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EXTENDED BALL CONVERGENCE FOR SEVENTH ORDER DERIVATIVE FREE CLASS OF ALGORITHMS FOR NONLINEAR EQUATIONS

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In the earlier work, expensive Taylor formula and conditions on derivatives up to the eighth order have been utilized to establish the convergence of a derivative free class of seventh order iterative algorithms. Moreover, no error distances or results on uniqueness of the solution were given. In this study, extended ball convergence analysis is derived for this class by imposing conditions on the first derivative. Additionally, we offer error distances and convergence radius together with the region of uniqueness for the solution. Therefore, we enlarge the practical utility of these algorithms. Also, convergence regions of a specific member of this class are displayed for solving complex polynomial equations. At the end, standard numerical applications are provided to illustrate the efficacy of our theoretical findings.

1. Introduction. Let us consider the equation

$$\mathscr{A}(w)=0, \tag{1}$$

where $\mathscr{A}\colon \mathscr{U}\subseteq \mathscr{X}_1\to \mathscr{X}_2$ is derivable according to Fréchet. The symbols $\mathscr{X}_1$, $\mathscr{X}_2$ denote Banach spaces and $\mathscr{U}$ stands for a non-empty, convex and open subset of $\mathscr{X}_1$. Solving nonlinear equations of the type (1) is a significant and difficult research topic in computational mathematics. This topic has a plethora of applications in scientific and technical disciplines. However, only in limited circumstances the solutions can be derived in closed form. Numerical approaches are thus the standard way to approximate the solutions. Newton's procedure is one of the earliest known algorithms for addressing nonlinear equations. When assuming the function is continuously differentiable and a decent starting estimate of solution is given, this algorithm is written as

$$w_{n+1}=w_n-\mathscr{A}'(w_n)^{-1}\mathscr{A}(w_n), \tag{2}$$

and it converges quadratically. The first derivative does not exist or is difficult to compute in many practical situations. A quadratically convergent Tarub's approach, free form derivative, is often used in such circumstances. This algorithm is expressed as follows.

$$w_{n+1}=w_n-[b_n,w_n;\mathscr{A}]^{-1}\mathscr{A}(w_n), \tag{3}$$

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where $[b_n, w_n; \mathscr{A}]^{-1}$ denotes the inverse of the first order divided difference $[b_n, w_n; \mathscr{A}]$ of $\mathscr{A}$ and $b_n = w_n + a\mathscr{A}(w_n)$, $a \neq 0$ is arbitrary.

Iterative approaches tend to converge to a solution at varying speed, thus researchers are continually working to improve the efficiency of iterative approaches. Increasing the order of convergence or minimizing the computational cost of an algorithm are both effective approaches to enhance its efficiency. Some of the higher order approaches that are already established may be found in literature [1, 2, 4, 6, 7, 8, 9, 12, 15, 19, 21, 26], along with the references therein. A derivative free fourth convergence order iterative algorithm is discussed in [16]. This algorithm needs computations of three values of $\mathscr{A}$, three divided differences and two inversions of matrix in each iteration. Grau-Sánchez et al. [14], by applying central divided differences to approximate derivatives, derived fourth and sixth convergence order algorithms. A sixth order family of derivative free algorithms is described in [24]. They used the generalizations of Steffenson-like algorithms [16] to derive specific cases of the family. A class of derivative free algorithms is provided by Wang and Zhang [27]. This family is seventh order convergent and its particular cases demand evaluations of four values of $\mathscr{A}$, five divided differences and three inversions of matrix in each iterations. Sharma and Arora [23] designed a derivative free seventh order algorithm that involves four values of $\mathscr{A}$, five divided differences and two inversions matrix. Another seveth order algorithm independent of derivative is constructed by Wang et al. [28], which needs evaluations of one inversion of matrix, five values of $\mathscr{A}$ and three divided differences in each iteration. More results on iterative schemes and their convergence can be found in [3, 5, 11, 10, 13, 17, 22, 25].

We discuss extended ball convergence of a seventh convergence order derivative free class of iterative algorithms proposed by Narang et al. [18]. This class of algorithms is expressed as follows

$$\begin{aligned} y_n &= w_n - A_n^{-1}\mathscr{A}(w_n), \\ z_n &= y_n - A_n^{-1}\mathscr{A}(y_n), \\ w_{n+1} &= z_n - \left(\frac{17}{4}I + A_n^{-1}B_n\left(-\frac{27}{4}I + A_n^{-1}B_n\left(\frac{19}{4}I - \frac{5}{4}A_n^{-1}B_n\right)\right)\right)A_n^{-1}\mathscr{A}(z_n), \end{aligned} \tag{4}$$

where $A_n = [b_n, s_n; \mathscr{A}]$, $B_n = [u_n, v_n; \mathscr{A}]$,

$$\begin{aligned} b_n = w_n + \alpha\mathscr{A}(w_n), s_n = w_n + \beta\mathscr{A}(w_n), \\ u_n = z_n + \gamma\mathscr{A}(z_n), v_n = z_n + \delta\mathscr{A}(z_n), \end{aligned}$$

α, β, γ, δ arbitrary and $[., ; \mathscr{A}] : \mathscr{U} \times \mathscr{U} \to L(\mathscr{X}_1, \mathscr{X}_1)$. Expensive Taylor formula and assumptions on derivative of the eighth order were utilized in [18] to obtain its seventh rate of convergence. Because of such convergence technique, the scope of application of this class is limited. In order to support our claim, we introduce the function

$$\mathscr{A}(w) = \begin{cases} w^3\ln(w^2) + w^5 - w^4, & \text{if} \quad w \neq 0, \\ 0, & \text{if} \quad w = 0. \end{cases} \tag{5}$$

where $\mathscr{X}_1 = \mathscr{X}_2 = \mathbb{R}$ and $\mathscr{A}$ is defined on $\mathscr{U} = [-\frac{1}{2}, \frac{3}{2}]$. Then, since $\mathscr{A}'''$ is not bounded, the existing convergence theorem [18] does not apply to this example. Additionally, no statements on error $\|w_n - w_*\|$, the convergence ball and accurate position of the solution w_* are discussed in [18]. Analyzing the convergence ball of an iterative algorithm is very beneficial for many different purposes including determining the radii of convergence balls, bounds on error $\|w_n - w_*\|$ and area of uniqueness for the solution w_*. Notably, the consequences of ball convergence are highly valuable since they offer insight on the complicated problem of picking initial choices. This leads us to analyze the convergence ball of algorithm (4) by imposing conditions on just the first derivative of $\mathscr{A}$. By using our work, one is able to compute the convergence radii and the estimations on error $\|w_n - w_*\|$. Our analysis also offers a precise position of the solution w_*.

The arrangement of the whole text can be summarized as: Section 1 serves as the introduction. For algorithm (4), the analysis of convergence ball is discussed in Section 2. Section 3 describes attraction basins for this algorithm. The results of numerical investigations are shown in Section 4.

2. Convergence. We rely on some scalar parameters and functions to show the local convergence analysis of method (4). Set $M = [0, \infty)$. Consider parameters a, b, c, $d \geq 0$ and p, $q > 0$.

We assume the following properties of functions:

(1) $\mu_0(t) - 1$ has a smallest root $\rho_0 \in M \setminus \{0\}$ for some function $\mu_0 \colon M \to M$, which is continuous and non-decreasing. Set $M_0 = [0, \rho_0)$.

(2) $\lambda_1(t) - 1$ has a smallest root $R_1 \in M_0 \setminus \{0\}$ for some function $\mu \colon [0, 2\rho_0) \to M$, which is continuous and non-decreasing and function $\lambda_1 \colon M_0 \to M$ is defined by

$$\lambda_1(t) = \frac{\mu(t)}{1 - \mu_0(t)}.$$

(3) $\lambda_2(t) - 1$ has a smallest root $R_2 \in M_0 \setminus \{0\}$ for some function $\mu_1 \colon M_0 \to M$, which is continuous and non-decreasing and function $\lambda_2 \colon M_0 \to M$ is defined by

$$\lambda_2(t) = \frac{\mu_1(t)\lambda_1(t)}{1 - \mu_0(t)}.$$

(4) $\lambda_3(t) - 1$ has a smallest root $R_3 \in M_0 \setminus \{0\}$ for some function $\mu_2 \colon M_0 \to M$, $\mu_3 : M_0 \to M$, which is continuous and non-decreasing and functions $h \colon M_0 \to M$, $\lambda_3 \colon M_0 \to M$ are defined by

$$h(t) = \frac{\mu_3(t)}{1 - \mu_0(t)}$$

and

$$\lambda_3(t) = \left[\frac{\mu_2(t)}{1 - \mu_0(t)} + \frac{p}{4}(5h^2(t) + 4h(t) + 4)\frac{h(t)}{1 - \mu_0(t)}\right]\lambda_2(t).$$

The parameter R defined by

$$R = \min\{R_m\}, \quad m = 1, 2, 3 \tag{6}$$

shall be shown to be a convergence radius for method (4). Set $M_1 = [0, R)$.

By the definition of method (4) we have that for all $t \in M_1$

$$0 \leq \mu_0(t) < 1, \tag{7}$$

$$0 \leq h(t) < 1 \tag{8}$$

and

$$0 \le \lambda_m(t) < 1, \quad m = 1, 2, 3. \tag{9}$$

By $S[w_*, \rho]$ we denote the closure of the open ball $S(w_*, \rho)$ having center $w_* \in \mathscr{X}_1$ and of radius $\rho > 0$. $L(\mathscr{X}_1, \mathscr{X}_2)$ stands for the set $\{\mathscr{Y} : \mathscr{X}_1 \to \mathscr{X}_2 \text{ is bounded and linear}\}$.

We assume from now on that w_* is a simple solution of equation $\mathscr{A}(w) = 0$, divided difference $[.,.;\mathscr{A}] \to L(\mathscr{X}_1, \mathscr{X}_1)$ exists and the functions λ_m are as defined previously. Moreover, the following hypotheses (H) are used.

(h_1) For each $w \in \mathscr{U}$

$$\|\mathscr{A}'(w_*)^{-1}([w + \alpha\mathscr{A}(w), w + \beta\mathscr{A}(w); \mathscr{A}] - \mathscr{A}'(w_*))\| \le \mu_0(\|w - w_*\|),$$
$$\|I + \alpha[w, w_*; \mathscr{A}]\| \le a$$

and

$$\|I + \beta[w, w_*; \mathscr{A}]\| \le b.$$

Set $\mathscr{U}_0 = S[w_*, \rho_0] \cap \mathscr{U}$.

(h_2) For each w, y, $z \in \mathscr{U}_0$

$$\|\mathscr{A}'(w_*)^{-1}([w + \alpha\mathscr{A}(w), w + \beta\mathscr{A}(w); \mathscr{A}] - [w, w_*; \mathscr{A}])\| \le \mu(\|w - w_*\|),$$
$$\|\mathscr{A}'(w_*)^{-1}([w + \alpha\mathscr{A}(w), w + \beta\mathscr{A}(w); \mathscr{A}] - [y, w_*; \mathscr{A}])\| \le \mu_1(\|w - w_*\|),$$
$$\|\mathscr{A}'(w_*)^{-1}([w + \alpha\mathscr{A}(w), w + \beta\mathscr{A}(w); \mathscr{A}] - [z, w_*; \mathscr{A}])\| \le \mu_2(\|w - w_*\|),$$
$$\|\mathscr{A}'(w_*)^{-1}([z + \gamma\mathscr{A}(z), z + \delta\mathscr{A}(z); \mathscr{A}] - [w + \alpha\mathscr{A}(w), w + \beta\mathscr{A}(w); \mathscr{A}])\| \le \mu_3(\|w - w_*\|),$$
$$\|I + \gamma[w, w_*; \mathscr{A}]\| \le c, \quad \|I + \delta[w, w_*; \mathscr{A}]\| \le d, \quad \|\mathscr{A}'(w_*)^{-1}[w, w_*; \mathscr{A}]\| \le p$$

and $\|[w, w_*; \mathscr{A}]\| \le q$.

(h_3) $S[w_*, R_*] \subset \mathscr{U}$, where $R_* = \max\{aR, bR, cR, dR, R\}$.

Next, the hypotheses (H) are used to show the main ball convergence result for method (4).

Theorem 1. *Under hypotheses (H) further suppose $w_0 \in S(w_*, R) \setminus \{w_*\}$. Then, the sequence $\{w_n\}$ starting from w_0 converges to w_*.*

Proof. Mathematical induction is used to show assertions

$$\{w_n\} \subset S(w_*, R), \tag{10}$$
$$\|y_n - w_*\| \le \lambda_1(\|w_n - w_*\|)\|w_n - w_*\| \le \|w_n - w_*\| < R, \tag{11}$$
$$\|z_n - w_*\| \le \lambda_2(\|w_n - w_*\|)\|w_n - w_*\| \le \|w_n - w_*\|, \tag{12}$$
$$\|w_{n+1} - w_*\| \le \lambda_3(\|w_n - w_*\|)\|w_n - w_*\| \le \|w_n - w_*\| \tag{13}$$

and

$$\lim_{n\to\infty} w_n = w_*, \tag{14}$$

with the radius R defined in (6) and functions λ_m as given before.

By (6), (7) and (h_1), we have for $w \in S(w_*, R) \setminus \{w_*\}$

$$\|w + \alpha\mathscr{A}(w) - w_*\| = \|(I + \alpha[w, w_*; \mathscr{A}])(w - w_*)\| \le$$
$$\le \|I + \alpha[w, w_*; \mathscr{A}]\| \; \|w - w_*\| \le aR \le R_*, \tag{15}$$
$$\|w + \beta\mathscr{A}(w) - w_*\| = \|(I + \beta[w, w_*; \mathscr{A}])(w - w_*)\| \le bR < R_* \tag{16}$$

and

$$\|\mathscr{A}'(w_*)^{-1}([w+\alpha\mathscr{A}(w), w+\beta\mathscr{A}(w);\mathscr{A}] - \mathscr{A}'(w_*))\| \le \mu_0(\|w-w_*\|) \le \mu_0(R) < 1. \quad (17)$$

It follows from (17) and a lemma attributed to Banach on invertible linear operators [3, 20] that $A_0^{-1} \in L(\mathscr{X}_1, \mathscr{X}_1)$ for $w = w_0 \in S(w_*, R) \setminus \{w_*\}$ and

$$\|A_0^{-1}\mathscr{A}'(w_*)\| \le \frac{1}{1-\mu_0(\|w_0-w_*\|)}. \quad (18)$$

Hence, iterates y_0, z_0, w_1 exists.

Using the first substep of method (4) we get in turn

$$y_0 - w_* = w_0 - w_* - A_0^{-1}\mathscr{A}(w_0) = A_0^{-1}(A_0 - [w_0, w_*;\mathscr{A}])(w_0 - w_*). \quad (19)$$

In view of (6), (9)(for $m = 1$), (18), (19), and (h_2) we have in turn

$$\|y_0 - w_*\| \le \frac{\mu(\|w_0-w_*\|)\|w_0-w_*\|}{1-\mu_0(\|w_0-w_*\|)} \le \lambda_1(\|w_0-w_*\|)\|w_0-w_*\| \le \|w_0-w_*\| < R, \quad (20)$$

showing $y_0 \in S(w_*, R)$ and (11) for $n = 0$. Similarly but using (9)(for $m = 2$) and the second substep of method (4), we obtain in turn that

$$z_0 - w_* = y_0 - w_* - A_0^{-1}\mathscr{A}(y_0) = A_0^{-1}(A_0 - [y_0, w_*;\mathscr{A}])(y_0 - w_*),$$

so

$$\|z_0 - w_*\| \le \frac{\mu_1(\|w_0-w_*\|)\|y_0-w_*\|}{1-\mu_0(\|w_0-w_*\|)} \le \lambda_2(\|w_0-w_*\|)\|w_0-w_*\| \le \|w_0-w_*\| < R, \quad (21)$$

showing $z_0 \in S(w_*, R)$ and (12) for $n = 0$. Moreover using the third substep of method (4) we can also write in turn

$$\begin{aligned} w_1 - w_* &= z_0 - w_* - A_0^{-1}\mathscr{A}(z_0) + \\ &+ \frac{1}{4}(5(A_0^{-1}B_0 - I)^2 - 4(A_0^{-1}B_0 - I) + 4)(A_0^{-1}B_0 - I)A_0^{-1}\mathscr{A}(z_0). \end{aligned} \quad (22)$$

Then, as with the previous estimates but using (9)(for $m = 3$), we get in turn that

$$\begin{aligned} \|w_1 - w_*\| \le \Bigg[&\frac{\mu_2(\|w_0-w_*\|)}{1-\mu_0(\|w_0-w_*\|)} + \\ &+\frac{p}{4}(5h^2(\|w_0-w_*\|) + 4h(\|w_0-w_*\|) + 4)\frac{h(\|w_0-w_*\|)}{1-\mu_0(\|w_0-w_*\|)}\Bigg]\|z_0-w_*\| \le \\ &\le \lambda_3(\|w_0-w_*\|)\|w_0-w_*\| \le \|w_0-w_*\|, \end{aligned} \quad (23)$$

showing $w_1 \in S(w_*, R)$ and (13) for $n = 0$. We also used the estimates

$$\|\mathscr{A}'(w_*)^{-1}([w_0+\alpha\mathscr{A}(w_0), w_0+\beta\mathscr{A}(w_0);\mathscr{A}] - [z_0, w_*;\mathscr{A}])\| \le \mu_2(\|w_0-w_*\|)$$

and

$$\begin{aligned} \|A_0^{-1}B_0 - I\| &\le \|A_0^{-1}\mathscr{A}'(w_*)\| \; \|\mathscr{A}'(w_*)^{-1}(B_0 - A_0)\| \le \\ &\le \frac{\mu_3(\|w_0-w_*\|)}{1-\mu_0(\|w_0-w_*\|)} = h(\|w_0-w_*\|). \end{aligned}$$

Hence, assertions (10)–(13) are shown for $n = 0$. If we simply replace w_0, y_0, z_0, w_1 by w_n, y_n, z_n, w_{n+1} in the previous calculations, we complete the induction for (10)–(11). Then, in view of the estimation

$$\|w_{n+1} - w_*\| \leq r\|w_n - w_*\| < R, \tag{24}$$

where $r = \lambda_3(\|w_0 - w_*\|)$ is in [0, 1), we get $w_{n+1} \in S(w_*, R)$ and $\lim_{n\to\infty} w_n = w_*$. □

Next, connecting the uniqueness of the solution we give a result not necessarily relying on the hypotheses (H).

Proposition 1. *Suppose that the equation $\mathscr{A}(w) = 0$ has a simple solution $w_* \in \mathscr{U}$, for all $w \in \mathscr{U}$*

$$\|\mathscr{A}'(w_*)^{-1}([w, w_*; \mathscr{A}] - \mathscr{A}'(w_*))\| \leq \mu_4(\|w - w_*\|) \tag{25}$$

and the function $\mu_4(t) - 1$ has the smallest positive root $\overline{\rho}$, where $\mu_4 \colon M \to M$ is a continuous and non-decreasing function. Set $\mathscr{U}_1 = S[w_*, \tilde{\rho}] \cap \mathscr{U}$ for $0 < \tilde{\rho} < \overline{\rho}$. Then, the only solution of equation $\mathscr{A}(w) = 0$ in the region $\mathscr{U}_1$ is w_*.

Proof. Set $T = [w^*, w_*; \mathscr{A}]$ for some $w^* \in \mathscr{U}_1$ with $\mathscr{A}(w^*) = 0$. Then, using (25), we get

$$\|\mathscr{A}'(w_*)^{-1}(T - \mathscr{A}'(w_*))\| \leq \mu_4(\|w^* - w_*\|) \leq \mu_4(\tilde{\rho}) < 1,$$

so, $T^{-1} \in L(\mathscr{X}_1, \mathscr{X}_1)$ and $w^* = w_*$ follows from $T(w^* - w_*) = \mathscr{A}(w^*) - \mathscr{A}(w_*)$. □

Remark. Let us consider choices

$$[w, y; \mathscr{A}] = \frac{1}{2}(\mathscr{A}'(w) + \mathscr{A}'(y)) \text{ or } [w, y; \mathscr{A}] = \int_0^1 \mathscr{A}'(w + \theta(y - w))\, d\theta$$

or the standard definition of the divided difference when $\mathscr{X}_1 = \mathbb{R}^i$ [18].
Moreover, suppose

$$\|\mathscr{A}'(w_*)^{-1}(\mathscr{A}'(w) - \mathscr{A}'(w_*))\| \leq \psi_0(\|w - w_*\|)$$

and

$$\|\mathscr{A}'(w_*)^{-1}(\mathscr{A}'(w) - \mathscr{A}'(y))\| \leq \psi(\|w - y\|),$$

where functions $\psi_0 : M \to M$, $\psi : M \to M$, are continuous and non-decreasing. Then, under the first or second choice above it can easily be seen that these hypotheses (H) require

$$\mu_0(t) = \frac{1}{2}(\psi_0(at) + \psi_0(bt)), \quad \mu(t) = \frac{1}{2}(\psi(|\alpha|qt) + \psi_0(bt)),$$
$$\mu_1(t) = \frac{1}{2}(\psi(at + \lambda_1(t)t) + \psi_0(bt)), \quad \mu_2(t) = \frac{1}{2}(\psi(at + \lambda_2(t)t) + \psi_0(bt))$$

and

$$\mu_3(t) = \frac{1}{2}\psi(c\lambda_2(t)t + at).$$

3. Attraction basins. We present the convergence regions of scheme (4) for obtaining solutions of several complex polynomial equations. The region $\mathscr{T} = [-2, 2] \times [-2, 2]$ on $\mathbb{C}$ is used with a grid of 200×200 points on $\mathscr{T}$. Suppose $\{z_i\}_{i=0}^{\infty}$ is constructed by algorithm (4) starting with $z_0 \in \mathbb{C}$. The set $\{z_0 \in \mathbb{C} : z_i \to z_* \textit{ as } i \to \infty\}$ serves as the attraction basin of a zero z_* of $\mathscr{N}(z)$, where $\mathscr{N}$ stands for a complex polynomial of degree higher than or equal to two. In order to produce convergence regions via attraction basins, each point $z_0 \in \mathscr{T}$ is taken as a starting choice and algorithm (4) is applied on ten complex functions. The stater z_0 remains in the basin of a zero z_* of a test polynomial if $\lim_{i\to\infty} z_i = z_*$. Then, z_0 is painted with a fixed color corresponding to z_*. In accordance with the iteration numbers, we apply light to dark colors for each starting choice z_0. The point $z_0 \in \mathscr{T}$ is displayed in black if it is not in the attraction basin of any zero of the test polynomial. We consider MATLAB 2019a for producing the fractal pictures. We set the tolerance $\|z_i - z_*\| < 10^{-6}$ to end the iteration. Otherwise, iteration procedure is executed up to 100 times.

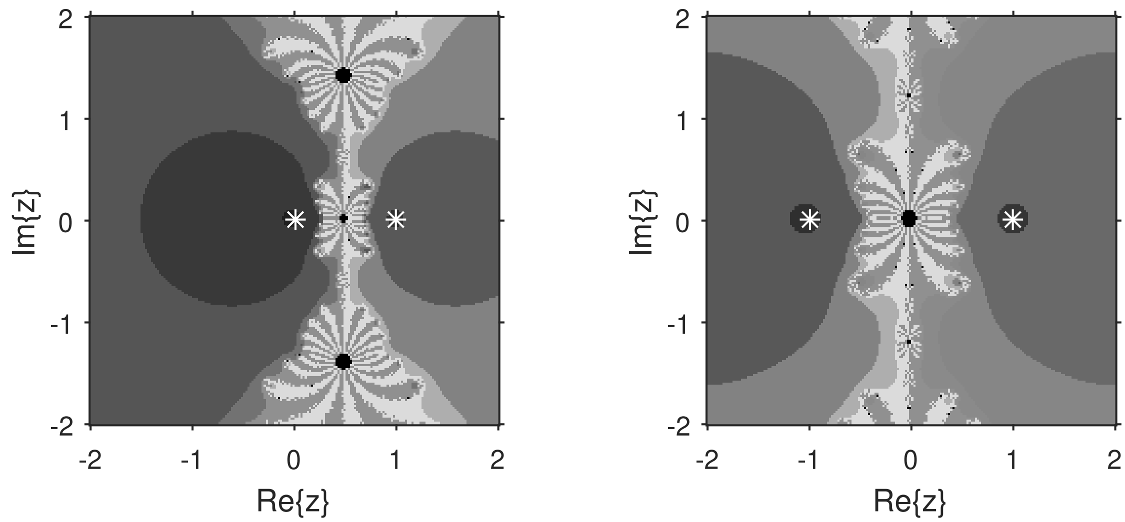
In the beginning, we choose the quadratic complex polynomials $\mathscr{N}_1(z) = z^2 - z$ and $\mathscr{N}_2(z) = z^2 - 1$ to create the attraction basins for (4). Fig. 1(a) shows the attraction basins related to the zeros 0 and 1 of $\mathscr{N}_1(z)$ in magenta and green colors, respectively. In Fig. 1(b), pink and green colors indicate the attraction basins of the zeros 1 and -1, respectively, of $\mathscr{N}_2(z)$. Further, we consider the complex polynomials $\mathscr{N}_3(z) = z^3 - z$ and $\mathscr{N}_4(z) = z^3 - 1$ of degree three. In Fig. 2(a), the attraction basins of the zeros 1, 0 and -1 of the polynomial $\mathscr{N}_3(z)$ are demonstrated in magenta, yellow and cyan colors, respectively. Fig. 2(b) represents the attraction basins associated to the zeros $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and 1 of $\mathscr{N}_4(z)$ in pink, blue and green, respectively. Next, the complex polynomials $\mathscr{N}_5(z) = z^4 - z$ and $\mathscr{N}_6(z) = z^4 - 1$ of degree four are taken to illustrate the attraction basins associated with their zeros. In Fig. 3(a), convergence to the zeros 0, $-\frac{1}{2} + 0.866025i$, 1 and $-\frac{1}{2} - 0.866025i$ of the polynomial $\mathscr{N}_5(z)$ is presented in green, red, blue and yellow, respectively. In Fig. 3(b), the basins of the solutions i, -1, $-i$ and 1 of $\mathscr{N}_6(z) = 0$ are respectively displayed in pink, blue, green and yellow zones. Furthermore, we choose polynomials $\mathscr{N}_7(z) = z^5 - z$ and $\mathscr{N}_8(z) = z^5 - 1$ of degree five. Fig. 4(a) displays the attraction basins related to the solutions 0, i, -1, 1 and $-i$ of $\mathscr{N}_7(z) = 0$ in green, magenta, red, blue and yellow colors, respectively. In Fig. 4(b), pink, yellow, red, cyan and green colors indicate the attraction basins of the zeros $-0.809016 + 0.587785i$, $0.309016 - 0.951056i$, $0.309016 + 0.951056i$, 1 and $-0.809016 - 0.587785i$, respectively, of $\mathscr{N}_8(z)$. At the end, two complex polynomials $\mathscr{N}_9(z) = z^6 - z$ and $\mathscr{N}_{10}(z) = z^6 - 1$ degree six are considered. Fig. 5(a) presents the attraction basins associated to the roots 1, $0.3090169 + 0.951056i$, 0, $0.3090169 - 0.951056i$, $-0.809016 + 0.587785i$ and $-0.809016 - 0.587785i$ of $\mathscr{N}_9(z) = 0$ in green, yellow, red, cyan, magenta and blue colors, respectively. In Fig. 5(b), the basins of the solutions $0.500000 - 0.866025i$, 1, $0.500000 + 0.866025i$, $-0.500000 - 0.866025i$, -1 and $-0.500000 + 0.866025i$ of $\mathscr{N}_{10}(z) = 0$ are painted in yellow, red, cyan, pink, green and blue, respectively.

4. Numerical examples. The convergence radius of the derivative free class of iterative algorithms (4) is produced by utilizing the presented analysis.

Example 1. Let $\mathscr{X}_1 = \mathscr{X}_2 = \mathbb{R}^3$ and $\mathscr{U} = S[0,1]$. Consider $\mathscr{A}$ on $\mathscr{U}$ for $w = (w_1, w_2, w_3)^t$ as

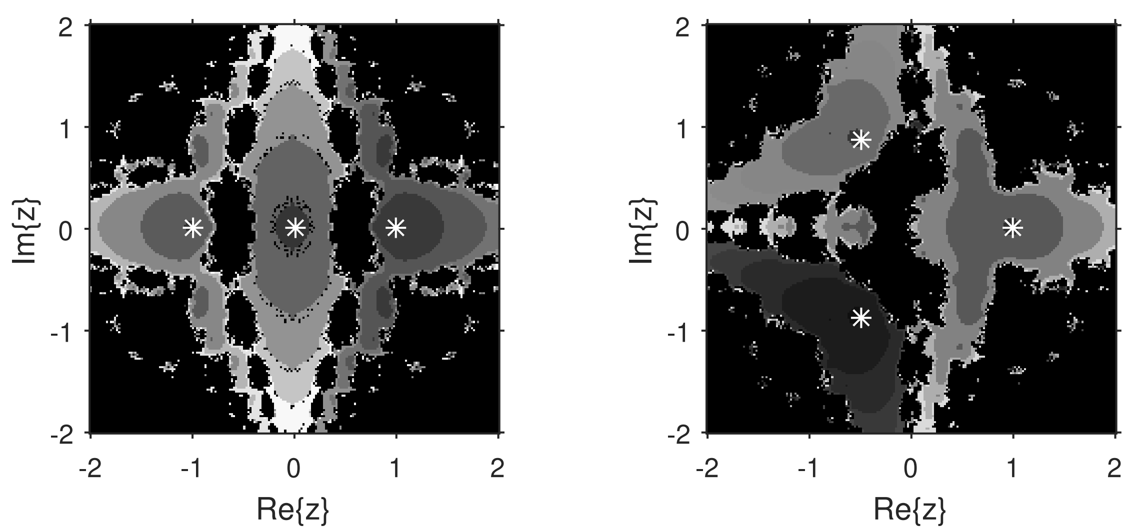
$$\mathscr{A}(w) = \Big(e^{w_1} - 1, \frac{e-1}{2} w_2^2 + w_2, w_3\Big)^t.$$

We have $w_* = (0,0,0)^t$. Also, $\alpha = \gamma = 1$, $\beta = \delta = -1$ $\psi_0(t) = (e-1)t$, $\psi_1(t) = e^{\frac{1}{e-1}} t$, $a = c = \frac{1}{2}(3 + e^{\frac{1}{e-1}})$ and $b = d = p = q = \frac{1}{2}(1 + e^{\frac{1}{e-1}})$. Using Theorem 1 the value of R is determined and presented in Table 1.



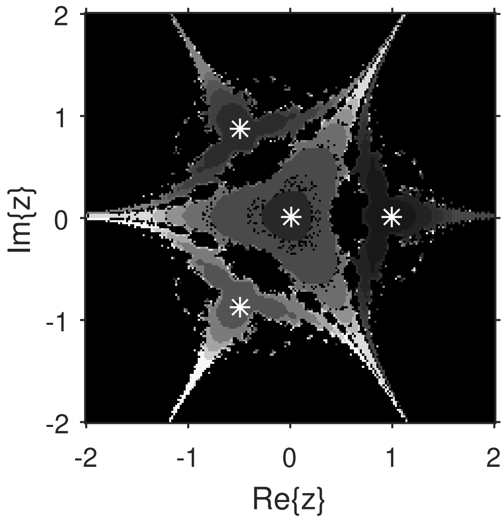
(a) $\mathscr{N}_1(z) = z^2 - z$ (б) $\mathscr{N}_2(z) = z^2 - 1$

Fig. 1: Attraction basins for (4) related to degree two complex polynomials

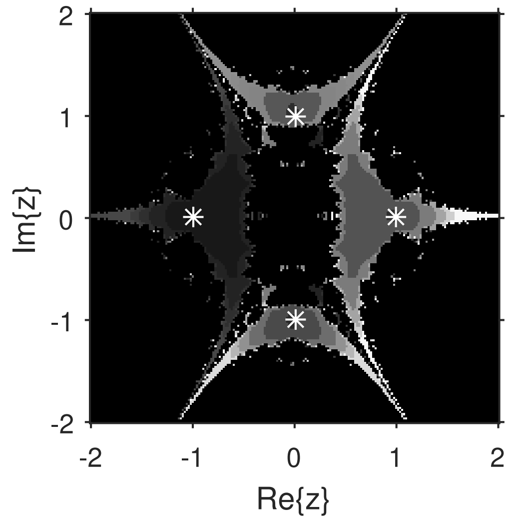


(a) $\mathscr{N}_3(z) = z^3 - z$ (б) $\mathscr{N}_4(z) = z^3 - 1$

Fig. 2: Attraction basins for (4) related to degree three complex polynomials

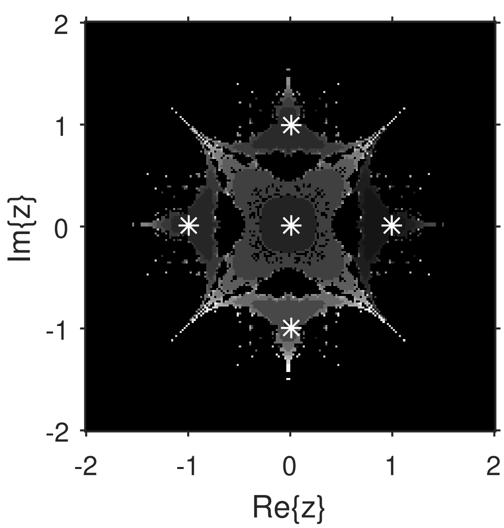


(a) $\mathscr{N}_5(z) = z^4 - z$

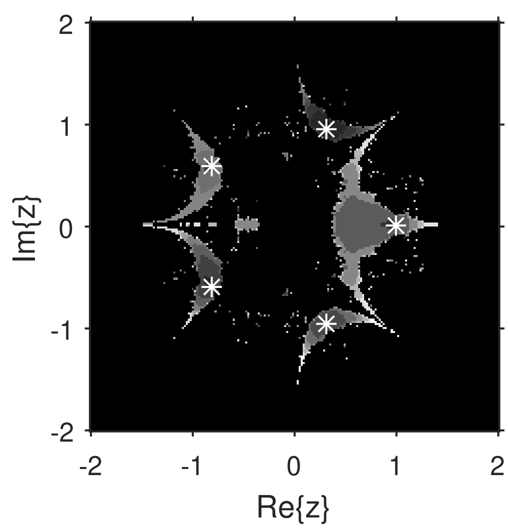


(б) $\mathscr{N}_6(z) = z^4 - 1$

Fig. 3: Attraction basins for (4) related to degree four complex polynomials

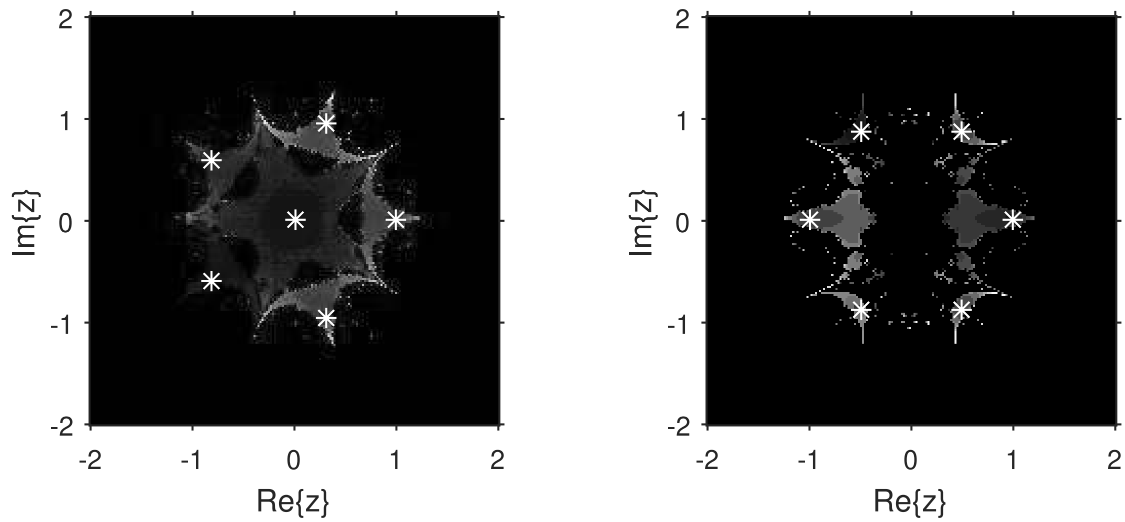


(a) $\mathscr{N}_7(z) = z^5 - z$



(б) $\mathscr{N}_8(z) = z^5 - 1$

Fig. 4: Attraction basins for (4) related to degree five complex polynomials



(a) $\mathscr{N}_9(z) = z^6 - z$ (б) $\mathscr{N}_{10}(z) = z^6 - 1$

Fig. 5: Attraction basins for (4) related to degree six complex polynomials

Table 1: Convergence radii for Example 1

Method (4)
$R_1 = 0.175373$
$R_2 = 0.156284$
$R_3 = 0.109914$
$R = 0.109914$

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