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AN EXACT CONSTANT IN THE ESTIMATION OF THE APPROXIMATION OF CLASSES OF PERIODIC FUNCTIONS OF TWO VARIABLES BY CESÀRO MEANS

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In the present work, we study problem related to the approximation of continuous 2π-periodic functions by linear means of their Fourier series. The simplest example of a linear approximation of a periodic function is the approximation of this function by partial sums of the Fourier series. However, as well known, the sequence of partial Fourier sums is not uniformly convergent over the class of continuous 2π-periodic functions. Therefore, a significant number of papers is devoted to the research of the approximative properties of different approximation methods, which are generated by some transformations of the partial sums of the Fourier series. The methods allow us to construct a sequence of trigonometrical polynomials that would be uniformly convergent for all functions $f \in C$. Particularly, Cesàro means and Féjer sums have been widely studied in past decades. One of the important problems in this field is the study of the exact constant in an inequality for upper bounds of linear means deviations of the Fourier sums on fixed classes of periodic functions. Methods of investigation of integral representations for trigonometric polynomial deviations are generated by linear methods of summation of the Fourier series. They were developed in papers of Nikolsky, Stechkin, Nagy and others.

The paper presents known results related to the approximation of classes of continuous functions by linear means of the Fourier sums and new facts obtained for some particular cases. In the paper, it is studied the approximation by the Cesàro means of Fourier sums in Lipschitz class. In certain cases, the exact inequalities are found for upper bounds of deviations in the uniform metric of the second order rectangular Cesàro means on the Lipschitz class of periodic functions in two variables.

1. Introduction. Let $H^{\alpha,\beta}$ be the class of functions $f(x;y)$ continuous on $\mathbb{R}^2$, 2π-periodic in each variable, and satisfying the condition

$$|f(x;y) - f(x_1;y_1)| \le |x - x_1|^\alpha + |y - y_1|^\beta, \quad (x;y),\ (x_1;y_1) \in T^2,$$

where $T^2 = [-\pi;\pi]^2$, and $0 < \alpha,\ \beta \le 1$ are absolute constants.

Let $\gamma,\ \delta \in \mathbb{N}$. The rectangular Cesàro means $\sigma^{\gamma,\delta}_{m,n}$ of order (γ, δ) and power (m, n) for function $f \in C$ are defined by relationship [8], [19, p. 130]

$$\sigma^{\gamma,\delta}_{m,n}[f](x;y) = \frac{1}{\pi^2} \iint\limits_{T^2} f(x+t;y+s) K^{\gamma}_{m}(t) K^{\delta}_{n}(s)\, dt\, ds,$$

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where

$$K_i^{\gamma}(t) = \frac{1}{2} + \sum_{k=1}^{i} \left(1 - \frac{k}{i+1}\right) \dots \left(1 - \frac{k}{i+\gamma}\right) \cos kt.$$

If γ, $\delta = 1$, then the rectangular Cesàro means $\sigma_{m,n}^{\gamma,\delta}[f]$ are the rectangular Fejér sums $\sigma_{m,n}[f]$ of the function f

$$\sigma_{m,n}[f](x;y) = \frac{1}{(m+1)(n+1)} \sum_{k=0}^{m} \sum_{j=0}^{n} S_{m,n}[f](x;y),$$

where

$$S_{m,n}[f](x;y) = \frac{1}{\pi^2} \iint\limits_{T^2} f(x+t;y+s) D_m(t) D_n(s)\, dt\, ds$$

is a rectangular partial sum of power (m,n) of the Fourier series for the function f, $D_i(t)$ is the Dirichlet kernel of power i.

In the one-dimensional case, the Cesàro means and their special cases (Fejér sums) have been extensively studied for past decades by many prominent experts in the theory of functions. In 1946, Nikolsky [12, 13] established the asymptotic equality

$$\sup_{f \in H^1} \|f - \sigma_n[f]\|_C = \frac{2}{\pi} \frac{\ln n}{n} + O\left(\frac{1}{n}\right), \quad n \to \infty.$$

Asymptotic equalities for upper bounds of the Cesàro means deviations σ_n^{δ} of any order $\delta > 0$ on the classes H^1 was obtained by Ryźankova [14]

$$\sup_{f \in H^1} \|f - \sigma_n^{\delta}[f]\|_C = \frac{2\delta}{\pi} \frac{\ln n}{n} + O\left(\frac{1}{n}\right), \quad n \to \infty.$$

For $\delta \in \mathbb{N}$, this equality was obtained by Nagy in [9, 10]. In the present paper, we want to find sharp estimates for upper bounds of linear methods deviations on a fixed class.

Let $U_n[f]$, $n = 1, 2, \dots$ be a sequence of linear polynomial operators defined on the set C and $\omega(f;\mu)$ be a modulus of continuity of function $f \in C$ for a given real number $\mu > 0$. The problem of finding the quantity

$$\sup_{\substack{f \in C \\ f \neq \text{const}}} \frac{\|f(x) - U_n[f]\|_C}{\omega(f;\mu_n)},$$

that defines the exact constant in the inequality

$$\|f(x) - U_n[f]\|_C \leq A\omega(f;\mu_n)$$

is one of the important problems in the theory of approximation. Problems of this type for different linear operators were considered by Wang Xing-hua [18], Stechkin [17], Schurer and Steutel [15, 16], and others [1, 2, 4, 6, 7]. In more general case for the matrix summation methods this problem was solved by Falaleev in [3].

In the multidimensional case, the search of the sharp constant becomes problematic. On a class of continuous functions in two variables sharp constant were obtained in [1] for Jackson polynomials:

$$\sup_{(m,n) \in \mathbb{N}^2} \sup_{f \in C} \frac{\|f - D_{m,n}[f]\|_C}{\max\left\{\omega_1(f; \frac{2\pi}{m}); \omega_2(f; \frac{2\pi}{n})\right\}} = \frac{8}{3} - \frac{45\sqrt{3}}{38\pi},$$

where

$$\omega_1(f;\delta)=\sup_{y}\sup_{|x-x_1|\le\delta}|f(x;y)-f(x_1;y)|,\quad \omega_2(f;\gamma)=\sup_{x}\sup_{|y-y_1|\le\gamma}|f(x;y)-f(x;y_1)|$$

are partial modules of continuity for the function $f\in C$.

The aim of the present paper is to present sharp constants in the estimation of the approximation of classes $H^{1,1}$ by the second order rectangular Cesàro means.

2. Result. Our main result is contained in the following theorem.

Theorem. *Let $f\in H^{1,1}$, and n, $m\in\mathbb{N}$. Then the inequality*

$$\|f-\sigma_{m,n}^{2,2}[f]\|_C\le\frac{6\pi^2-16}{3\pi\ln 2}\max\left\{\frac{\ln(m+1)}{m+1};\frac{\ln(n+1)}{n+1}\right\}\tag{1}$$

is true. The constant $\frac{6\pi^2-16}{3\pi\ln 2}$ is sharp.

Proof. First, we prove an auxiliary statement.

Lemma. *Let $(m,n)\in\mathbb{N}^2$. Then*

$$\sup_{f\in H^{1,1}}\|f-\sigma_{m,n}^{2,2}[f]\|_C=\frac{1}{\pi}\int_{-\pi}^{\pi}|s|K_n^2(s)\,ds+\frac{1}{\pi}\int_{-\pi}^{\pi}|t|K_m^2(t)\,dt.\tag{2}$$

Proof. The lemma can be proved by the procedure proposed in [12]. Since $f\in H^{1,1}$ and $K_i^2(t)\ge 0$, $i=0,1,\dots$ [3], we have

$$|f(x;y)-\sigma_{m,n}^{2,2}[f](x;y)|\le\frac{1}{\pi^2}\iint_{T^2}|f(x;y)-f(x+t;y+s)|K_m^2(t)K_n^2(s)\,dt\,ds\le$$

$$\le\frac{1}{\pi^2}\iint_{T^2}(|t|+|s|)K_m^2(t)K_n^2(s)\,dt\,ds=\frac{1}{\pi}\int_{-\pi}^{\pi}|s|K_n^2(s)\,ds+\frac{1}{\pi}\int_{-\pi}^{\pi}|t|K_m^2(t)\,dt.$$

Denote by f^* 2π-periodic in each variable extension on $\mathbb{R}^2$ of function $|x|+|y|$. We get $f^*\in H^{1,1}$, and

$$|f^*(0;0)-\sigma_{m,n}^{2,2}[f^*](0;0)|=\frac{1}{\pi^2}\iint_{T^2}(|t|+|s|)K_m^2(t)K_n^2(s)\,dt\,ds.$$

Hence,

$$\sup_{f\in H^{1,1}}\|f-\sigma_{m,n}^{2,2}[f]\|_C=\frac{1}{\pi^2}\iint_{T^2}(|t|+|s|)K_m^2(t)K_n^2(s)\,dt\,ds.$$

Using the last relation, we obtain the lemma statement. □

Further, we denote

$$\Omega(j):=\frac{1}{\pi}\int_{-\pi}^{\pi}|t|K_j^2(t)\,dt.$$

We have

$$\Omega(j)=\frac{1}{\pi(j+1)(j+2)}\int\limits_0^\pi t\Big[(j+1)(j+2)+2\sum_{k=1}^{j}(j-k+1)(j-k+2)\cos kt\Big]dt=$$

$$=\frac{1}{\pi(j+1)(j+2)}\Big[(j+1)(j+2)\frac{\pi^2}{2}+2\sum_{k=1}^{j}(j-k+1)(j-k+2)\int\limits_0^\pi t\cos kt\Big]dt=$$

$$=\frac{1}{\pi(j+1)(j+2)}\Big[(j+1)(j+2)\frac{\pi^2}{2}+2\sum_{k=1}^{j}(j-k+1)(j-k+2)\frac{(-1)^k-1}{k^2}\Big].$$

Taking into account

$$\sum_{k=1}^{j}(j-k+1)(j-k+2)\frac{(-1)^k-1}{k^2}=$$

$$=-2\sum_{k=0}^{[\frac{j+1}{2}]-1}\Big[(j+1)(j+2)-2(j+1)(2k+1)+2k(2k+1)\Big]\frac{1}{(2k+1)^2},$$

we get

$$\Omega(j)=\frac{4}{\pi(j+1)(j+2)}\Big[(j+1)(j+2)\frac{\pi^2}{8}-\sum_{k=0}^{[\frac{j+1}{2}]-1}\Big(\frac{(j+1)(j+2)}{(2k+1)^2}-\frac{2j+3-(2k+1)}{(2k+1)}\Big)\Big]. \quad (3)$$

By the well-known formula [5, p. 21]

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi^2}{8},$$

we can write

$$\Omega(j)=\frac{4}{\pi(j+1)(j+2)}\Big[(j+1)(j+2)\sum_{k=[\frac{j+1}{2}]}^{\infty}\frac{1}{(2k+1)^2}+(2j+3)\sum_{k=0}^{[\frac{j+1}{2}]-1}\frac{1}{2k+1}-[\frac{j+1}{2}]\Big]. \quad (4)$$

Denote

$$\lambda(j):=\frac{(j+1)}{\ln(j+1)}\Omega(j). \quad (5)$$

Combining (4) and (5), we obtain

$$\lambda(j)=\frac{4}{\pi(j+2)\ln(j+1)}\Big[(j+1)(j+2)\sum_{k=[\frac{j+1}{2}]}^{\infty}\frac{1}{(2k+1)^2}+(2j+3)\sum_{k=0}^{[\frac{j+1}{2}]-1}\frac{1}{2k+1}-[\frac{j+1}{2}]\Big]. \quad (6)$$

To estimate the first sums in (6), we use the following relationship

$$\frac{1}{(2k+1)^2}<\frac{1}{2k}-\frac{1}{2k+1}.$$

Hence,

$$\sum_{k=[\frac{j+1}{2}]}^{\infty} \frac{1}{(2k+1)^2} < \sum_{k=[\frac{j+1}{2}]}^{\infty} \left(\frac{1}{2k} - \frac{1}{2k+1}\right) = \frac{1}{2}\frac{1}{[\frac{j+1}{2}]} < \frac{1}{j-1}, \quad j = 2, 3, \dots .$$

Taking into account [11, p. 208], we have

$$\sum_{k=0}^{[\frac{j+1}{2}]-1} \frac{1}{2k+1} < 1 + \frac{1}{2}\ln j, \quad j = 2, 3, \dots .$$

Therefore,

$$\lambda(j) < \frac{2}{\pi}\left(2 + \frac{1}{(j-1)\ln(j+1)} + \frac{4j+5}{(j+2)\ln(j+1)} - \frac{1}{j+2}\right), \quad j = 2, 3, \dots . \tag{7}$$

Let us consider the function

$$\gamma(x) := \frac{2}{\pi}\left(2 + \frac{1}{(x-1)\ln(x+1)} + \frac{4x+5}{(x+2)\ln(x+1)} - \frac{1}{x+2}\right), \quad x \geq 2.$$

The function $\gamma(x)$ is decomposed as follows

$$\gamma(x) = \varphi^{(1)}(x) + \varphi^{(2)}(x),$$

where

$$\varphi^{(1)}(x) := \frac{4}{\pi} + \frac{2}{\pi(x-1)\ln(x+1)}, \quad \varphi^{(2)}(x) := \frac{2}{\pi(x+2)}\left(\frac{4x+5}{\ln(x+1)} - 1\right).$$

Denote $y(x) = \frac{4x+5}{\ln(x+1)} - 1$. We obtain

$$\varphi'(x) = \frac{2}{\pi}\left(\frac{y(x)}{x+2}\right)' = \frac{2}{\pi}\frac{y'(x)(x+2) - y(x)}{(x+2)^2}.$$

By the standard calculations we have $y'(x)(x+2) - y(x) < 0$ as $x \geq 2$. Applying these facts, we conclude that $\gamma(x)$ is a monotone decreasing function.

In view of (7), we get

$$\lambda(j) < \frac{18}{5\pi} + \frac{3.9}{\pi\ln 2}, \quad j \geq 3.$$

Using the relations (3), (5), one has

$$\lambda(1) > \lambda(2) > \frac{18}{5\pi} + \frac{3.9}{\pi\ln 2}.$$

Hence,

$$\sup_{j\in\mathbb{N}} \lambda(j) = \lambda(1) = \frac{3\pi^2 - 8}{3\pi\ln 2}. \tag{8}$$

The rate of uniformly approximation of functions from the class $H^{1,1}$ by the Cesàro means $\sigma_{m,n}^{2,2}[f]$ is determined by inequality [8]

$$\|f - \sigma_{m,n}^{2,2}[f]\|_C \leq A \max_{m,n\in\mathbb{N}}\left\{\frac{\ln(m+1)}{m+1}; \frac{\ln(n+1)}{n+1}\right\}.$$

In this inequality the constant

$$A^* = \sup_{n,m\in\mathbb{N}} \frac{\sup\limits_{f\in H^{1,1}} \|f - \sigma^{2,2}_{m,n}(f)\|_C}{\max\{\frac{\ln(m+1)}{m+1}; \frac{\ln(n+1)}{n+1}\}} \tag{9}$$

is the best constant in the class $H^{1,1}$.

Taking into account (2), (5), and (8), we have

$$A^* = \sup_{n,m\in\mathbb{N}} \frac{\Omega(m)+\Omega(n)}{\max\{\frac{\ln(m+1)}{m+1}; \frac{\ln(n+1)}{n+1}\}} =$$

$$= \sup_{n,m\in\mathbb{N}} \left[\frac{\Omega(m)}{\max\{\frac{\ln(m+1)}{m+1}; \frac{\ln(n+1)}{n+1}\}} + \frac{\Omega(n)}{\max\{\frac{\ln(m+1)}{m+1}; \frac{\ln(n+1)}{n+1}\}} \right] \le$$

$$\le \sup_{n,m\in\mathbb{N}} \left[\frac{\Omega(m)}{\frac{\ln(m+1)}{m+1}} + \frac{\Omega(n)}{\frac{\ln(n+1)}{n+1}} \right] = \sup_{m\in\mathbb{N}} \frac{(m+1)\Omega(m)}{\ln(m+1)} + \sup_{n\in\mathbb{N}} \frac{(n+1)\Omega(n)}{\ln(n+1)} = 2\lambda(1). \tag{10}$$

Contrariwise, using (2) we find

$$\frac{\sup\limits_{f\in H^{1,1}} \|f - \sigma^{2,2}_{1,1}[f]\|_C}{\max\{\frac{\ln 2}{2}; \frac{\ln 2}{2}\}} = \frac{\Omega(1)+\Omega(1)}{\frac{\ln 2}{2}} = 2\lambda(1). \tag{11}$$

Combining (9)–(11), we have $A^* = 2\lambda(1)$. The proof is completed.

□

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