ON THE GROWTH OF SERIES IN SYSTEMS OF FUNCTIONS AND LAPLACE-STIELTJES TYPE INTEGRALS


For a regularly convergent in $\mathbb{C}$ series $A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$ in the system $f(\lambda_n z)$, where $f(z) = \sum_{k=0}^{\infty} f_k z^k$ is an entire transcendental function and $(\lambda_n)$ is a sequence of positive numbers increasing to $+\infty$, it is investigated the relationship between the growth of functions $A$ and $f$ in terms of a generalized order. It is proved that if $a_n \geq 0$ for all $n \geq n_0$,

$$\ln \lambda_n = o(\beta^{-1}(\alpha(\frac{1}{m_{\lambda_n} \ln \frac{1}{a_n}})))$$

for each $c \in (0, +\infty)$ and $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ then

$$\lim_{r \to +\infty} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)} = \lim_{r \to +\infty} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)},$$

where $M_f(r) = \max\{|f(z)|: |z| = r\}$. $\Gamma_f(r) := \frac{dM_f(r)}{\ln x}$ and positive continuous on $(x_0, +\infty)$ functions $\alpha$ and $\beta$ are such that $\beta((1 + o(1))x) = (1 + o(1))\beta(x)$, $\alpha(x) = (1 + o(1))\alpha(x)$ and $\frac{d\beta^{-1}(\alpha(x))}{d\ln x} = O(1)$ as $x \to +\infty$ for each $c \in (0, +\infty)$. A similar result is obtained for the Laplace-Stieltjes type integral $I(r) = \int_0^{\infty} a(x)f(rx)dF(x)$.

1. Introduction. Let

$$f(z) = \sum_{k=0}^{\infty} f_k z^k$$

be an entire function, $M_f(r) = \max\{|f(z)|: |z| = r\}$ and $(\lambda_n)$ be a sequence of positive numbers increasing to $+\infty$. Suppose that the series

$$A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$$

in the system $f(\lambda_n z)$ regularly convergent in $\mathbb{C}$, i.e., $\sum_{n=1}^{\infty} |a_n|M_f(r\lambda_n) < +\infty$ for all $r \in [0, +\infty)$. Many authors have studied the representation of analytic functions by series in the system $f(\lambda_n z)$. We will specify here only on the monographs of A.F. Leont’ev [1] and B.V. Vinnitskyi [2], where references are to other works. Since series (2) regularly convergent in $\mathbb{C}$, the function $A$ is entire. To study its growth, we will use generalized orders. For this purpose, as in [3] by $L$ we denote the class of continuous non-negative on $(-\infty, +\infty)$ functions $\alpha$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$

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Lemma 3. \(\text{Lemma 3}\).

Lemma 1 ([1]). If \(a \in L_s, \beta \in L^0\) and \(\frac{d\beta^{-1}(\alpha(x))}{dx} = O(1)\) as \(x \to +\infty\) for each \(c \in (0, +\infty)\), i.e. \(\alpha\) is a slowly increasing function. Clearly, \(L_s \subseteq L^0\). For \(a \in L\) and \(\beta \in L\) quantity \(\varrho_{a, \beta}[f] = \lim_{r \to +\infty} \frac{\alpha(r)}{\beta[r]}\) is called generalized \((\alpha, \beta)\)-order of the entire function \(f\) ([3]). Note that functions of form (2) were also studied in [4].

Lemma 1 ([1]). If \(a \in L_s, \beta \in L^0\) and \(\frac{d\beta^{-1}(\alpha(x))}{dx} = O(1)\) as \(x \to +\infty\) for each \(c \in (0, +\infty)\) then

\[\varrho_{a, \beta}[f] = \lim_{k \to +\infty} \frac{\alpha(k)}{\beta(\frac{k}{\ln(1/k)})}.\]  

(3)

Using Lemma 1 here we establish a relationship between the growth of the entire functions \(f\) and \(F\) in terms of generalized orders.

2. Main result. Suppose that \(a_n \geq 0\) for all \(n \geq 1\). Since

\[A(z) = \sum_{n=1}^{\infty} a_n \sum_{k=0}^{\infty} f_k(z\lambda_n)^k = \sum_{k=0}^{\infty} f_k \left( \sum_{n=1}^{\infty} a_n \lambda_n^k \right) z^k,\]

in view of Cauchy’s inequality we have

\[M_A(r) \geq |f_k| \left( \sum_{n=1}^{\infty} a_n \lambda_n^k \right) r^k \geq a_n |f_k| (\lambda_n r)^k\]

for all \(n \geq 1, k \geq 0\) and \(r \in [0, +\infty)\). Hence it follows that \(M_A(r) \geq |f_k| \mu_D(k)r^k\), where \(\mu_D(\sigma) = \max\{|a_n| \exp\{\sigma \ln \lambda_n\} : n \geq 1\}\) is the maximal term of entire Dirichlet series

\[D(\sigma) = \sum_{n=1}^{\infty} |a_n| \exp\{\sigma \ln \lambda_n\}.\]  

(4)

Therefore, \(M_A(r) \geq \mu_G(r)\), where \(\mu_G(r) = \max\{|f_k| \mu_D(k)r^k : k \geq 0\}\) is the maximal term of the series

\[G(r) = \sum_{k=0}^{\infty} |f_k| \mu_D(k)r^k.\]  

(5)

To obtain the estimate \(M_A(r)\) from above, in addition to Lemma 1, the following two well-known lemmas will be required.

Lemma 2. If a function \(f\) is transcendental then the function \(\ln M_f(r)\) is logarithmically convex and, thus,

\[\Gamma_f(r) := \frac{d \ln M_f(r)}{dr} \to +\infty, \quad r \to +\infty,\]

(in points where the derivative does not exist, under \(\frac{d \ln M_f(r)}{dr}\) we mean the right-hand derivative).

Lemma 3. If a function \(f\) is transcendental then

\[M_f(r) \leq \sum_{k=0}^{\infty} |f_k|(2r)^k 2^{-k} \leq 2\mu_f(2r).\]

Lemma 4 ([5]). If \(\beta \in L\) and \(B(\delta) = \lim_{x \to +\infty} \frac{\beta((1+\delta) x)}{\beta(x)}\), \(\delta > 0\), then in order that \(\beta \in L^0\), it is necessary and sufficient that \(B(\delta) \to 1\) as \(\delta \to +0\).
Since series (2) regularly convergent in $\mathbb{C}$, for every $r \in [0, +\infty)$ and $\tau > 0$ we have

$$M_A(r) \leq \sum_{n=1}^{\infty} |a_n| M_f(r \lambda_n) \leq \mu_A((1 + \tau)r) \sum_{n=1}^{\infty} \frac{M_f(r \lambda_n)}{M_f((1 + \tau)r \lambda_n)},$$

(6)

where $\mu_A(r) = \max\{|a_n| M_f(r \lambda_n): n \geq 1\}$.

Then by Lemma 2 for $r \geq 1$ we have

$$\ln M_f((1 + \tau)r \lambda_n) - \ln M_f(r \lambda_n) = \int_{r \lambda_n}^{(1 + \tau)r \lambda_n} \frac{d \ln M_f(x)}{d \ln x} d \ln x = \int_{r \lambda_n}^{(1 + \tau)r \lambda_n} \Gamma_f(x) d \ln x \geq \Gamma_f(r \lambda_n) \ln(1 + \tau) \geq \Gamma_f(\lambda_n) \ln(1 + \tau)$$

Therefore, if $\ln n \leq q \Gamma_f(\lambda_n)$ for all $n \geq n_0$ and $\ln(1 + \tau) > q$ then

$$\sum_{n=n_0}^{\infty} \frac{M_f(r \lambda_n)}{M_f((1 + \tau)r \lambda_n)} \leq \sum_{n=n_0}^{\infty} \exp\{-\Gamma_f(\lambda_n) \ln(1 + \tau)\} \leq \sum_{n=n_0}^{\infty} \exp\{-\frac{\ln(1 + \tau)}{q} \ln n\} < +\infty$$

and (6) for $r \geq 1$ implies

$$M_A(r) \leq T \mu_A((1 + \tau)r), \quad T = \text{const} > 0.$$  (7)

Also we have

$$\mu_A(r) \leq \max\{|a_n| \sum_{k=0}^{\infty} |f_k|(r \lambda_n)^k: n \geq 1\} \leq \sum_{k=0}^{\infty} \max\{|a_n| \lambda_n^k: n \geq 1\}|f_k|r^k =\sum_{k=0}^{\infty} \mu_D(k)|f_k|r^k \leq \mu_G(2r) \sum_{k=0}^{\infty} 2^{-k} = 2\mu_G(2r).$$  (8)

From (7) and (8) we get the estimate $M_A(r) \leq 2T \mu_G(2(1 + \tau)r)$ for $r \geq 1$ and, thus,

$$\ln \mu_G(r) \leq \ln M_A(r) \leq \ln \mu_G(2(1 + \tau)r) + \ln(2T), \quad r \geq 1.$$  (9)

Now we can prove such a theorem.

**Theorem 1.** Let $f$ be an entire transcendental function, $a_n \geq 0$ for all $n \geq 1$ and series (2) regularly convergent in $\mathbb{C}$. Suppose that the functions $\alpha$ and $\beta$ satisfy the conditions of Lemma 1, $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ and for each $c \in (0, +\infty)$

$$\ln \lambda_n = o\left(\beta^{-1}(c \alpha)\left(\frac{1}{\ln \lambda_n} \ln \frac{1}{a_n}\right)\right), \quad n \to \infty.$$  (10)

Then $\varrho_{\alpha, \beta}[A] = \varrho_{\alpha, \beta}[f]$.

**Proof.** Since $\mu_D(\sigma) \to +\infty$ as $\sigma \to +\infty$, we have $\mu_D(k) \geq 1$ for $k \geq k_0$. For simplicity, we assume that $k_0 = 0$. Then $\mu_G(r) = \max\{|f_k|\mu_D(k)r^k: k \geq 0\} \geq \max\{|f_k|r^k: k \geq 0\} = \mu_f(r)$, whence in view of (9) and Lemma 3 it follows that $\varrho_{\alpha, \beta}[f] \leq \varrho_{\alpha, \beta}[F]$. 


On the other hand, in view of (9) \( g_{\alpha,\beta}[A] \leq g_{\alpha,\beta}[G] \). By Lemma 1

\[
\varrho_{\alpha,\beta}[G] = \lim_{k \to +\infty} \beta \left( \frac{\lambda}{\mu_{\nu_{D}^{(\alpha)(\beta)}}} \right) \leq \lim_{k \to +\infty} \beta \left( \frac{\lambda}{\mu_{\nu_{D}^{(\alpha)(\beta)}}} \right).
\]

If \( g_{\alpha,\beta}[f] < +\infty \) then by Lemma 1 for every \( \varrho > g_{\alpha,\beta}[f] \) and all \( k \geq k_{0}(\varrho) \) we have \( \alpha(k) \leq \varrho \beta(\frac{1}{k} \ln \frac{1}{\lambda_{k}}) \) and, thus,

\[
\frac{1}{k} \ln \frac{1}{\lambda_{k}} \geq \beta^{-1} \left( \frac{\alpha(k)}{\varrho} \right), \quad k \geq k_{0}(\varrho).
\]

Let \( \nu_{D}(\sigma) = \max \{ n : |a_{n}| \exp \{ \sigma \ln \lambda_{n} \} = \mu_{D}(\sigma) \} \) be the central index of series (4). Then ([6, p.17])

\[
\int_{\sigma_{0}}^{\sigma} \ln \nu_{D}(x)dx, \quad \sigma_{0} \leq \sigma.
\]

From condition (10) with \( c = 1/\varrho \) we get

\[
\ln a_{n} \leq - \ln \lambda_{n} \alpha^{-1} \left( \varrho \beta \left( \frac{\ln \lambda_{n}}{\epsilon} \right) \right)
\]

for each \( \epsilon > 0 \) and all \( n \geq n_{0}(\epsilon) \). Therefore, for all \( \sigma \geq \sigma_{0} = \sigma_{0}(\epsilon) \)

\[
\ln \mu_{D}(\sigma) = \ln a_{n} \nu_{D}(\sigma) + \sigma \ln \lambda_{n} \nu_{D}(\sigma) \leq - \ln \lambda_{n} \nu_{D}(\sigma) \alpha^{-1} \left( \varrho \beta \left( \frac{\ln \lambda_{n} \nu_{D}(\sigma)}{\epsilon} \right) \right) + \sigma \ln \lambda_{n} \nu_{D}(\sigma) = \ln \lambda_{n} \nu_{D}(\sigma) \left( \sigma - \alpha^{-1} \left( \varrho \beta \left( \frac{\ln \lambda_{n} \nu_{D}(\sigma)}{\epsilon} \right) \right) \right).
\]

Since \( \mu_{D}(\sigma) \to +\infty \) as \( \sigma \to +\infty \), hence it follows that \( \sigma - \alpha^{-1}(\varrho \beta(\ln \lambda_{n} \nu_{D}(\sigma)/\epsilon)) \geq 0 \), i.e. \( \ln \lambda_{n} \nu_{D}(\sigma) \leq \epsilon \beta^{-1}(\sigma/\varrho) \) for \( \sigma \geq \sigma_{0} \). Therefore, in view of (13)

\[
\ln \mu_{D}(\sigma) \leq \ln \mu_{D}(\sigma_{0}) + \epsilon \int_{\sigma_{0}}^{\sigma} \beta^{-1} \left( \frac{\alpha(x)}{\varrho} \right) dx \leq \ln \mu_{D}(\sigma_{0}) + \epsilon \beta^{-1} \left( \frac{\alpha(\sigma)}{\varrho} \right)
\]

and, thus,

\[
\frac{\ln \mu_{D}(k)}{k} \leq \epsilon + \epsilon \beta^{-1} \left( \frac{\alpha(k)}{\varrho} \right), \quad k \geq k_{0}(\epsilon).
\]

From (11), (12) and (14) we obtain

\[
\varrho_{\alpha,\beta}[G] \leq \lim_{k \to +\infty} \beta \left( \frac{\alpha(k)}{\varrho} \right) = \lim_{k \to +\infty} \beta \left( \frac{\alpha(k)}{\varrho} \right) = \frac{\alpha(k)}{\varrho} \leq \varrho B(\varepsilon),
\]

where by Lemma 4 \( B(\varepsilon) = \lim_{k \to +\infty} \frac{\beta(x)}{\varrho(1-\varepsilon)x} \to 1 \) as \( \varepsilon \to 0 \). Thus, \( \varrho_{\alpha,\beta}[G] \leq \varrho \) and since \( \varrho \) is arbitrary, we obtain the inequality \( \varrho_{\alpha,\beta}[G] \leq \varrho_{\alpha,\beta}[f] \) which is obvious when \( \varrho_{\alpha,\beta}[f] = +\infty \). Finally, (9) implies the inequality \( \varrho_{\alpha,\beta}[A] \leq \varrho_{\alpha,\beta}[G] \leq \varrho_{\alpha,\beta}[f] \). \( \square \)
The functions $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ satisfy the conditions of Theorem 1. Therefore, Theorem 1 implies the following statement.

**Corollary 1.** Let an entire transcendental function $f$ have the order $\rho[f] := \lim_{r \to +\infty} \frac{\ln \ln M_f(r)}{\ln r} = \rho \in (0, +\infty)$ and

$$0 < \underline{\sigma}_f := \lim_{r \to +\infty} \frac{\ln M_f(r)}{r^\rho} \leq \overline{\sigma}_f := \lim_{r \to +\infty} \frac{\ln M_f(r)}{r^\rho} < +\infty. \quad (15)$$

Suppose that $a_n \geq 0$ for all $n \geq 1$ and series (2) regularly convergent in $C$. If $\ln n = O(\lambda_n^\rho)$ and $\ln \lambda_n = o(\ln(\ln(1/a_n)))$ as $n \to \infty$ then $\rho[A] = \rho[f]$.

Indeed, it is clear that

$$\ln M_f(r) = \ln M_f(r_0) + \int_{r_0}^r \frac{\Gamma_f(t)}{t} \, dt, \quad 0 \leq r_0 \leq r < +\infty.$$

Therefore, if we put

$$\tau = \lim_{r \to +\infty} \frac{\Gamma_f(r)}{r^\rho}, \quad \overline{\tau} = \lim_{r \to +\infty} \frac{\Gamma_f(r)}{r^\rho},$$

then using results from [7] we get

$$\underline{\tau} \leq \rho \overline{\sigma} \leq \tau \left(1 + \ln \frac{\overline{\tau}}{\underline{\tau}}\right) \leq \overline{\tau} \leq e\rho \overline{\sigma}.$$

Hence in view of (15) it follows that $\overline{\tau} < +\infty$ and $\underline{\tau} > 0$. Therefore, $\Gamma_f(r) \asymp r^\rho$ as $r \to +\infty$ and, thus, the conditions $\ln n = O(\lambda_n^\rho)$ and $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ are equivalent.

We remark also that condition (10) now looks like $\ln \lambda_n = o(\ln(\ln(1/a_n)))$, i.e. $\ln \lambda_n = o(\ln(\ln(1/a_n)))$ as $n \to \infty$.

All conditions of Theorem 1 are satisfied and Theorem 1 implies Corollary 1.

**3. Growth of Laplace-Stieltjes type integrals.** Let $V$ be the class of nonnegative nondecreasing unbounded continuous on the right functions $F$ on $[0, +\infty)$. We assume that $f$ is an entire transcendental function and $f_k \geq 0$ for all $k \geq 0$ and a positive on $[0, +\infty)$ function $a$ is such that the Laplace-Stieltjes type integral

$$I(r) = \int_0^\infty a(x)f(rx) \, dF(x) \quad (16)$$

exists for every $r \in [0, +\infty)$. The asymptotical behavior of such integrals in the case when $f(x) = e^x$ is studied in the monograph [8] (see also [9, 10, 11]), as well as for the case of positive functions $f$ such that the function $\ln f$ is convex on $(0, +\infty)$ in [12].

Suppose that $x_0 > 1$ is such that $\int_1^{x_0} a(x) \, dF(x) \geq c > 0$. Then

$$I(r) \geq \int_1^{x_0} a(x)f(rx) \, dF(x) \geq f(r)c. \quad (17)$$
On the other hand, as in the proof of Theorem 1 for \( r \geq 1 \) we have \( \ln f((1 + \tau)rx) - \ln f(rx) \geq \Gamma_f(x)\ln(1 + \tau) \). Therefore, if \( \mu_f(r) = \max\{a(x)f(rx) : x \geq 0\} \) is the maximum of the integrand, \( \ln F(x) \leq q\Gamma_f(x) \) and \( \ln(1 + \tau) > q \)

\[
I(r) = \int_0^\infty a(x)f((1 + \tau)rx)\frac{f(rx)}{f((1 + \tau)rx)}dF(x) \leq \mu_f((1 + \tau)r) \int_0^\infty \frac{f(rx)}{f((r + \tau)rx)}dF(x) \leq \\
\leq \mu_f((1 + \tau)r) \int_0^\infty e^{-\Gamma_f(x)\ln(1+\tau)}dF(x) = \\
= \mu_f((1 + \tau)r) \left( T_1 + \ln(1 + \tau) \int_0^\infty F(x)e^{-\Gamma_f(x)\ln(1+\tau)}d\Gamma_f(x) \right) \leq \\
\leq \mu_f((1 + \tau)r) \left( T_1 + \ln(1 + \tau) \int_0^\infty e^{-\Gamma_f(x)((\ln(1+\tau)-q)}d\Gamma_f(x) \right) \leq T_2\mu_f(r + \tau). \quad (18)
\]

where \( T_j = \text{const} > 0 \). Also, as above, we have

\[
\mu_f(r) = \max \left\{ a(x) \sum_{k=0}^\infty f_k(xr)^k : x \geq 0 \right\} \leq \\
\leq \sum_{k=0}^\infty \max\{a(x)x^k : x \geq 0\} f_k r^k = G_1(r) := \sum_{k=0}^\infty \mu_j(k) f_k r^k, \quad (19)
\]

where \( \mu_j(\sigma) = \max\{a(x)e^{\sigma \ln x} : x \geq 0\} \) is the maximum of the integrand for Laplace-Stieltjes integral

\[
J(\sigma) = \int_0^\infty a(x)e^{\sigma \ln x}dF(x).
\]

Now we prove the following analog of Theorem 1.

**Theorem 2.** Let \( F \in V, f \) be an entire transcendental function and \( f_k \geq 0 \) for all \( k \geq 0 \). Suppose that \( \ln F(x) \leq q\Gamma_f(x) \) for some \( q > 0 \) and all \( x \geq 0 \), the functions \( \alpha \) and \( \beta \) satisfy the conditions of Lemma 1 and for each \( c \in (0, +\infty) \)

\[
\ln x = o \left( \beta^{-1} \left( c\alpha \left( \frac{1}{\ln x} \ln \frac{1}{a(x)} \right) \right) \right), \quad x \to +\infty. \quad (20)
\]

Then \( \varrho_{a,\beta}[I] = \varrho_{a,\beta}[f] \).

**Proof.** From (17) it follows that \( \varrho_{a,\beta}[f] \leq \varrho_{a,\beta}[I] \).

On the other hand, in view of (18) and (19) \( \varrho_{a,\beta}[I] \leq \varrho_{a,\beta}[G_1] \). By Lemma 1

\[
\varrho_{a,\beta}[G_1] = \lim_{k \to +\infty} \frac{\alpha(k)}{\beta \left( k \ln \frac{1}{G_k} - \frac{\ln \mu_j(k)}{k} \right)}. \quad (21)
\]

If \( \varrho_{a,\beta}[f] < +\infty \) then as above we get (12).
As in [8, p.24], let \( \nu_j(\sigma) \) be the central point of \( \mu_j(\sigma) \). Then [8, p.26]

\[
\ln \mu_j(\sigma) = \ln \mu_j(\sigma_0) + \int_{\sigma_0}^{\sigma} \ln \nu_j(x) \, dx, \quad \sigma_0 \leq \sigma. \tag{22}
\]

From condition (20) with \( c = 1/\theta \) we get \( \ln a(x) \leq -\ln x \alpha^{-1} \left( \frac{\ln x}{\varepsilon} \right) \) for each \( \varepsilon > 0 \) and all \( x \geq x_0(\varepsilon) \). Therefore, as in the proof of Theorem 1, for all \( \sigma \geq \sigma_0 = \sigma_0(\varepsilon) \) we have

\[
\ln \mu_j(\sigma) \leq \ln \nu_j(\sigma) \left( \sigma - \alpha^{-1} \left( \frac{\ln \nu_j(\sigma)}{\varepsilon} \right) \right),
\]

whence it follows that \( \ln \nu_j(\sigma) \leq \varepsilon \beta^{-1} (\alpha(\sigma)/\theta) \) for \( \sigma \geq \sigma_0 \). Therefore, in view of (22) \( \ln \mu_j(\sigma) \leq \ln \mu_j(\sigma_0) + \varepsilon \sigma \beta^{-1} (\alpha(\sigma)/\theta) \) and, thus,

\[
\frac{\ln \mu_D(k)}{k} \leq \varepsilon + \varepsilon \beta^{-1} \left( \frac{\alpha(k)}{\theta} \right), \quad k \geq k_0(\varepsilon). \tag{23}
\]

From (21), (12) and (23) as in the proof of Theorem 1 we get \( \varrho_{\alpha,\beta}[I] \leq \varrho_{\alpha,\beta}[G_1] \leq \varrho_{\alpha,\beta}[f] \). \( \square \)

For the functions \( \alpha(x) = \ln^+ x \) and \( \beta(x) = x^+ \) Theorem 2 implies the following statement.

**Corollary 2.** Let an entire transcendental function (1) with \( f_k \geq 0 \) satisfy condition (15). If \( \ln F(x) = O(x^\rho) \) and \( \ln x = o(\ln \ln(1/a(x))) \) as \( x \to +\infty \) then \( \varrho[I] = \varrho[f] \).

**4. Remarks.** The conditions \( \ln n = O(\lambda^g_n) \) and \( \ln \lambda_n = o(\ln \ln(1/a_n)) \) as \( n \to \infty \) in Corollary 1 and their analogues \( \ln F(x) = O(x^\rho) \) and \( \ln x = o(\ln \ln(1/a(x))) \) as \( x \to +\infty \) in Corollary 2 are natural. Let us show this by the example of the function \( A_\rho(z) = \sum_{n=1}^{\infty} a_n E_\rho(z\lambda_n) \), where

\[
E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k/\rho)}, \quad 0 < \rho < +\infty,
\]

is the Mittag-Leffler function. The properties of this function have been used in many problems in the theory of entire functions. We only need the following property of the Mittag-Leffler function: if \( 0 < \rho < +\infty \) then [13, p.115]

\[
M_{E_\rho}(r) = E_\rho(r) = (1 + o(1)) \rho e^{r^\rho}, \quad r \to +\infty.
\]

Hence it follows that \( \varrho[E_\rho] = \rho \) and \( \varrho[A_\rho] = \varrho[A_\rho^*] \), where \( A_\rho^*(r) = \sum_{n=1}^{\infty} a_n \exp\{r^\rho \lambda_n^g\} \). We put \( r^\rho = \sigma \) and \( \lambda_n^g = \mu_n \). Then \( A_\rho^*(r) = D_\rho(\sigma) = \sum_{n=1}^{\infty} a_n e^{\sigma \mu_n} \) and \( \varrho[A_\rho^*] = \varrho[D_\rho] \), where

\[
\varrho[D_\rho] = \lim_{\sigma \to +\infty} \frac{\ln \ln D_\rho(\sigma)}{\ln \sigma}
\]

is the logarithmic order of Dirichlet series \( D_\rho \). It is known [14] that if \( \ln n = O(\mu_n) \) as \( n \to \infty \) then \( \varrho[D_\rho] = p_l + 1 \), where

\[
p_l = \lim_{n \to +\infty} \frac{\ln \mu_n}{\ln \left( \frac{1}{\mu_n} \ln \frac{1}{a_n} \right)}.
\]
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Therefore, if \( \ln n = O(\lambda^n) \) as \( n \to \infty \) and \( p_l = 0 \) then \( \varrho[A_\varrho] = \varrho = \varrho[E_\varrho] \). Finally, \( p_l = 0 \) if and only if \( \ln \mu_n = o(\ln(1/a_n)) \), i.e. \( \ln \lambda_n = o(\ln(1/a_n)) \) as \( n \to \infty \).

By a similar method, studying the growth of an integral \( I_\varrho(r)(r) = \int_0^\infty a(x)E_\varrho(rx)dF(x) \) can be reduced to studying the growth of the integral \( J(\sigma) = \int_0^\infty a_1(x)e^{\sigma x}dF_1(x) \) and then use the formula [8, p.83]

\[
\lim_{\sigma \to +\infty} \frac{\ln J(\sigma)}{\ln \sigma} = \lim_{x \to +\infty} \frac{\ln x}{\ln \left( \frac{1}{x} \ln \left( \frac{1}{a_1(x)} \right) \right)} + 1,
\]

provided

\[
\lim_{x \to +\infty} \frac{\ln \ln F_1(x)}{\ln x} \leq 1.
\]

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