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ON THE GROWTH OF SERIES IN SYSTEMS OF FUNCTIONS AND LAPLACE-STIELTJES TYPE INTEGRALS

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For a regularly convergent in $\mathbb{C}$ series $A(z)=\sum_{n=1}^{\infty}a_nf(\lambda_nz)$ in the system $f(\lambda_nz)$, where $f(z)=\sum_{k=0}^{\infty}f_kz^k$ is an entire transcendental function and (λ_n) is a sequence of positive numbers increasing to $+\infty$, it is investigated the relationship between the growth of functions A and f in terms of a generalized order. It is proved that if $a_n\ge 0$ for all $n\ge n_0$,

$$\ln\lambda_n=o\big(\beta^{-1}\big(c\alpha(\tfrac{1}{\ln\lambda_n}\ln\tfrac{1}{a_n})\big)\big)$$

for each $c\in(0,+\infty)$ and $\ln n=O(\Gamma_f(\lambda_n))$ as $n\to\infty$ then

$$\varlimsup_{r\to+\infty}\frac{\alpha(\ln M_A(r))}{\beta(\ln r)}=\varlimsup_{r\to+\infty}\frac{\alpha(\ln M_f(r))}{\beta(\ln r)},$$

where $M_f(r)=\max\{|f(z)|\colon |z|=r\}$, $\Gamma_f(r):=\frac{d\ln M_f(r)}{d\ln r}$ and positive continuous on $(x_0,+\infty)$ functions α and β are such that $\beta((1+o(1))x)=(1+o(1))\beta(x)$, $\alpha(cx)=(1+o(1))\alpha(x)$ and $\frac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,+\infty)$. A similar result is obtained for the Laplace-Stieltjes type integral $I(r)=\int\limits_0^{\infty}a(x)f(rx)dF(x)$.

1. Introduction. Let

$$f(z)=\sum_{k=0}^{\infty}f_kz^k \tag{1}$$

be an entire function, $M_f(r)=\max\{|f(z)|\colon |z|=r\}$ and (λ_n) be a sequence of positive numbers increasing to $+\infty$. Suppose that the series

$$A(z)=\sum_{n=1}^{\infty}a_nf(\lambda_nz) \tag{2}$$

in the system $f(\lambda_nz)$ regularly convergent in $\mathbb{C}$, i. e. $\sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n)<+\infty$ for all $r\in[0,+\infty)$. Many authors have studied the representation of analytic functions by series in the system $f(\lambda_nz)$. We will specify here only on the monographs of A.F. Leont'ev [1] and B.V. Vinnitskyi [2], where references are to other works. Since series (2) regularly convergent in $\mathbb{C}$, the function A is entire. To study its growth, we will use generalized orders. For this purpose, as in [3] by L we denote the class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\ge 0$ for $x\le x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\le x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$

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and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each $c\in(0,+\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si}\subset L^0$. For $\alpha\in L$ and $\beta\in L$ quantity $\varrho_{\alpha,\beta}[f]=\varlimsup\limits_{r\to+\infty}\frac{\alpha(\ln M_f(r))}{\beta(\ln r)}$ is called generalized (α,β)-order of the entire function f ([3]). Note that functions of form (2) were also studied in [4].

Lemma 1 ([1])**.** *If $\alpha\in L_{si}$, $\beta\in L^0$ and $\frac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,+\infty)$ then*

$$\varrho_{\alpha,\beta}[f]=\varlimsup_{k\to+\infty}\frac{\alpha(k)}{\beta\left(\frac{1}{k}\ln\frac{1}{|f_k|}\right)}. \tag{3}$$

Using Lemma 1 here we establish a relationship between the growth of the entire functions f and F in terms of generalized orders.

2. Main result. Suppose that $a_n\ge0$ for all $n\ge1$. Since

$$A(z)=\sum_{n=1}^{\infty}a_n\sum_{k=0}^{\infty}f_k(z\lambda_n)^k=\sum_{k=0}^{\infty}f_k\Big(\sum_{n=1}^{\infty}a_n\lambda_n^k\Big)z^k,$$

in view of Cauchy's inequality we have

$$M_A(r)\ge|f_k|\Big(\sum_{n=1}^{\infty}a_n\lambda_n^k\Big)r^k\ge a_n|f_k|(\lambda_nr)^k$$

for all $n\ge1$, $k\ge0$ and $r\in[0,+\infty)$. Hence it follows that $M_A(r)\ge|f_k|\mu_D(k)r^k$, where $\mu_D(\sigma)=\max\{|a_n|\exp\{\sigma\ln\lambda_n\}\colon n\ge1\}$ be the maximal term of entire Dirichlet series

$$D(\sigma)=\sum_{n=1}^{\infty}|a_n|\exp\{\sigma\ln\lambda_n\}. \tag{4}$$

Therefore, $M_A(r)\ge\mu_G(r)$, where $\mu_G(r)=\max\{|f_k|\mu_D(k)r^k\colon k\ge0\}$ is the maximal term of the series

$$G(r)=\sum_{k=0}^{\infty}|f_k|\mu_D(k)r^k. \tag{5}$$

To obtain the estimate $M_A(r)$ from above, in addition to Lemma 1, the following two well-known lemmas will be required.

Lemma 2. *If a function f is transcendental then the function $\ln M_f(r)$ is logarithmically convex and, thus,*

$$\Gamma_f(r):=\frac{d\ln M_f(r)}{d\ln r}\nearrow+\infty,\quad r\to+\infty,$$

(in points where the derivative does not exist, under $\frac{d\ln M_f(r)}{d\ln r}$ we mean the right-hand derivative).

Lemma 3. *If a function f is transcendental then*

$$M_f(r)\le\sum_{k=0}^{\infty}|f_k|(2r)^k2^{-k}\le2\mu_f(2r).$$

Lemma 4 ([5])**.** *If $\beta\in L$ and $B(\delta)=\varlimsup\limits_{x\to+\infty}\frac{\beta((1+\delta)x)}{\beta(x)}$, $\delta>0$, then in order that $\beta\in L^0$, it is necessary and sufficient that $B(\delta)\to1$ as $\delta\to+0$.*

Since series (2) regularly convergent in $\mathbb{C}$, for every $r \in [0, +\infty)$ and $\tau > 0$ we have

$$M_A(r) \le \sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) \le \mu_A((1+\tau)r) \sum_{n=1}^{\infty} \frac{M_f(r\lambda_n)}{M_f((1+\tau)r\lambda_n)}, \tag{6}$$

where $\mu_A(r) = \max\{|a_n| M_f(r\lambda_n) \colon n \ge 1\}$.

Then by Lemma 2 for $r \ge 1$ we have

$$\ln M_f((1+\tau)r\lambda_n) - \ln M_f(r\lambda_n) = \int\limits_{r\lambda_n}^{(1+\tau)r\lambda_n} \frac{d\ln M_f(x)}{d\ln x} d\ln x = \int\limits_{r\lambda_n}^{(1+\tau)r\lambda_n} \Gamma_f(x) d\ln x \ge$$
$$\ge \Gamma_f(r\lambda_n) \ln(1+\tau) \ge \Gamma_f(\lambda_n) \ln(1+\tau)$$

Therefore, if $\ln n \le q\Gamma_f(\lambda_n)$ for all $n \ge n_0$ and $\ln(1+\tau) > q$ then

$$\sum_{n=n_0}^{\infty} \frac{M_f(r\lambda_n)}{M_f((1+\tau)r\lambda_n)} \le \sum_{n=n_0}^{\infty} \exp\{-\Gamma_f(\lambda_n)\ln(1+\tau)\} \le \sum_{n=n_0}^{\infty} \exp\left\{-\frac{\ln(1+\tau)}{q} \ln n\right\} < +\infty$$

and (6) for $r \ge 1$ implies

$$M_A(r) \le T\mu_A((1+\tau)r), \quad T = \text{const} > 0. \tag{7}$$

Also we have

$$\mu_A(r) \le \max\left\{|a_n| \sum_{k=0}^{\infty} |f_k| (r\lambda_n)^k \colon n \ge 1\right\} \le \sum_{k=0}^{\infty} \max\{|a_n|\lambda_n^k \colon n \ge 1\} |f_k| r^k =$$
$$= \sum_{k=0}^{\infty} \mu_D(k) |f_k| r^k \le \mu_G(2r) \sum_{k=0}^{\infty} 2^{-k} = 2\mu_G(2r). \tag{8}$$

From (7) and (8) we get the estimate $M_A(r) \le 2T\mu_G(2(1+\tau)r)$ for $r \ge 1$ and, thus,

$$\ln \mu_G(r) \le \ln M_A(r) \le \ln \mu_G(2(1+\tau)r) + \ln(2T), \quad r \ge 1. \tag{9}$$

Now we can prove such a theorem.

Theorem 1. *Let f be an entire transcendental function, $a_n \ge 0$ for all $n \ge 1$ and series (2) regularly convergent in $\mathbb{C}$. Suppose that the functions α and β satisfy the conditions of Lemma 1, $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ and for each $c \in (0, +\infty)$*

$$\ln \lambda_n = o\left(\beta^{-1}\left(c\alpha\left(\frac{1}{\ln \lambda_n} \ln \frac{1}{a_n}\right)\right)\right), \quad n \to \infty. \tag{10}$$

Then $\varrho_{\alpha,\beta}[A] = \varrho_{\alpha,\beta}[f]$.

Proof. Since $\mu_D(\sigma) \to +\infty$ as $\sigma \to +\infty$, we have $\mu_D(k) \ge 1$ for $k \ge k_0$. For simplicity, we assume that $k_0 = 0$. Then $\mu_G(r) = \max\{|f_k|\mu_D(k)r^k \colon k \ge 0\} \ge \max\{|f_k|r^k \colon k \ge 0\} = \mu_f(r)$, whence in view of (9) and Lemma 3 it follows that $\varrho_{\alpha,\beta}[f] \le \varrho_{\alpha,\beta}[F]$.

On the other hand, in view of (9) $\varrho_{\alpha,\beta}[A] \le \varrho_{\alpha,\beta}[G]$. By Lemma 1

$$\varrho_{\alpha,\beta}[G] = \varlimsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left(\frac{1}{k}\ln\frac{1}{\mu_D(k)|f_k|}\right)} = \varlimsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left(\frac{1}{k}\ln\frac{1}{|f_k|} - \frac{\ln\mu_D(k)}{k}\right)}. \tag{11}$$

If $\varrho_{\alpha,\beta}[f] < +\infty$ then by Lemma 1 for every $\varrho > \varrho_{\alpha,\beta}[f]$ and all $k \ge k_0(\varrho)$ we have $\alpha(k) \le \varrho\beta(\frac{1}{k}\ln\frac{1}{|f_k|})$ and, thus,

$$\frac{1}{k}\ln\frac{1}{|f_k|} \ge \beta^{-1}\left(\frac{\alpha(k)}{\varrho}\right), \quad k \ge k_0(\varrho). \tag{12}$$

Let $\nu_D(\sigma) = \max\{n\colon |a_n|\exp\{\sigma\ln\lambda_n\} = \mu_D(\sigma)\}$ be the central index of series (4). Then ([6, p.17])

$$\ln\mu_D(\sigma) = \ln\mu_D(\sigma_0) + \int_{\sigma_0}^{\sigma} \ln\lambda_{\nu_D(x)}dx, \quad \sigma_0 \le \sigma. \tag{13}$$

From condition (10) with $c = 1/\varrho$ we get

$$\ln a_n \le -\ln\lambda_n\alpha^{-1}\left(\varrho\beta\left(\frac{\ln\lambda_n}{\varepsilon}\right)\right)$$

for each $\varepsilon > 0$ and all $n \ge n_0(\varepsilon)$. Therefore, for all $\sigma \ge \sigma_0 = \sigma_0(\varepsilon)$

$$\ln\mu_D(\sigma) = \ln a_{\nu_D(\sigma)} + \sigma\ln\lambda_{\nu_D(\sigma)} \le -\ln\lambda_{\nu_D(\sigma)}\alpha^{-1}\left(\varrho\beta\left(\frac{\ln\lambda_{\nu_D(\sigma)}}{\varepsilon}\right)\right) + \sigma\ln\lambda_{\nu_D(\sigma)} =$$
$$= \ln\lambda_{\nu_D(\sigma)}\left(\sigma - \alpha^{-1}\left(\varrho\beta\left(\frac{\ln\lambda_{\nu_D(\sigma)}}{\varepsilon}\right)\right)\right).$$

Since $\mu_D(\sigma) \to +\infty$ as $\sigma \to +\infty$, hence it follows that $\sigma - \alpha^{-1}(\varrho\beta(\frac{\ln\lambda_{\nu_D(\sigma)}}{\varepsilon})) \ge 0$, i. e. $\ln\lambda_{\nu_D(\sigma)} \le \varepsilon\beta^{-1}(\frac{\alpha(\sigma)}{\varrho})$ for $\sigma \ge \sigma_0$. Therefore, in view of (13)

$$\ln\mu_D(\sigma) \le \ln\mu_D(\sigma_0) + \varepsilon\int_{\sigma_0}^{\sigma}\beta^{-1}\left(\frac{\alpha(x)}{\varrho}\right)dx \le \ln\mu_D(\sigma_0) + \varepsilon\sigma\beta^{-1}\left(\frac{\alpha(\sigma)}{\varrho}\right)$$

and, thus,

$$\frac{\ln\mu_D(k)}{k} \le \varepsilon + \varepsilon\beta^{-1}\left(\frac{\alpha(k)}{\varrho}\right), \quad k \ge k_0(\varepsilon). \tag{14}$$

From (11), (12) and (14) we obtain

$$\varrho_{\alpha,\beta}[G] \le \varlimsup_{k\to+\infty}\frac{\alpha(k)}{\beta\left(\beta^{-1}\left(\frac{\alpha(k)}{\varrho}\right) - \varepsilon - \varepsilon\beta^{-1}\left(\frac{\alpha(k)}{\varrho}\right)\right)} = \varlimsup_{k\to+\infty}\frac{\alpha(k)}{\beta\left((1-\varepsilon)\beta^{-1}\left(\frac{\alpha(k)}{\varrho}\right)\right)} =$$
$$= \varlimsup_{k\to+\infty}\frac{\alpha(k)}{\beta\left(\beta^{-1}\left(\frac{\alpha(k)}{\varrho}\right)\right)}\frac{\beta\left(\beta^{-1}\left(\frac{\alpha(k)}{\varrho}\right)\right)}{\beta\left((1-\varepsilon)\beta^{-1}\left(\frac{\alpha(k)}{\varrho}\right)\right)} \le \varrho B(\varepsilon),$$

where by Lemma 4 $B(\varepsilon) = \varlimsup_{k\to+\infty}\frac{\beta(x)}{\beta((1-\varepsilon)x)} \to 1$ as $\epsilon \to 0$. Thus, $\varrho_{\alpha,\beta}[G] \le \varrho$ and since ϱ is arbitrary, we obtain the inequality $\varrho_{\alpha,\beta}[G] \le \varrho_{\alpha,\beta}[f]$ which is obvious when $\varrho_{\alpha,\beta}[f] = +\infty$. Finally, (9) implies the inequality $\varrho_{\alpha,\beta}[A] \le \varrho_{\alpha,\beta}[G] \le \varrho_{\alpha,\beta}[f]$. □

The functions $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ satisfy the conditions of Theorem 1. Therefore, Theorem 1 implies the following statement.

Corollary 1. *Let an entire transcendental function f have the order* $\varrho[f] := \varlimsup\limits_{r\to+\infty} \frac{\ln\ln M_f(r)}{\ln r} = = \varrho \in (0, +\infty)$ *and*

$$0 < \underline{\sigma}_f := \varliminf_{r\to+\infty} \frac{\ln M_f(r)}{r^\varrho} \le \overline{\sigma}_f := \varlimsup_{r\to+\infty} \frac{\ln M_f(r)}{r^\varrho} < +\infty. \tag{15}$$

Suppose that $a_n \ge 0$ for all $n \ge 1$ and series (2) regularly convergent in $\mathbb{C}$. If $\ln n = O(\lambda_n^\varrho)$ and $\ln \lambda_n = o(\ln\ln(1/a_n))$ as $n \to \infty$ then $\varrho[A] = \varrho[f]$.

Indeed, it is clear that

$$\ln M_f(r) = \ln M_f(r_0) + \int\limits_{r_0}^{r} \frac{\Gamma_f(t)}{t} dt, \quad 0 \le r_0 \le r < +\infty.$$

Therefore, if we put

$$\underline{\tau} = \varliminf_{r\to+\infty} \frac{\Gamma_f(r)}{r^\varrho}, \quad \overline{\tau} = \varlimsup_{r\to+\infty} \frac{\Gamma_f(r)}{r^\varrho}$$

then using results from [7] we get

$$\underline{\tau} \le \varrho\underline{\sigma} \le \underline{\tau}\left(1 + \ln\frac{\overline{\tau}}{\underline{\tau}}\right) \le \overline{\tau} \le e\varrho\overline{\sigma}.$$

Hence in view of (15) it follows that $\overline{\tau} < +\infty$ and $\underline{\tau} > 0$. Therefore, $\Gamma_f(r) \asymp r^\varrho$ as $r \to +\infty$ and, thus, the conditions $\ln n = O(\lambda_n^\varrho)$ and $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ are equivalent.

We remark also that condition (10) now looks like $\ln \lambda_n = o(\ln(\frac{1}{\ln\lambda_n} \ln \frac{1}{a_n}))$, i. e. $\ln \lambda_n = o(\ln\ln(1/a_n))$ as $n \to \infty$.

All conditions of Theorem 1 are satisfied and Theorem 1 implies Corollary 1.

3. Growth of Laplace-Stieltjes type integrals. Let V be the class of nonnegative nondecreasing unbounded continuous on the right functions F on $[0, +\infty)$. We assume that f is an entire transcendental function and $f_k \ge 0$ for all $k \ge 0$ and a positive on $[0, +\infty)$ function a is such that the Laplace-Stieltjes type integral

$$I(r) = \int\limits_0^\infty a(x) f(rx) dF(x) \tag{16}$$

exists for every $r \in [0, +\infty)$. The asymptotical behavior of such integrals in the case when $f(x) = e^x$ is studied in the monograph [8] (see also [9, 10, 11]), as well as for the case of positive functions f such that the function $\ln f$ is convex on $(0, +\infty)$ in [12].

Suppose that $x_0 > 1$ is such that $\int_1^{x_0} a(x) dF(x) \ge c > 0$. Then

$$I(r) \ge \int\limits_1^{x_0} a(x) f(rx) dF(x) \ge f(r)c. \tag{17}$$

On the other hand, as in the proof of Theorem 1 for $r \ge 1$ we have $\ln f((1+\tau)xr) - \ln f(rx) \ge \Gamma_f(x)\ln(1+\tau)$. Therefore, if $\mu_I(r) = \max\{a(x)f(rx)\colon x \ge 0\}$ is the maximum of the integrand, $\ln F(x) \le q\Gamma_f(x)$ and $\ln(1+\tau) > q$

$$I(r) = \int\limits_0^\infty a(x)f((1+\tau)rx)\frac{f(rx)}{f((1+\tau)rx)}dF(x) \le \mu_I((1+\tau)r)\int\limits_0^\infty \frac{f(rx)}{f((r+\tau)x)}dF(x) \le$$

$$\le \mu_I((1+\tau)r)\int\limits_0^\infty e^{-\Gamma_f(x)\ln(1+\tau)}dF(x) =$$

$$= \mu_I((1+\tau)r)\left(T_1 + \ln(1+\tau)\int\limits_0^\infty F(x)e^{-\Gamma_f(x)\ln(1+\tau)}d\Gamma_f(x)\right) \le$$

$$\le \mu_I((1+\tau)r)\left(T_1 + \ln(1+\tau)\int\limits_0^\infty e^{-\Gamma_f(x)((\ln(1+\tau)-q)}d\Gamma_f(x)\right) \le T_2\mu_I(r+\tau). \tag{18}$$

where $T_j = \text{const} > 0$. Also, as above, we have

$$\mu_I(r) = \max\left\{a(x)\sum_{k=0}^\infty f_k(xr)^k\colon x \ge 0\right\} \le$$

$$\le \sum_{k=0}^\infty \max\{a(x)x^k\colon x \ge 0\}f_k r^k = G_1(r) := \sum_{k=0}^\infty \mu_J(k)f_k r^k, \tag{19}$$

where $\mu_J(\sigma) = \max\{a(x)e^{\sigma\ln x}\colon x \ge 0\}$ is the maximum of the integrand for Laplace-Stieltjes integral

$$J(\sigma) = \int_0^\infty a(x)e^{\sigma\ln x}dF(x).$$

Now we prove the following analog of Theorem 1.

Theorem 2. *Let $F \in V$, f be an entire transcendental function and $f_k \ge 0$ for all $k \ge 0$. Suppose that $\ln F(x) \le q\Gamma_f(x)$ for some $q > 0$ and all $x \ge 0$, the functions α and β satisfy the conditions of Lemma 1 and for each $c \in (0, +\infty)$*

$$\ln x = o\left(\beta^{-1}\left(c\alpha\left(\frac{1}{\ln x}\ln\frac{1}{a(x)}\right)\right)\right), \quad x \to +\infty. \tag{20}$$

Then $\varrho_{\alpha,\beta}[I] = \varrho_{\alpha,\beta}[f]$.

Proof. From (17) it follows that $\varrho_{\alpha,\beta}[f] \le \varrho_{\alpha,\beta}[I]$.

On the other hand, in view of (18) and (19) $\varrho_{\alpha,\beta}[I] \le \varrho_{\alpha,\beta}[G_1]$. By Lemma 1

$$\varrho_{\alpha,\beta}[G_1] = \varlimsup_{k\to+\infty}\frac{\alpha(k)}{\beta\left(\frac{1}{k}\ln\frac{1}{|f_k|} - \frac{\ln\mu_J(k)}{k}\right)}. \tag{21}$$

If $\varrho_{\alpha,\beta}[f] < +\infty$ then as above we get (12).

As in [8, p.24], let $\nu_j(\sigma)$ be the central point of $\mu_J(\sigma)$. Then [8, p.26]

$$\ln \mu_J(\sigma) = \ln \mu_J(\sigma_0) + \int\limits_{\sigma_0}^{\sigma} \ln \nu_J(x)dx, \quad \sigma_0 \le \sigma. \tag{22}$$

From condition (20) with $c = 1/\varrho$ we get $\ln a(x) \le -\ln x\alpha^{-1}\left(\varrho\beta\left(\frac{\ln x}{\varepsilon}\right)\right)$ for each $\varepsilon > 0$ and all $x \ge x_0(\varepsilon)$. Therefore, as in the proof of Theorem 1, for all $\sigma \ge \sigma_0 = \sigma_0(\varepsilon)$ we have

$$\ln \mu_J(\sigma) \le \ln \nu_J(\sigma)\left(\sigma - \alpha^{-1}\left(\varrho\beta\left(\frac{\ln \nu_J(\sigma}{\varepsilon}\right)\right)\right),$$

whence it follows that $\ln \nu_J(\sigma) \le \varepsilon\beta^{-1}(\alpha(\sigma)/\varrho)$ for $\sigma \ge \sigma_0$. Therefore, in view of (22) $\ln \mu_J(\sigma) \le \ln \mu_J(\sigma_0) + \varepsilon\sigma\beta^{-1}(\alpha(\sigma)/\varrho)$ and, thus,

$$\frac{\ln \mu_D(k)}{k} \le \varepsilon + \varepsilon\beta^{-1}\left(\frac{\alpha(k)}{\varrho}\right), \quad k \ge k_0(\varepsilon). \tag{23}$$

From (21), (12) and (23) as in the proof of Theorem 1 we get $\varrho_{\alpha,\beta}[I] \le \varrho_{\alpha,\beta}[G_1] \le \varrho_{\alpha,\beta}[f]$. □

For the functions $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ Theorem 2 implies the following statement.

Corollary 2. *Let an entire transcendental function* (1) *with* $f_k \ge 0$ *satisfy condition* (15). *If* $\ln F(x) = O(x^\varrho)$ *and* $\ln x = o(\ln\ln(1/a(x))$ *as* $x \to +\infty$ *then* $\varrho[I] = \varrho[f]$.

4. Remarks. The conditions $\ln n = O(\lambda_n^\varrho)$ and $\ln \lambda_n = o(\ln\ln(1/a_n))$ as $n \to \infty$ in Corollary 1 and their analogues $\ln F(x) = O(x^\varrho)$ and $\ln x = o(\ln\ln(1/a(x))$ as $x \to +\infty$ in Corollary 2 are natural. Let us show this by the example of the function $A_\varrho(z) = \sum\limits_{n=1}^{\infty} a_n E_\varrho(z\lambda_n)$, where

$$E_\varrho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + k/\varrho)}, \; 0 < \varrho < +\infty,$$

is the Mittag-Leffler function. The properties of this function have been used in many problems in the theory of entire functions. We only need the following property of the Mittag-Leffler function: if $0 < \varrho < +\infty$ then [13, p.115]

$$M_{E_\varrho}(r) = E_\varrho(r) = (1 + o(1))\varrho e^{r^\varrho}, \quad r \to +\infty.$$

Hence it follows that $\varrho[E_\varrho] = \varrho$ and $\varrho[A_\varrho] = \varrho[A_\varrho^*]$, where $A_\varrho^*(r) = \sum_{n=1}^{\infty} a_n \exp\{r^\varrho\lambda_n^\varrho\}$. We put $r^\varrho = \sigma$ and $\lambda_n^\varrho = \mu_n$. Then $A_\varrho^*(r) = D_\varrho(\sigma) = \sum_{n=1}^{\infty} a_n e^{\sigma\mu_n}$ and $\varrho[A_\varrho^*] = \varrho\varrho_l[D_\varrho]$, where

$$\varrho_l[D_\varrho] = \varlimsup_{\sigma\to+\infty} \frac{\ln\ln D_\varrho(\sigma)}{\ln \sigma}$$

is the logarithmic order of Dirichlet series D_ϱ. It is known [14] that if $\ln n = O(\mu_n)$ as $n \to \infty$ then $\varrho_l[D_\varrho] = p_l + 1$, where

$$p_l = \varlimsup_{n\to+\infty} \frac{\ln \mu_n}{\ln\left(\frac{1}{\mu_n}\ln\frac{1}{a_n}\right)}.$$

Therefore, if $\ln n = O(\lambda^{\varrho})$ as $n\to\infty$ and $p_l = 0$ then $\varrho[A_{\varrho}] = \varrho = \varrho[E_{\varrho}]$. Finally, $p_l = 0$ if and only $\ln\mu_n = o(\ln\ln(1/a_n))$, i. e. $\ln\lambda_n = o(\ln\ln(1/a_n))$ as $n\to\infty$.

By a similar method, studying the growth of an integral $I_{\varrho}(r)(r) = \int_0^{\infty} a(x)E_{\varrho}(rx)dF(x)$ can be reduced to studying the growth of the integral $J(\sigma) = \int_0^{\infty} a_1(x)e^{x\sigma}dF_1(x)$ and then use the formula [8, p.83]

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\ln J(\sigma)}{\ln\sigma} = \varlimsup_{x\to+\infty}\frac{\ln x}{\ln\left(\frac{1}{x}\ln\frac{1}{a_1(x)}\right)} + 1,$$

provided

$$\varlimsup_{x\to+\infty}\frac{\ln\ln F_1(x)}{\ln x} \le 1.$$

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