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A. A. Dmytruk, A. I. Gatalevych, M. I. Kuchma

STABLE RANGE CONDITIONS FOR ABELIAN AND DUO RINGS

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The paper deals with the following question: when is the classical ring of quotients of a duo ring exist and idempotents in the classical ring of quotients $Q_{Cl}(R)$ there are idempotents in R? We introduce the concepts of a ring of (von Neumann) regular range 1, a ring of semihereditary range 1, a ring of regular range 1. We find relationships between the introduced classes of rings and known ones for abelian and duo rings. We proved that semihereditary local duo ring is a ring of semihereditary range 1. Also it was proved that a regular local Bezout duo ring is a ring of stable range 2. In particular , the following Theorem 1 is proved: For an abelian ring R the following conditions are equivalent: 1. R is a ring of stable range 1; 2. R is a ring of von Neumann regular range 1.

The paper also introduces the concept of Gelfand element and a ring of Gelfand range 1 for the case of a duo ring. We proved that Hermite duo ring of Gelfand range 1 is an elementary divisor ring (Theorem 3).

Definition of stable range came to the ring theory from the K-theory and was very useful for solving some open problems and tasks. This definition was introduced by Bass [3]. Through last few decades a lot of authors, e.g. Ara [1], Chen [4], Goodearl [8], Lam [16], McGovern [13], Menal [15], Zabavsky [19] and many others, have been studying the influence of stable range on the properties of rings and their behavior in the solutions of different ring-theoretical problems. Moreover, there was noticed that rings with some properties have constraints on the stable range, and the nature of these properties can vary. From here, we can conclude that there are very close relationships between the stable range and other properties of rings. Now, the stable range is very popular and widely used in problems of diagonal reduction of matrices (e.g., McGovern [14], Zabavsky [19], and Zabavsky [20]). There are suggested some generalizations of this concept, among which is a notion of stable range. The similar problems were considered by Zabavsky in [21].

Throughout this article, all rings are assumed to be associative with unit and $1 \neq 0$. The set of nonzero divisors (also called regular elements) of R is denoted by $\mathrm{R}(R)$, the set of units by $\mathrm{U}(R)$, and the set of idempotents by $\mathrm{B}(R)$. The Jacobson radical of the ring R is denoted by $\mathrm{J}(R)$. The classical ring of quotients of the ring R is denoted by $Q_{Cl}(R)$.

A ring R is said to be *a duo ring* if every right or left one-sided ideal in R is two-sided. Such rings were investigated by E. Feller [6] and G. Thierrin [17] Trivial examples of duo rings are, of course, commutative rings and division rings. Nontrivial duo rings are not difficult to come by (e.g., any noncommutative special primary ring is duo, since the only right or

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left ideals are powers of the unique maximal ideal). In fact some interesting examples of duo rings have already occurred in the literature: M. Auslander and O. Goldman have shown in [2, p. 13] that there exist noncommutative maximal orders which are both duo rings and Noetherian domains. Further investigations of such rings have been carried out by G. Maury in [12].

Recall that a ring R is said to be a *right (left) Bezout ring* if every finitely-generated right (left) ideal is principle. The right and left Bezout ring is called *a Bezout ring* [10]. A ring R is said to be an *abelian ring* if for every idempotent $e = e^2 \in R$ holds $ae = ea$ for any element $a \in R$. In other words, any idempotent $e = e^2$ of ring R is central.

We should notice that the class of abelian rings contains the class of right (left) duo rings. Indeed, let $e = e^2 \in R$ and $a \in R$. Due to the definition of duo rings, we have the equality $ea = a'e$ for some element $a' \in R$. Then we have the equality $ea = (a'e)e = eae$. For symmetry $ae = eae$. From here we obtaine that $ea = ae$.

Also, the example of an abelian ring is any reduced ring. In [9], Corollary 2.2, it was shown that R is abelian if and only if $eg = ge$ for all idempotents e and all units g of R. A direct sum of division rings is an example of abelian regular ring, which is not a division ring.

A row $(a_1; a_2; \ldots; a_n)$ over a ring R is called unimodular if $a_1R + a_2R \cdots + a_nR = R$. If $(a_1; a_2; \ldots; a_n)$ is a unimodular n-row over a ring R, then we say that $(a_1; a_2; \ldots; a_n)$ is reducible if there exists $(n-1)$-row $(b_1; b_2; \ldots; b_{n-1})$ such that the $(n-1)$-row $(a_1 + a_nb_1; a_2 + a_nb_2; \ldots; a_{n-1} + a_nb_{n-1})$ is unimodular. A ring R is said to have a stable range n if n is the least positive integer such that every unimodular $(n+1)$-row is reducible.

A ring R is said to be a *ring of an idempotent stable range 1* if for any $a, b \in R$ such that $Ra + Rb = R$, there exists an idempotent $e \in R$ such that $a + eb$ is a unit of R. An obvious example of a ring of idempotent regular range 1 is a ring of idempotent stable range 1, i.e a commutative clean ring. An element a is called an *element of stable range 1* if for any element $b \in R$ such that $aR + bR = R$, there exists $t \in R$ such that $(a + bt)R = R$. An element a of a ring R is called an *element of almost stable range 1* if the quotient-ring R/aR is a ring of stable range 1. A duo ring in which every nonzero element is an element of almost stable range 1 is called a *ring of almost stable range 1*. An element a of a ring R is *(von Neumann) regular element*, if $axa = a$ for some element $x \in R$.

Similar to article [21], we give the following definitions.

A ring R is said to have a *von Neumann regular range 1* if for any $a, b \in R$ such that $aR + bR = R$, there exists $y \in R$ such that $a + by$ is a (von Neumann) regular element of R.

The obvious example of a ring of von Neumann regular range 1 is a ring of stable range 1.

An element a of a ring R is called a *left (right) semihereditary element* if $Ra(aR)$ is projective.

A ring R is said to have a *semihereditary range 1* if for any $a, b \in R$ such that $aR + bR = R$, there exists $y \in R$ such that $a + by$ is a semihereditary right element of R.

Obviously, an example of a ring of semihereditary range 1 is a ring of stable range 1 and a commutative semihereditary ring. A special place in the class of rings of semihereditary range 1 is taken by semihereditary local rings.

A duo ring R is called a *semihereditary (regular) local ring* if for any $a, b \in R$ such that $aR + bR = R$, element a or element b is a semihereditary (regular) element of R.

A ring R is said to have *regular range 1* if for any $a, b \in R$ such that $aR + bR = R$, there exists $y \in R$ such that $a + by$ is a regular element of R.

Moreover, let us introduce some propositions about relationships between duo rings and other types of rings.

Due to Cohn [5], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Reduced rings are clearly reversible.

Proposition 1. *Let R be a reversible Bezout duo ring. If $\varphi \in \mathrm{B}(R)(Q_{Cl}(R))$ then $\varphi \in \mathrm{B}(R)$.*

Proof. By Tuganbayev [18] the classical quotient ring exists. Let $\varphi \in \mathrm{B}(R)(Q_{Cl}(R))$ and $\varphi = es^{-1}$, where s is a regular element of R. Let $eR + sR = \delta R$, then $e = e_0\delta$, $s = s_0\delta$, and $eu + sv = \delta$ for some elements $e_0, s_0, u, v \in R$. Since s is a regular element, δ is a regular element as a divisor of s. Since $eu + sv = \delta$, then $\delta(e_0u + s_0v - 1) = 0$. Since $\delta \neq 0$ and δ is a regular element of R, we have $e_0u + s_0v - 1 = 0$. Then $es^{-1} = e_0s_0^{-1}$, where $e_0R + s_0R = R$. Since $e_0s_0^{-1} \in \mathrm{B}(Q_{Cl}(R))$, we have $e_0s_0^{-1}e_0 = e_0$ and $(e_0s_0^{-1} - 1)e_0 = 0$. Since R is reversible, $Q_{Cl}(R)$ is reversible by Theorem 2.6 from [11]. Now we obtain $e_0(e_0s_0^{-1} - 1) = 0$. Therefore, $e_0^2s_0^{-1} = e_0$ and $e_0^2 = e_0s_0$

Since $e_0u + s_0v = 1$, we have $e_0^2u + e_0s_0v = e_0$ and $s_0(e_0u + s_0v) = e_0$. Note that all idempotents in duo rings are central. Hence $e_0s_0^{-1} \in R$. □

Proposition 2. *Let R be a Bezout duo ring and a is a (von Neumann) regular element of R. Then $a = eu$, where $e \in \mathrm{B}(R)$ and $u \in U(R)$.*

Proof. Let $axa = a$. This implies $axax = ax$, i.e. $e = ax \in \mathrm{B}(R)$ and $e \in aR$. Since $axa = a$, we have $ea = a$, i.e. $a \in eR$, and $aR = eR$. Consider the element $u = (1 - e) + a$. Since $u(1 - e) = 1 - e$, we have $uR + eR = R$. We proved that $eR = aR$, then $uR + aR = R$. Since $ue = ((1 - e) + a)e = ae = a$, we deduce $aR \subset uR$. Obviously, the equality $uR + aR = R$ and inclusion $aR \subset uR$ in a duo ring is possible if $u \in U(R)$.

Then we have $ue = a$. □

Proposition 3. *Let R be a duo ring. Then a is a semihereditary element if and only if $a = er$, where $e \in \mathrm{B}(R)$ and $r \in \mathrm{Re}(R)$.*

Proof. Let $\varphi R = \{x | xa = 0\}$ and $\varphi \in \mathrm{B}(R)$. Since $\varphi a = 0$, we have $(1 - \varphi)a = a$. Let $r = a - \varphi$ and $rx = 0$. Since $ax = \varphi x$ and $(1 - \varphi)a = a$, we obtain $(1 - \varphi)ax = \varphi x$ and $(1 - \varphi)\varphi x = 0$. Then $\varphi x = 0$ and $ax = 0$. Since $ax = 0$, we have $x \in \varphi R$, i.e. $x = x\varphi$. Since $x\varphi = 0$, we get $x = 0$. Then we see that r is a regular element of R. In fact, $r(1 - \varphi) = a(1 - \varphi) - \varphi(1 - \varphi) = a(1 - \varphi) = a$, i.e. $a = r(1 - \varphi)$. Put $1 - \varphi = e$, we have $a = re$, where $e \in \mathrm{B}(R)$ and $r \in \mathrm{Re}(R)$. Obviously, $\{x | x(re) = 0\} = (1 - e)R$. □

Proposition 4. *A semihereditary local duo ring is a ring of semihereditary range 1.*

Proof. Let R be a semihereditary local duo ring and $aR + bR = R$. If a is a semihereditary element, the representation $a + b_0$ is as required. If a is not semihereditary, by the condition $aR + (a + b)R = R$, the element $a + b_1$ is semihereditary. □

Proposition 5. *A regular local Bezout duo ring is a ring of stable range 2.*

Proof. Let R be a regular local Bezout ring. Let a, b be nonzero elements of R. Since R is a duo Bezout ring, we have $aR + bR = dR$. Then we have $au + bv = d$, $a = da_0$, $b = db_0$ for some elements $a_0, b_0, u, v \in R$. Since $d(a_0u + b_0v - 1) = 0$, by the definition of a ring R, we see that either $a_0u + b_0v$ or $a_0u + b_0v - 1$ is a regular element of R. If $a_0u + b_0v - 1$ is a regular element, by $d(a_0u + b_0v - 1) = 0$ we have $d = 0$, i.e. $a = b = 0$ and this is impossible. Let $a_0u + b_0v = r$ be a regular element of R. Let $a_0R + b_0R = \delta R$. If $\delta \notin U(R)$, we have $a_0x + b_0y = \delta$, $a_0 = \delta a_1$, $b_0 = \delta b_1$ for some elements $a_1, b_1, x, y \in R$. This implies $\delta(a_1u + b_1v) = a_0u + b_0v = r$. Since $r \in \mathrm{Re}(R)$, we deduce $\delta \in \mathrm{Re}(R)$.

This implies $\delta(a_1x + b_1y - 1) = 0$ and, since $\delta \neq 0$, we have $a_1x + b_1y - 1 = 0$, i.e. $a_1R + b_1R = R$. Thus, we have $a = d\delta a_1$, $b = d\delta b_1$, $a_1R + b_1R = R$. By Kaplansky [10] , R is an Hermite ring and by [15] we obtain that R is a ring of stable range 2. □

And now we can state the type conditions for abelian rings.

Theorem 1. *For an abelian ring R the following conditions are equivalent:*
1. *R is a ring of stable range 1;*
2. *R is a ring of von Neumann regular range 1.*

Proof. (2) ⇒ (1).

Let $aR + bR = R$. Since R is a ring of von Neumann regular range 1, we have $a + bx = eu(e^2 = e,\ u \in U(R))$. Then $ea + ebex = eu$ is invertible in eR. Also $(1 - e)a = (1 - e)bx$. Thus, the condition $aR + bR = R$ implies that $(1 - e)b$ is invertible in $(1 - e)R$. As a result, $(1 - e)a + (1 - e)by$ is invertible in $(1 - e)R$ for some $y \in (1 - e)R$.

If $z = ex + y$, then $a + bz$ is invertible in R.

(1) ⇒ (2) Obvious. □

Theorem 2. *For an abelian ring R the following conditions are equivalent:*
1. *R is a ring of regular range 1;*
2. *R is a ring of semihereditary range 1.*

Proof. Let $aR + bR = R$. Since R has semihereditary range 1, we have $a + bx = er(e^2 = e, r$ is not a zero divisor). Then $ea + ebex = er$ is not a zero divisor in eR. By analogy with Theorem 1, $(1 - e)b$ is invertible in $(1 - e)R$. As a result, $(1 - e)a + (1 - e)by$ is not a zero divisor in $(1 - e)R$ for some $y \in R$.

If $z = ex + y$, then $a + bz$ is not a zero divisor in R. □

Recall that matrix A admits diagonal reduction if there exists unimodular matrices P, Q such that $PAQ = diag(d_1, d_2, ...)$, where $Rd_i \cap d_iR \supseteq Rd_{i+1}R$. If every matrix over R admits diagonal reduction, we call R an *elementary divisor ring* ([10]).

We call R a right Hermite ring if every 1 by 2 matrix admits diagonal reduction; R is a left Hermite ring if 2 by 1 matrices admit diagonal reduction, and if both then R is an Hermite ring. [10]

Obviously, an elementary divisor ring is Hermite and it is easy to see that an Hermite ring is Bezout [10]. Examples that neither implication is revertible are provided by Gillmann and Henriksen in [7]. In the case of commutative rings there are many developments on this rings, while noncommutative rings are little investigated and fragmented. A general picture is far from its full description.

It is an open *problem*: when a ring of stable range 1 is an elementary divisor ring?

Next we describe a new class of noncommutative elementary divisor rings.

An element a of a duo ring R is said to be a *Gelfand element* if for any elements $b, s \in R$ such that $aR + bR + cR = R$ there exist such elements $r, s \in R$ that $a = rs, rR + bR = R$ and $rR + sR = R$.

A duo ring R is said to be a *ring of Gelfand range 1* if for any elements $a, b \in R$ such that $aR + bR = R$ there exists such element $t \in R$ that $a + bt$ is a Gelfand element of R.

Theorem 3. *Let R be a Hermite duo ring of Gelfand range 1. Then R is an elementary divisor ring.*

Proof. Let $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ and $aR + bR + cR = R$. For the proof of our statement, according to [10], it is sufficient to show that matrix A admits diagonal reduction.

Let $aR + cR = dR$, i.e. $au + cv = d$ for some elements $u, v \in R$. From the condition $aR + bR + cR = R$, it follows that $bR + dR = R$. Since R is a ring of Gelfand range 1, we obtain that $b + (au + cv)t = k$ is a Gelfand element for some element t. Thus

$$\begin{pmatrix} 1 & 0 \\ t'u' & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ vt & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ k & c \end{pmatrix}$$

where $t'u'a = t'au = aut$. Then, obviously, $aR + kR + cR = R$ and r is a Gelfand element.

Then $k = rs$, where $rR + aR = R$ and $sR + cR = R$. Let $p \in R$ be a such element that $sp + cl = 1$ for some element $l \in R$. Hence $rsp + rcl = r$ and $kp + cr'l = r$ for some element $r' \in R$. Denoting $r'l = q$, we obtain $(kp + cq)R + aR = R$. Suppose $pR + qR = \delta R$, i.e. $p = p_1\delta$, $q = q_1\delta$ and $\delta = px + qy$, $p_1R + q_1R = R$ for some elements $x, y, p_1, q_1 \in R$. Then from $pR \subset p_1R$ and $pR + cR = R \Rightarrow p_1R + cR = R$, and from $p_1R + q_1R = R \Rightarrow p_1R + (p_1k + q_1c)R = R$.

Since $pk + qc = \delta(p_1k + q_1c)$ and $(pk + qc)R + aR = R$, we obtain $(p_1d + q_1c)R + aR = R$. As well as $p_1R + (p_1d + q_1c)R = R$, finally we have $p_1aR + (p_1k + q_1c)R = R$. By using the ideas from [22] for duo rings, the matrix $\begin{pmatrix} a & 0 \\ k & c \end{pmatrix}$ admits diagonal reduction. Hence, obviously, the matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ admits diagonal reduction.

The theorem is proved. □

Corollary 1 ([23], Th. 21)**.** *Let R be a commutative Bezout ring of stable range 2 and of Gelfand range 1. Then R is an elementary divisor ring.*

Proof. A commutative Bezout ring of stable range 2 is a Hermite ring [19]. This completes the proof. □

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Ivan Franko National University of Lviv
Lviv, Ukraine
tolikd1488@gmail.com
gatalevych@ukr.net

Lviv Polytechnic National University
Lviv, Ukraine
markuchma@ukr.net

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