Y. Bouhafsi, M. Ech-chad, M. Missouri, A. Zouaki

REMARKS ON THE RANGE AND THE KERNEL OF GENERALIZED DERIVATION

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Let $L(H)$ denote the algebra of operators on a complex infinite dimensional Hilbert space H and let $\mathcal{J}$ denote a two-sided ideal in $L(H)$. Given $A, B \in L(H)$, define the generalized derivation $\delta_{A,B}$ as an operator on $L(H)$ by

$$\delta_{A,B}(X) = AX - XB.$$

We say that the pair of operators (A, B) has the Fuglede-Putnam property $(PF)_{\mathcal{J}}$ if $AT = TB$ and $T \in \mathcal{J}$ implies $A^*T = TB^*$. In this paper, we give operators A, B for which the pair (A, B) has the property $(PF)_{\mathcal{J}}$. We establish the orthogonality of the range and the kernel of a generalized derivation $\delta_{A,B}$ for non-normal operators $A, B \in L(H)$. We also obtain new results concerning the intersection of the closure of the range and the kernel of $\delta_{A,B}$.

1. Introduction. Let H be a separable infinite dimensional complex Hilbert space, and let $L(H)$ denote the algebra of all bounded linear operators acting on H into itself. Given $A, B \in L(H)$, we define the generalized derivation $\delta_{A,B}(X)\colon L(H) \longrightarrow L(H)$ by $\delta_{A,B}(X) = AX - XB$, and the elementary operator $\Delta_{A,B}(X)\colon L(H) \longrightarrow L(H)$ by $\Delta_{A,B}(X) = AXB - X$. We simply write δ_A for $\delta_{A,A}$ and Δ_A denote $\Delta_{A,A}$.

Let $\mathcal{J}$ denote a two sided ideal of $L(H)$. We say that the pair of operators (A, B) satisfies the Fuglede-Putnam property $(PF)_{\mathcal{J}}$ if $\ker(\delta_{A,B}|\mathcal{J}) \subseteq \ker(\delta_{A^*,B^*}|\mathcal{J})$, where $\ker(\delta_{A,B}|\mathcal{J})$ denote the kernel of the restriction of $\delta_{A,B}$ to $\mathcal{J}$.

In this paper, we give some pairs of operators (A, B) having the Fuglede-Putnam property $(PF)_{\mathcal{J}}$. It is proved that if A is a left invertible by a contraction and B is a contraction or, if A is invertible and B be such that $\|A^{-1}\|.\|B\| \le 1$, then the pair (A, B) satisfies the Fuglede-Putnam property $(FP)_{\mathcal{J}}$.

Let F and G be two subspaces of a normed linear space E with norm $\|.\|$. The subspace F is said to be orthogonal to the subspace G, in the sense of Birkhof-James, if $\|x+y\| \ge \|y\|$ for all $x \in F$ and for all $y \in G$. This asymmetric definition of orthogonality agrees with the usual definition of orthogonality in the case in which $E = H$ is a Hilbert space.
In [1], J. Anderson proved that if A and T are operators in $L(H)$, such that A is normal and $AT = TA$ then for all $X \in L(H)$

$$\|\delta_A(X) + T\| \ge \|T\|.$$

In view of the previous definition, the above inequality says that the range $R(\delta_A)$ is orthogonal to the kernel $\ker(\delta_A)$ of δ_A.

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Let A, B, T be operators in $L(H)$ such that A and B are normal and $AT = TB$. On $H \oplus H$, if we apply Anderson's Theorem to the operators $A \oplus B$, $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$, then we get the inequality

$$\|\delta_{A,B}(X) + T\| \geq \|T\|.$$

The range-kernel orthogonality of elementary operators has been considered in a number of papers (see for examples [2], [5], [6], [7], [9], [10], [11], [12], [19], [20], [21]).

We investigate the orthogonality of the range and the kernel of a generalized derivation with respect to the usual operator norm. By using a very simple argument, we give pairs of operators (A, B) such that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$. Furthermore, it is proved that if A is a dominant (respectively, M-hyponormal) and essentially normal (essentially isometric, respectively) operator, then

$$\|\delta_A(X) + T\| \geq \|T\|$$

for all $X \in L(H)$, and for all hyponormal operator T in the commutant $\{A\}'$ of A. Also, we establish the orthogonality of the range and the kernel of a derivation δ_A, induced by a rationally cyclic subnormal operator A.

We obtain some new results concerning the intersection of the closure of the range and the kernel of the generalized derivation $\delta_{A,B}$. Also, it is showed that if A is a cyclic subnormal operator with no point spectrum, then A commute with nonzero compact operator. We present a pair of operators A, B for which $\overline{R(\delta_{A,B})}^w \cap \ker(\delta_{A^*,B^*}) = \{0\}$, where $\overline{R(\delta_{A,B})}^w$ is the weak closure of $R(\delta_{A,B})$.

Notations. Let $K(H)$ be the ideal of compact operators, and let $C_1(H)$ be the ideal of trace class operators. The trace function is defined on $C_1(H)$ by $\text{tr}(T) = \sum_n (Te_n, e_n)$, where (e_n) is any complete orthonormal sequence in H. The weak continuous linear functionals on $L(H)$ are those of the form $f_T(X) = \text{tr}(XT)$, where T is a finite rank operator. Let $\pi\colon L(H) \longrightarrow L(H)/K(H)$ denote the Calkin map, and let $\mathcal{C}(H) = L(H)/K(H)$ denote the Calkin algebra.

Given $X \in L(H)$, we shall denote the kernel, the orthogonal complement of the kernel and the closure of the range of X by $\ker(X)$, $\ker^{\perp}(X)$, and $\overline{R(X)}$, respectively. The spectrum, the essential spectrum, the left essential spectrum, the point spectrum and the the spectral radius of X will be denoted by $\sigma(X)$, $\sigma_e(X)$, $\sigma_{le}(X)$, $\sigma_p(X)$, $r(X)$. By $X|M$ we will denote the restriction of X to an invariant subspace M.

2. Main Results.

Definition 1 ([6], Definition 2)**.** Let $A, B \in L(H)$ and $\mathcal{J}$ be a two-sided ideal of $L(H)$. The pair (A, B) is said to *possess the Fuglede-Putnam property* $(FP)_{\mathcal{J}}$ if $AT = TB$ and $T \in \mathcal{J}$ implies $A^*T = TB^*$. i.e. $\ker(\delta_{A,B}|\mathcal{J}) \subseteq \ker(\delta_{A^*,B^*}|\mathcal{J})$.

Before giving our results we need the following lemmas.

Lemma 1 ([21], Theorem 2.2)**.** *Let A and B be contractions and T a compact operator such that $ATB = T$. Then $A^*TB^* = T$.*

Lemma 2 ([12], Lemma 3.4)**.** *Let A and B be contractions, such that $\Delta_{A,B}(T) = 0$ for some $T \in L(H)$. Then $\|\Delta_{A,B}(X) + T\| \geq \|T\|$, for all $X \in L(H)$.*

Theorem 1. *Let $A, B \in L(H)$. If one of the following assertions:*

(i) A is a left invertible by a contraction and B is a contraction,

(ii) A is a contraction and B is a right invertible by a contraction,

(iii) A is invertible and B be such that $\|A^{-1}\|.\|B\| \leq 1$,

is verified, then the pair of operators (A, B) satisfies the Fuglede-Putnam property $(FP)_{\mathcal{J}}$.

Proof. (i) Let $T \in \ker(\delta_{A,B}) \cap \mathcal{J}$. We have A is a left invertible by a contraction, then there exists $C \in L(H)$ such that $CA = I$ and $\|C\| \leq 1$. Since $AT = TB$, hence it follows that $T = CTB$. It results from Lemma 1 that $T = C^*TB^*$. Consequently, we get $A^*T = TB^*$ and the pair (A, B) has the property $(FP)_{\mathcal{J}}$.
(ii) The second assertion is an immediate consequence of the first, by taking adjoint.
(iii) Let $T \in \mathcal{J}$ such that $AT = TB$. Since A is invertible, then $T = A^{-1}TB$. We can write

$$T = \sqrt{\frac{\|B\|}{\|A^{-1}\|}}.A^{-1}T\sqrt{\frac{\|A^{-1}\|}{\|B\|}}.B,$$

The operators $A_1 = \sqrt{\frac{\|B\|}{\|A^{-1}\|}}.A^{-1}$ and $B_1 = \sqrt{\frac{\|A^{-1}\|}{\|B\|}}.B$ are contractions. We obtain that $T = A_1TB_1$ and T is compact. It holds From Lemma 1 that $T = A_1^*TB_1^*$. Hence we deduce that $A^*T = TB^*$. □

The following definition generalizes the idea of orthogonality in Hilbert space.

Definition 2 ([17])**.** Let E be a normed linear space and $\mathbb{C}$ be the complex numbers.

1) We say that $x \in E$ is *orthogonal* to $y \in E$ if $\|x - \lambda y\| \geq \|\lambda y\|$ for all $\lambda \in \mathbb{C}$.
2) Let F and G be two subspaces in E. If $\|x + y\| \geq \|y\|$ for all $x \in F$ and for all $y \in G$, then F is said to be *orthogonal* to G.

The following theorem generalizes a well-known result of J. Anderson [1, Theorem 1.4].

Theorem 2. *Let $A, B \in L(H)$. Suppose that A and B satisfy one of the following cases:*

(i) A is left invertible by a contraction and B is a contraction.

(ii) A is a contraction and B is right invertible by a contraction.

(iii) A is invertible and B be such that $\|A^{-1}\|.\|B\| \leq 1$.

Then, we have $\|\delta_{A,B}(X) + T\| \geq \|T\|$, for all $T \in \ker(\delta_{A,B})$ and for all $X \in L(H)$.

Proof. (i) Given $T \in L(H)$ such that $AT = TB$. Since A is left invertible by a contraction, then there exists $C \in L(H)$ for which $CA = I$ and $\|C\| \leq 1$. It follows that $T = CTB$. By applying Lemma 2, it holds that $\|\Delta_{C,B}(X) + T\| \geq \|T\|$ for all $X \in L(H)$. From this we get $\|\Delta_{C,B}(-AY) + T\| \geq \|T\|$ for all $Y \in L(H)$. Consequently, we have $\|\delta_{A,B}(Y) + T\| \geq \|T\|$ for all $Y \in L(H)$. This implies that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$.
(ii) We notice that the seconde assertion is a direct consequence of the first.
(iii) Suppose that A is invertible such that $\|A^{-1}\|.\|B\| \leq 1$. Let $T \in \ker(\delta_{A,B})$, then we have $T = A^{-1}TB$. It can be easily seen that

$$T = \sqrt{\frac{\|B\|}{\|A^{-1}\|}}.A^{-1}T\sqrt{\frac{\|A^{-1}\|}{\|B\|}}.B.$$

Consider the operators $A_1 = \sqrt{\frac{\|B\|}{\|A^{-1}\|}}.A^{-1}$ and $B_1 = \sqrt{\frac{\|A^{-1}\|}{\|B\|}}.B$. Since $\|A^{-1}\|.\|B\| \le 1$, it follows that A_1 and B_1 are contractions and $T = A_1 T B_1$. Hence, by another application of the Lemma 2, we obtain $\|\Delta_{A_1,B_1}(X)+T\| \ge \|T\|$ for all $X \in L(H)$. By setting $Y = -A^{-1}X$, then we have $\|\delta_{A,B}(Y)+T\| \ge \|T\|$ for all $Y \in L(H)$. □

The following definitions are well-known.

Definition 3. An operator $A \in L(H)$ is called *subnormal*, if there exists a Hilbert space K and a normal operator $N \in L(K)$, such that H is a subspace of K and $A = N|H$. Operator N is called a *normal extension of* A.

Definition 4. An operator $A \in L(H)$, is called *cyclic* if for some $x \in H$ we get

$$\overline{\{p(A)x \colon p \in \mathbb{C}[Z]\}} = H.$$

Vector x is called a *cyclic vector of* A.

The following results has a crucial role in the sequel.

Theorem 3 ([8], Theorem 2.3). *Let $A \in L(H)$. Then, we have the following properties:*

1) *If A is a cyclic subnormal operator, then $\overline{R(\delta_A)} \cap \{A\}' = \{0\}$.*
2) *If $p(A)$ is a cyclic subnormal operator for some polynomial p, then every operator in $\overline{R(\delta_A)} \cap \{A\}'$ is nilpotent.*

Theorem 4 ([18], Theorem 1). *If K is compact and S is any operator, then all solutions X of the equation $X = KXS$ have finite rank.*

Now, we are in a position to prove the following propositions.

Proposition 1. *Let $A \in L(H)$ be a cyclic subnormal operator with no point spectrum. Then A commute with nonzero compact operator.*

Proof. Let T nonzero compact operator such that $AT = TA$. Then T is subnormal by Yoshino's result ([23]). But any compact subnormal operator is normal. Hence $AT = TA$ implies $AT^* = T^*A$. It follows that $A(T^*T) = (T^*T)A$. We have $T \neq 0$, thus T^*T has a positive eigenvalue λ.

Since T^*T and A commutes, the corresponding finite dimensional eigenspace $\ker(T^*T-\lambda)$ is invariant under A, and A has point spectrum, contrary to assumption. □

Proposition 2. *Let $A, B \in L(H)$. Suppose that one of the following conditions holds:*

(i) *A, B^* are cyclic subnormal operators.*
(ii) *A is cyclic subnormal and B is normal.*
(iii) *A is cyclic subnormal and B is isometric.*

Then every operator

$$T \in \overline{R(\delta_{A\oplus B})} \cap \{\{A \oplus B\}' \cup \{(A \oplus B)^*\}'\},$$

is nilpotent of index less than 2.

Proof. We consider the case in which A, B^* are cyclic subnormal operators. Assume that $T \in \overline{R(\delta_{A\oplus B})} \cap \{A \oplus B\}'$. Then there exists a sequence $(X_n)_n$ in $L(H)$ such that

$$(A \oplus B)X_n - X_n(A \oplus B) \longrightarrow T \in \{A \oplus B\}',$$

On $H = H_\circ \oplus H_1$, let

$$T = \begin{pmatrix} T_\circ & T_1 \\ T_2 & T_3 \end{pmatrix} \text{ and } X_n = \begin{pmatrix} Y_n & Z_n \\ U_n & V_n \end{pmatrix}.$$

Then an elementary calculations shows that

$$AY_n - Y_nA \longrightarrow T_\circ \in \{A\}', \quad BV_n - V_nB \longrightarrow T_3 \in \{B\}',$$
$$AZ_n - Z_nB \longrightarrow T_1 \in \ker(\delta_{A,B}), BU_n - U_nA \longrightarrow T_2 \in \ker(\delta_{B,A}).$$

A is a cyclic subnormal operator, hence it results from Theorem 3, that $T_\circ = 0$. It follows from Theorem 2.5 ([4]) that B^* is D-symmetric, which means that $\overline{R(\delta_{B^*})} = \overline{R(\delta_B)}$. This implies that $T_3^* \in \overline{R(\delta_{B^*})} \cap \{B^*\}'$. By applying Theorem 3, we get $T_3 = 0$.

Since A, B^* are cyclic subnormal operators, it follows from Theorem 1 ([15]), that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$. From this, we obtain $T_1 = 0$. Consequently

$$T = \begin{pmatrix} 0 & 0 \\ T_2 & 0 \end{pmatrix}$$

is nilpotent of index less than 2. We leave the proof of the other cases to the reader. □

Proposition 3. *Let $A, B \in L(H)$. If A is invertible and B is compact, then*

$$\overline{R(\delta_{A,B})}^w \cap \ker(\delta_{A^*,B^*}) = \{0\}.$$

Proof. Suppose that $T \in \overline{R(\delta_{A,B})}^w \cap \ker(\delta_{A^*,B^*})$. We have $A^*T = TB^*$, this implies that $BT^* = T^*A$, and so $BT^*A^{-1} = T^*$. It follows from Theorem 4 that T^* has finite rank. Then it results that $f_{T^*}(T) = \text{tr}(T^*T) = 0$, that is $T = 0$. □

We will need the following definitions.

Definition 5 ([19], Definition 1)**.** An operator $A \in L(H)$ is called *dominan, by J. Stampfli and B. Wadhwa*, if for all complex λ, $R(A - \lambda) \subseteq R(A^* - \overline{\lambda})$, or equivalently, if there is a real number $M_\lambda \geq 1$ such that

$$\|(A - \lambda)^*x\| \leq M_\lambda \|(A - \lambda)x\| \quad (\forall x \in H).$$

If there exists a real number M such that $M_\lambda \leq M$ for all λ, the dominant operator A is said to be M-hyponormal. If $M = 1$, then A is hyponormal.

Definition 6 ([22])**.** An operator $A \in L(H)$ is called *finite*, if $\|AX - XA + I\| \geq 1$ for each $X \in L(H)$.

The following theorem allows a stronger deduction for dominant operators.

Theorem 5. *Let $A \in L(H)$ be dominant (respectively, M-hyponormal) and essentially normal operator(essentially isometric, respectively) . If T is a hyponormal operator such that $AT = TA$, then*

$$\|\delta_A(X) + T\| \geq \|T\|$$

for every $X \in L(H)$.

Proof. Let us first suppose that T is a compact operator. Note that any compact hyponormal operator is normal. Then we have T is normal in the commutant of A. Since A is dominant it results from ([17]) that

$$\|\delta_A(X) + T\| \geq \|T\|$$

for all $X \in L(H)$.

We now wish to consider the case when T is not compact.

Let T be hyponormal such that $AT = TA$. We have $r(T) = \|T\|$, then there exists some scalar $\lambda \in \partial\sigma(T)$ which satisfies $\|T\| = |\lambda|$. Hence it will suffice to show that

$$\|\delta_A(X) + T\| \geq |\lambda| \ \ (\forall X \in L(H)) \ , \ (\forall \lambda \in \partial\sigma(T)).$$

It is well known that $\partial\sigma(T) \subseteq \sigma_p(T) \cup \sigma_{le}(T)$. Let $\lambda \in \partial\sigma(T)$ we consider two cases:

Case 1: If $\lambda \in \sigma_p(T)$ such that $M = \ker(T - \lambda)$ is finite dimensional.

The subspace M is invariant under T and A, and the restriction $A|M$ is dominant. Since M is finite dimensional, it follows that $A|M$ is normal, then M reduces A. On $H = M \oplus M^{\perp}$, we get decompositions of operators respectively

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \ \text{and} \ T = \begin{pmatrix} \lambda & 0 \\ 0 & * \end{pmatrix}.$$

By setting $X = \begin{pmatrix} X_\circ & X_1 \\ X_2 & X_3 \end{pmatrix}$, we have

$$\|\delta_A(X) + T\| = \left\| \begin{pmatrix} BX_\circ - X_\circ B + \lambda & * \\ * & * \end{pmatrix} \right\| \geq \|BX_\circ - X_\circ B + \lambda\|.$$

B is a finite operator, this implies $\|BX_\circ - X_\circ B + \lambda\| \geq |\lambda|$. Consequently, we obtain

$$\|\delta_A(X) + T\| \geq |\lambda|, \ \ (\forall X \in L(H)).$$

Case 2: If $\lambda \in \sigma_{le}(T)$. Suppose that T has isolated eigenvalues of finite multiplicity.

Let

$$E = \bigvee_{\mu \in \Pi_{\circ\circ}(T)} \ker(T - \mu),$$

where $\Pi_{\circ\circ}(T)$ is the set of all isolated eigenvalues of T with finite multiplicity.

Since T is hyponormal, it results that E reduces T. On $H = E \oplus E^{\perp}$, we can write $T = T_\circ \oplus T_1$.

The condition $AT = TA$ implies $\pi(A)\pi(T) = \pi(T)\pi(A)$. Furthermore A is essentially normal(resp. essentially isometric), then $R(\delta_{\pi(A)})$ is orthogonal to $\ker(\delta_{\pi(A)})$. Anderson's result ([1]) applied to the Calkin algebra guarantees that

$$\|\delta_A(X) + T\| \geq \|\delta_{\pi(A)}(\pi(X)) + \pi(T)\| \geq \|\pi(T)\|.$$

On the other hand, it is easily seen that $\|\pi(T)\| \geq \|\pi(T_1)\|$. Since T_1 is hyponormal and has no isolated eigenvalues of finite multiplicity, it follows from ([14]) that $\|\pi(T_1)\| = r(\pi(T_1))$.

Consequently, we have

$$\|\delta_A(X) + T\| \geq |\lambda|, \ \ (\forall X \in L(H)).$$

The case T has no isolated eigenvalues of finite multiplicity, follows from a similar argument as seen above for T_1. □

The next Corollary is an immediate consequence of the above theorem.

Corollary 1. *Let $A \in L(H)$ be a rationally cyclic subnormal operator. If $AT = TA$ for some $T \in L(H)$, then*

$$\|\delta_A(X) + T\| \geq \|T\|$$

for all $X \in L(H)$.

Proof. Indeed, if A is a rationally cyclic hyponormal operator, then it results from ([3]) that $A^*A - AA^* \in C_1(H)$. Hence, A is a hyponormal and essentially normal operator. Since $T \in \{A\}'$ and A is a rationally cyclic subnormal operator, it follows by Yoshino's results ([23]) that T is also subnormal. Hence, it suffices to apply the preceding Theorem. □

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Chouaib Doukkali University, Faculty of Science, Department of Mathematics
El Jadida, Morocco
Département de Mathématiques
Centre Régional des Métiers de l'Education et de la Formation
Marrakech-Safi, Marocco
ybouhafsi@yahoo.fr

Ibn Tofail University, Faculty of Science
Kénitra, Morocco
m.echchad@yahoo.fr
mohamed-tri3@hotmail.com
zouadil-2007@hotmail.com

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