that $t(\beta)$ strictly increases on $(1,+\infty)$ from $(q-p)/p$ to $+\infty$. Indeed, it is clear that $\lim\limits_{\beta\to+\infty} t(\beta) = +\infty$, and L'Hôspital's rule gives

$$\lim_{\beta\to1+0} t(\beta) = \lim_{\beta\to1+0} \beta^p\frac{\beta^{q-p}-1}{\beta^p-1} = \lim_{\beta\to1+0} \frac{(q-p)\beta^{q-p-1}}{p\beta^{p-1}} = \frac{q-p}{p}.$$

Further, we show that the function

$$t\left(\tau^{1/p}\right) = \frac{\tau^{q/p}-\tau}{\tau-1}$$

is strictly increasing on the interval $(1,+\infty)$. Indeed, we have

$$\frac{d}{d\tau}\left(t\left(\tau^{1/p}\right)\right) = \frac{1}{(\tau-1)^2}\left(\left(\frac{q}{p}-1\right)\tau^{q/p} - \frac{q}{p}\tau^{q/p-1}+1\right).$$

Since

$$\left.\left(\left(\frac{q}{p}-1\right)\tau^{q/p} - \frac{q}{p}\tau^{q/p-1}+1\right)\right|_{\tau=1} = 0$$

and for $\tau>1$ we have

$$\frac{d}{d\tau}\left(\left(\frac{q}{p}-1\right)\tau^{q/p} - \frac{q}{p}\tau^{q/p-1}+1\right) = \frac{q}{p}\left(\frac{q}{p}-1\right)\tau^{q/p-2}(\tau-1) > 0,$$

the function $t = t\left(\tau^{1/p}\right)$ is strictly increasing on the interval $(1,+\infty)$ from $(q-p)/p$ to $+\infty$. It follows that the inverse function $\beta=\beta(t)$ is also strictly increasing on $((q-p)/p,+\infty)$ from 1 to $+\infty$.

In these notations we have

$$C_{0<p<q}(\beta(t)) = \frac{p^{1/q}}{q^{1/p}}(q-p)^{1/p-1/q}\left(1+\frac{1}{t}\right)^{1/p} t^{1/q}.$$

In order to establish the strict increase of the function $C_{0<p<q}(\beta)$ on $(1,+\infty)$, one has to check whether

$$\varphi(t) := \left(1+\frac{1}{t}\right)^{1/p} t^{1/q} = t^{1/q-1/p}(1+t)^{1/p}$$

strictly increases on $((q-p)/p, +\infty)$. The computation of the corresponding derivative for $t > q\,(1/p - 1/q) = (q-p)/p$ gives

$$\varphi'(t) = t^{1/q-1/p-1}(1+t)^{1/p-1}\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{q}t\right) > 0,$$

and we are done.

Case 2. Let $p < q < 0$. Using the representation

$$C_{p<q<0}(\beta) = \frac{(-p)^{1/q}}{(-q)^{1/p}}(q-p)^{1/p-1/q}\left(\frac{1-\beta^q}{\beta^q-\beta^p}\right)^{1/p}\left(1+\frac{1-\beta^q}{\beta^q-\beta^p}\right)^{-1/q}$$

and notation $t = t(\beta) = (\beta^q - \beta^p)\,/\,(1-\beta^q)$, we show that $t(\beta)$ strictly decreases on $(1, +\infty)$ from $(p-q)/q$ to 0. Indeed, it is clear that $\lim\limits_{\beta\to+\infty} t(\beta) = 0$, and the L'Hôspital rule gives

$$C_{p<q<0}(\beta) = \lim_{\beta\to 1+0} t(\beta) = \lim_{\beta\to 1+0} \beta^p \frac{\beta^{q-p}-1}{1-\beta^q} = \lim_{\beta\to 1+0} \frac{(q-p)\beta^{q-p-1}}{-q\beta^{q-1}} = \frac{p-q}{q}.$$

Further, we show that the function $t\left(\tau^{1/q}\right) = \left(\tau - \tau^{p/q}\right)/\left(1-\tau\right)$ strictly decreases on $(1, +\infty)$. Indeed, we have

$$\frac{d}{d\tau}\left(t\left(\tau^{1/q}\right)\right) = \frac{1}{(1-\tau)^2}\left(\left(\frac{p}{q}-1\right)\tau^{p/q} - \frac{p}{q}\tau^{p/q-1} + 1\right).$$

Since

$$\left.\left(\left(\frac{p}{q}-1\right)\tau^{p/q} - \frac{p}{q}\tau^{p/q-1} + 1\right)\right|_{\tau=1} = 0$$

and for $\tau < 1$ it holds

$$\frac{d}{d\tau}\left(\left(\frac{p}{q}-1\right)\tau^{p/q} - \frac{p}{q}\tau^{p/q-1} + 1\right) = \frac{p}{q}\left(\frac{p}{q}-1\right)\tau^{p/q-2}(\tau-1) < 0,$$

the function $t = t\left(\tau^{1q}\right)$ strictly decreases on $(0,1)$ from $(p-q)/q$ to 0. It follows that the inverse function $\beta = \beta(t)$ strictly decreases on $(0, (p-q)/q)$ from $+\infty$ to 1.

In these notations we have

$$C_{p<q<0}(\beta(t)) = \frac{(-p)^{1/q}}{(-q)^{1/p}}(q-p)^{1/p-1/q}t^{-1/p}\left(1+\frac{1}{t}\right)^{-1/q}.$$

In order to show the strict increase of $C_{p<q<0}(\beta)$ on $(1, +\infty)$, it suffices to verify that

$$\varphi(t) := t^{-1/p}\left(1+\frac{1}{t}\right)^{-1/q} = t^{1/q-1/p}(1+t)^{-1/q}$$

strictly decreases on $(0, (p-q)/q)$. The computation of the corresponding derivative for $0 < t < (p-q)/q$ gives

$$\varphi'(t) = t^{1/q-1/p-1}(1+t)^{1/q-1}\left(\frac{1}{q} - \frac{1}{p} - \frac{1}{p}t\right) < 0,$$

and we are done.

Case 3. Let $p < 0 < q$. Using the representation

$$C_{p<0<q}(\beta) = \frac{(-p)^{1/q}}{q^{1/p}}(q-p)^{1/p-1/q}\left(1-\frac{1-\beta^p}{\beta^q-\beta^p}\right)^{1/p}\left(\frac{1-\beta^p}{\beta^q-\beta^p}\right)^{-1/q}$$

and the notation $t = t(\beta) = (\beta^q - \beta^p)\,/\,(1-\beta^p)$, we show that $t(\beta)$ strictly increases on $(1,+\infty)$ from $(p-q)/p$ to $+\infty$. It is clear that $\lim\limits_{\beta\to+\infty} t(\beta) = +\infty$, and the L'Hôspital rule gives

$$\lim_{\beta\to1+0} t(\beta) = \lim_{\beta\to1+0} \beta^p\frac{\beta^{q-p}-1}{1-\beta^p} = \lim_{\beta\to1+0}\frac{(q-p)\beta^{q-p-1}}{-p\beta^{p-1}} = \frac{p-q}{p}.$$

Further, we show that the function $t\left(\tau^{1/p}\right) = \left(\tau^{q/p}-\tau\right)/(1-\tau)$ strictly decreases on $(0,1)$. Indeed, we have

$$\frac{d}{d\tau}\left(t\left(\tau^{1/p}\right)\right) = \frac{1}{(1-\tau)^2}\left(\left(1-\frac{q}{p}\right)\tau^{q/p}+\frac{q}{p}\tau^{q/p-1}-1\right).$$

Since

$$\left.\left(\left(1-\frac{q}{p}\right)\tau^{q/p}+\frac{q}{p}\tau^{q/p-1}-1\right)\right|_{\tau=1} = 0$$

and for $\tau > 1$ one has

$$\frac{d}{d\tau}\left(\left(1-\frac{q}{p}\right)\tau^{q/p}+\frac{q}{p}\tau^{q/p-1}-1\right) = \frac{q}{p}\left(1-\frac{q}{p}\right)\tau^{q/p-2}(\tau-1) < 0,$$

the function $t = t\left(\tau^{1/p}\right)$ strictly decreases on $(0,1)$ from $+\infty$ to $(p-q)/p$. It follows that the inverse function $\beta = \beta(t)$ strictly increases on $((p-q)/p,+\infty)$ from 1 to $+\infty$.

In these notations we have

$$C_{p<0<q}(\beta(t)) = \frac{(-p)^{1/q}}{q^{1/p}}(q-p)^{1/p-1/q}\left(1-\frac{1}{t}\right)^{1/p}t^{1/q}.$$

In order to prove the strict increasing of $C_{p<0<q}(\beta)$ on $(1,+\infty)$, one has to verify that

$$\varphi(t) := \left(1-\frac{1}{t}\right)^{1/p}t^{1/q} = t^{1/q-1/p}(t-1)^{1/p}$$

strictly increases on $((p-q)/p,+\infty)$. Computing the corresponding derivative on $t > (p-q)/p$ gives

$$\varphi'(t) = t^{1/q-1/p-1}(t-1)^{1/p-1}\left(-\frac{1}{q}+\frac{1}{p}+\frac{1}{q}t\right) > 0,$$

and we are done.

Case 4. Let $0 < q$. Setting $t = t(\beta) = (\ln\beta)/\left(\beta^q-1\right)$, we get

$$\lim_{\beta\to1+0} t(\beta) = \frac{1}{q}, \qquad \lim_{\beta\to+\infty} t(\beta) = 0.$$

We show that $t(\beta)$ strictly decreases on $(1,+\infty)$ from $1/q$ to 0. For this we compute the derivative

$$\frac{d}{d\beta}(t(\beta)) = \frac{-\beta^{q-1}}{(\beta^q-1)^2}\left(q\ln\beta+\frac{1}{\beta^q}-1\right).$$

It remains to show that the term $q\ln\beta+\beta^{-q}-1$ is positive. Write $\varphi(\beta):=q\ln\beta+\beta^{-q}-1$ and note that $\varphi(1)=0$ and

$$\varphi'(\beta)=q\cdot\frac{1}{\beta}-q\cdot\beta^{-q-1}=\frac{q}{\beta}\left(1-\frac{1}{\beta^{q}}\right)>0.$$

Hence, $\varphi(\beta)>0$ and $t(\beta)$ strictly decreases on $(1,+\infty)$ from $1/q$ to 0. Therefore, the reverse function $\beta=\beta(t)$ strictly decreases on $(1/q,0)$ from $+\infty$ to 1. Thus in order to establish the strict increase of the function $C_{0<q}(\beta)$, one has to show that $C_{0<q}(\beta(t))$ strictly decreases on $(0,1/q)$. However, since

$$C_{0<q}(\beta(t))=(\mathrm{e}q)^{-1/q}\exp\left(t-\frac{1}{q}\ln t\right),$$

it suffices to show the strict decreasing of $\psi(t):=t-(\ln t)/q$ on $(0,1/q)$. It follows from the negativity of $\psi'(t)=1-1/(qt)$ for $t\in(0,1/q)$.

Case 5. Let $p<0$. Set $t=t(\beta)=(\ln\beta)/\left(1-\beta^{p}\right)$ and note

$$\lim_{\beta\to1+0}t(\beta)=-\frac{1}{p},\qquad\lim_{\beta\to+\infty}t(\beta)=+\infty.$$

We show that $t(\beta)$ strictly increases on $(1,+\infty)$ from $-1/p$ to $+\infty$. Compute the derivative

$$\frac{d}{d\beta}(t(\beta))=\frac{\beta^{p-1}}{(1-\beta^{p})^{2}}\left(p\ln\beta+\frac{1}{\beta^{p}}-1\right)$$

and show that the term $p\ln\beta+\beta^{-p}-1$ is positive. Writing $\varphi(\beta):=p\ln\beta+\beta^{-p}-1$, we note that $\varphi(1)=0$ and

$$\varphi'(\beta)=p\cdot\frac{1}{\beta}-p\cdot\beta^{-p-1}=\frac{p}{\beta}\left(1-\frac{1}{\beta^{p}}\right)>0.$$

Thus $\varphi(\beta)>0$, so that $t(\beta)$ strictly increases on $(1,+\infty)$ from $-1/p$ to $+\infty$. Therefore, the inverse function $\beta=\beta(t)$ strictly increases on $(-1/p,+\infty)$ from 1 to $+\infty$. In order to prove the strict increase of $C_{0<q}(\beta)$, it suffices to show that $C_{0<q}(\beta(t))$ strictly increases on $(-1/p,+\infty)$. However, since

$$C_{p<0}(\beta(t))=(-\mathrm{e}p)^{1/p}\exp\left(t+\frac{1}{p}\ln t\right),$$

one can show that $\psi(t):=t+(\ln t)/p$ strictly decreases on $(-1/p,+\infty)$. But this directly follows from the positivity of $\psi'(t)=1+1/(pt)$ for $t\in(-1/p,+\infty)$. □

Proof of Theorem 3. For any $p<q$, the continuity of $\theta_{p<q}(\beta)$ is clear. Let $p\cdot q\neq0$. In order to find the limit of $\theta_{p<q}(\beta)$ as $\beta\to1+0$ we again use the L'Hôspital rule. Thus, we obtain

$$\lim_{\beta\to1+0}\theta_{p<q}(\beta)=\lim_{\beta\to1+0}\frac{pq\beta^{q-1}-pq\beta^{q-1}}{(q-p)\left(p\beta^{p-1}\left(\beta^{q}-1\right)+q\beta^{q-1}\left(\beta^{p}-1\right)\right)}=$$
$$=\lim_{\beta\to1+0}\left\{\left[pq(q-1)\beta^{q-2}-pq(p-1)\beta^{p-2}\right]\times\left[(q-p)\left(p(p-1)\beta^{p-2}\left(\beta^{q}-1\right)+\right.\right.\right.$$
$$\left.\left.\left.+p\beta^{p-1}q\beta^{q-1}+q(q-1)\beta^{q-2}\left(\beta^{p}-1\right)+q\beta^{q-1}p\beta^{p-1}\right)\right]^{-1}\right\}=\frac{pq(q-p)}{(q-p)(pq+pq)}=\frac{1}{2}.$$

On the other hand, if $\beta \to +\infty$, then

$$\theta_{0<p<q}(\beta) \sim \frac{p\beta^q}{(q-p)\beta^{q+p}} \to 0, \quad \theta_{p<q<0}(\beta) \sim \frac{-p+q}{q-p} = 1,$$
$$\theta_{p<0<q}(\beta) \sim \frac{p\beta^q}{(p-q)\beta^q} \to \frac{p}{p-q} \in (0,1).$$

Consider now the case $p \cdot q = 0$. The application of the L'Hôspital rule as $\beta \to 1+0$ gives

$$\theta_{0<q}(\beta) = \frac{\beta^q - 1 - q\ln\beta}{q\ln\beta\,(\beta^q-1)} \sim \frac{\beta^q - 1 - q\ln\beta}{q^2(\beta-1)^2},$$
$$\lim_{\beta\to 1+0} \theta_{0<q}(\beta) = \frac{1}{q^2}\lim_{\beta\to 1+0}\frac{q\beta^{q-1}-\frac{q}{\beta}}{2(\beta-1)} = \frac{1}{2q}\lim_{\beta\to 1+0}\frac{\beta^q-1}{\beta-1} = \frac{1}{2},$$
$$\theta_{p<0}(\beta) = \frac{\beta^p - 1 - p\ln\beta}{p\ln\beta\,(\beta^p-1)} \sim \frac{\beta^p - 1 - p\ln\beta}{p^2(\beta-1)^2},$$
$$\lim_{\beta\to 1+0} \theta_{p<0}(\beta) = \frac{1}{p^2}\lim_{\beta\to 1+0}\frac{p\beta^{p-1}-\frac{p}{\beta}}{2(\beta-1)} = \frac{1}{2p}\lim_{\beta\to 1+0}\frac{\beta^p-1}{\beta-1} = \frac{1}{2}.$$

However, if $\beta \to +\infty$, then it is clear that $\lim\limits_{\beta\to+\infty} \theta_{0<q}(\beta) = 0, \quad \lim\limits_{\beta\to+\infty} \theta_{p<0}(\beta) = 1.$

It remains to study the monotonicity of function $\theta_{p<q}(\beta)$ on the interval $(1,+\infty)$. Let $p \cdot q \neq 0$. Then the character of the monotonicity of $\theta_{p<q}(\beta)$ is the same as of the auxiliary function $\varphi(\beta) := \frac{p}{\beta^p-1} - \frac{q}{\beta^q-1}$. Computing the derivative

$$\varphi'(\beta) = \frac{q^2\beta^{q-1}}{(\beta^q-1)^2} - \frac{p^2\beta^{p-1}}{(\beta^p-1)^2} = \frac{1}{\beta}\cdot\left(\frac{q^2\beta^q}{(\beta^q-1)^2} - \frac{p^2\beta^p}{(\beta^p-1)^2}\right),$$

we have to determine the sign of $\varphi'(\beta)$. Fix $\beta > 1$ and consider another auxiliary function $\varphi_1(t) := \frac{t^2\beta^t}{(\beta^t-1)^2}, \quad t \in (-\infty,+\infty)$. Set $u = \beta^t > 0$, i.e. $t = (\ln u)/(\ln\beta)$ and obtain

$$\varphi_1\left(\frac{\ln u}{\ln\beta}\right) = \frac{1}{\ln^2\beta}\cdot\frac{u\ln^2 u}{(u-1)^2} \equiv \frac{1}{\ln^2\beta}\cdot\varphi_2(u), \quad u > 0.$$

In order to study the monotonicity of $\varphi_2(u) := \frac{u\ln^2 u}{(u-1)^2}$ we compute $\varphi_2(1\pm 0) = 1$, $\varphi_2(0+) = 0$, $\varphi_2(+\infty) = 0$, and

$$\varphi_2'(u) = -\frac{(u+1)\ln u}{(u-1)^3}\left(\ln u - 2\cdot\frac{u-1}{u+1}\right).$$

Using the notation $\varphi_3(u) := \ln u - 2\cdot\frac{u-1}{u+1}$ we show that

$$\varphi_3(1) = 0, \quad \varphi_3'(u) = \frac{1}{u} - \frac{4}{(u+1)^2} = \frac{(u-1)^2}{u(u+1)^2} > 0,$$

which yields $\operatorname{sign}\varphi_3(u) = \operatorname{sign}(u-1)$. This means that the function $\varphi_2(u)$ strictly increases on $(0,1)$ and $(1,+\infty)$ from 0 to 1 and from 1 to 0, respectively. Therefore, $\varphi'(\beta) < 0$ if $0 < p < q$ and $\varphi'(\beta) > 0$ if $p < q < 0$, and assertions *a*) and *b*) of Theorem 3 are proven.

In order to consider the case *c*), i.e. the situation $p < 0 < q$, we set $r = -p > 0$ and note that $\frac{p^2\beta^p}{(\beta^p-1)^2} = \frac{r^2\beta^r}{(\beta^r-1)^2}$.

Now we can use the already proven strict decreasing of $\varphi_1(t) = t^2\beta^t\left(\beta^t - 1\right)^{-2}$ on $(0, +\infty)$ and obtain: 1. $\varphi'(\beta) < 0$ for $q > r = -p$, 2. $\varphi'(\beta) > 0$ for $q < r = -p$, 3. $\varphi'(\beta) \equiv 0$ for $q = r = -p$, which implies assertion c).

In order to show assertion d), we determine the derivative

$$\frac{d}{d\beta}\theta_{0<q}(\beta) = \frac{1}{q\left(\ln\beta\left(\beta^q - 1\right)\right)^2}\times$$
$$\times\left[\left(q\beta^{q-1} - \frac{q}{\beta}\right)\ln\beta\left(\beta^q - 1\right) - \left(\beta^q - 1 - q\ln\beta\right)\left(\frac{1}{\beta}\left(\beta^q - 1\right) + \ln\beta\cdot q\beta^{q-1}\right)\right].$$

The expression in the square brackets above can be written as

$$\varphi_q(\beta) := \frac{1}{\beta}\left[q^2\beta^q\ln^2\beta - \left(\beta^q - 1\right)^2\right].$$

Introducing the notation $t = \beta^q > 1$, we observe that the inequalities $\varphi_q(\beta) < 0$ and $\psi(t) := \ln t - \sqrt{t} + \frac{1}{\sqrt{t}} < 0 \quad (t > 1)$ are equivalent. However, the last one immediately follows from the obvious relations $\psi(1) = 0$, $\psi'(t) = 1/t - 1/(2\sqrt{t}) - 1/(2t\sqrt{t}) < 0$ which are valid both for $t > 1$ and $t \in (0, 1)$.

In the proof of assertion e), we obtain

$$\frac{d}{d\beta}\theta_{p<0}(\beta) = \frac{\varphi_p(\beta)}{p\left(\ln\beta\left(\beta^p - 1\right)\right)^2}.$$

In order to show the inequality $\varphi_p(\beta) < 0$, we again set $t = \beta^p \in (0, 1)$ and note that for $0 < t < 1$ the inequality $\varphi_p(\beta) < 0$ is equivalent to $\psi(t) > 0$, and the latter is obviously true. $\square$

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