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ON THE DISTRIBUTION OF UNIQUE RANGE SETS AND ITS ELEMENTS OVER THE EXTENDED COMPLEX PLANE

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In the paper, we discussed the distribution of unique range sets and its elements over the extended complex plane from a different point of view and obtained some new results regarding the structure and position of unique range sets. These new results have immense applications like classifying different subsets of $\mathbb{C}$ to be or not to be a unique range set, exploring the fact that every bi-linear transformation preserves unique range sets for meromorphic functions, providing simpler and shorter proofs of existence of some unique range sets, unfolding the fact that zeros or poles of any meromorphic function lie in a unique range set, in particular, identifying the Fundamental Theorem of Algebra to a more specific region and many more applications. We have also posed some open questions to unveil the mysterious arrangement of the elements of unique range sets.

1. Introduction. Throughout the paper by any meromorphic function we mean it is defined in $\mathbb{C}$ and non-constant unless otherwise stated. By $\overline{\mathbb{C}}$, $\mathbb{R}$ and $\mathbb{R}^+$ we mean the extended complex plane, set of all real numbers and set of all positive real numbers, respectively.

For a meromorphic function f and $a \in \mathbb{C}$ we define

$$E_f(a) = \{(z,p) \in \mathbb{C} \times \mathbb{N} : f(z) = a \text{ with multiplicity } p\}, \quad \overline{E}_f(a) = \{z : f(z) = a\}.$$

For $a = \infty$ we define $E_f(\infty) = E_{\frac{1}{f}}(0)$ and $\overline{E}_f(\infty) = \overline{E}_{\frac{1}{f}}(0)$.

Let $S \subseteq \overline{\mathbb{C}}$. Then we define

$$E_f(S) = \bigcup_{a \in S} E_f(a), \quad \overline{E}_f(S) = \bigcup_{a \in S} \overline{E}_f(a).$$

For two arbitrary meromorphic functions f, g and a set $S \subseteq \overline{\mathbb{C}}$ if $E_f(S) = E_g(S)$, then we say f, g *share the set S counting multiplicities* or CM in brief. If $\overline{E}_f(S) = \overline{E}_g(S)$, then we say f, g *share the set S ignoring multiplicities* or IM in brief.

Moreover, if $E_f(S) = E_g(S)$ implies $f \equiv g$, then S is called a *unique range set for meromorphic functions* [5,9,15] or URSM in short and if $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$, then S is called a *unique range set for meromorphic functions ignoring multiplicity* or URSM-IM in short. By $\mathcal{S}$ and $\mathcal{S}_I$ we denote the class of unique range sets for meromorphic functions and unique range set for meromorphic functions ignoring multiplicity, respectively. For example, $S \in \mathcal{S}$ means that S is a unique range set for meromorphic functions.

The interest to this topic come from the paper of F. Gross [7] and F. Gross, C. C. Yang [8]. There was formulated the well-known *Gross problem*: Can one find two (or possibly

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even one) finite sets S_j ($j = 1, 2$) such that any two entire functions f and g satisfying $f^{-1}(S_j) = g^{-1}(S_j)$ for $j = 1$, 2 must be identical? If "yes", then how small can they be?

Throughout the last four decades a lot of research in this direction have been done by various authors. Many mathematicians investigated various partial cases of the problem [1,2, 4,6,8] because the final answer to the problem is quite far away. One of the approaches uses the notion of unique range set [3,10,11,16] to consider the Gross problem for meromorphic functions. Recently, the author [12,13] used the notion to study a narrow formulation of the Gross problem for powers of meromorphic functions.

In due course of time, the research in this area has been splitted in two directions. One is to find out the smallest possible $S \in \mathcal{S}$ ($S \in \mathcal{S}_I$) and the other is to find out the characterization of $\mathcal{S}$ ($\mathcal{S}_I$). But in both the cases the authors always took finite $S \in \mathcal{S}$ or $S \in \mathcal{S}_I$ into consideration. More precisely, they always considered some specific polynomials whose zero sets form a set $S \in \mathcal{S}$ ($S \in \mathcal{S}_I$) to obtain the best possible answers in both the above directions. Since from the very definition of a set $S \in \mathcal{S}$ ($S \in \mathcal{S}_I$) it is a subset of $\overline{\mathbb{C}}$, so it is natural that ∞ may belong to S. But till date we have no such example or theory. So natural query arises whether there exist any sets $S \in \mathcal{S}$ ($S \in \mathcal{S}_I$) containing ∞ or not. In this paper, we unveil this fact in affirmative.

The present paper attempts to provide a qualitative description of unique range sets, to find out their geometric nature and their properties using the Möbius transformation.

Not only that, since every $S \in \mathcal{S}$ ($S \in \mathcal{S}_I$) is a subset of $\overline{\mathbb{C}}$, so these sets may or may not be infinite. And if there exists $S \in \mathcal{S}$ ($S \in \mathcal{S}_I$) containing infinite number of elements, then what can be said about the characteristics of those sets. In this paper, we would like to explore the characteristics of any kind of $S \in \mathcal{S}$ ($S \in \mathcal{S}_I$) irrespective of it's finiteness or infiniteness.

We have also discussed some applications of our results which shows that applying our results one can have very simple and short proofs of existence of some URSM's (URSM-IM's).

Furthermore, we have also posed different natural questions throughout the paper which are open problems.

2. Results on unique range set for meromorphic functions. Let a set $S \in \mathcal{S}$ be finite or infinite. Then we see that the elements of $\mathcal{S}$ are not symmetric in nature. Because if for every $a \in S$, we have $-a \in S$, then f and $-f$ share S CM. Hence $S \notin \mathcal{S}$.

So natural query arises whether the elements of a set $S \in \mathcal{S}$ can be reciprocal in nature or not. In relation to that for a set $S \subseteq \overline{\mathbb{C}}$, finite or infinite, we define

$$S^* = \left\{\frac{1}{a_k} : a_k \in S\right\}.$$

Because $S \subseteq \overline{\mathbb{C}}$, then by the definition, $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$. Thus, the definition of S^* is correct.

Let us pose the first question of this section as follows.

Question 1. *Let $S \in \mathcal{S}$. Does $S^* \in \mathcal{S}$?*

In the following theorem we answer Question 1 in affirmative.

Theorem 1. $S \in \mathcal{S} \iff S^* \in \mathcal{S}$.

Proof. ($\Longrightarrow$) Firstly we suppose that $S \in \mathcal{S}$. Let f and g be two meromorphic functions such that $E_f(S^*) = E_g(S^*)$. Now we consider the following four cases.

Case 1. Let $0, \infty \notin S$. Since $E_f(S^*) = E_g(S^*)$, we get $\bigcup_{a_i \in S} E_f(\frac{1}{a_i}) = \bigcup_{a_i \in S} E_g(\frac{1}{a_i})$. Now

$$E_f\left(\frac{1}{a_i}\right) = \{(z,p) : f(z) - \frac{1}{a_i} = 0\} = \Big\{(z,p) : \frac{1}{f_1(z)} - \frac{1}{a_i} = 0\Big\} = \Big\{(z,p) : \frac{a_i - f_1(z)}{a_i f_1(z)} = 0\Big\},$$

where $f_1(z) = \frac{1}{f(z)}$. Clearly a_i points of f_1 can not coincide with zeros of f_1. Hence,

$$E_f(\tfrac{1}{a_i}) = \{(z,p) : f_1(z) - a_i = 0\} = E_{f_1}(a_i).$$

Similarly we shall obtain $E_g(\frac{1}{a_i}) = \{(z,p) : g_1(z) - a_i = 0\} = E_{g_1}(a_i)$, where $g_1(z) = \frac{1}{g(z)}$. Therefore,

$$\bigcup_{a_i \in S} E_{f_1}(a_i) = \bigcup_{a_i \in S} E_f(\tfrac{1}{a_i}) = \bigcup_{a_i \in S} E_g(\tfrac{1}{a_i}) = \bigcup_{a_i \in S} E_{g_1}(a_i).$$

That is, $E_{f_1}(S) = E_{g_1}(S)$. Since $S \in \mathcal{S}$, we have $f_1 \equiv g_1$ and hence $f \equiv g$.

Case 2. Let $0 \in S$ and $\infty \notin S$. Then $\infty \in S^*$. Now by definition we always have $E_f(\infty) = E_{f_1}(0)$, where $f_1(z) = \frac{1}{f(z)}$. Similarly we shall obtain $E_g(\infty) = E_{g_1}(0)$, where $g_1(z) = \frac{1}{g(z)}$. Now for $\frac{1}{a_i} \in S^*$, where $a_i \neq 0$; we proceed same as Case 1 of this theorem and obtain

$$\bigcup_{a_i \in S\setminus\{0\}} E_{f_1}(a_i) = \bigcup_{a_i \in S\setminus\{0\}} E_f(\tfrac{1}{a_i}), \qquad \bigcup_{a_i \in S\setminus\{0\}} E_{g_1}(a_i) = \bigcup_{a_i \in S\setminus\{0\}} E_g(\tfrac{1}{a_i}).$$

That is

$$\bigcup_{a_i \in S} E_{f_1}(a_i) = \bigcup_{a_i \in S\setminus\{0\}} E_f(\tfrac{1}{a_i}) \bigcup E_f(\infty) = \bigcup_{a_i \in S\setminus\{0\}} E_g(\tfrac{1}{a_i}) \bigcup E_g(\infty) = \bigcup_{a_i \in S} E_{g_1}(a_i),$$

as $E_f(S^*) = E_g(S^*)$. Hence the above equality says $E_{f_1}(S) = E_{g_1}(S)$. Since $S \in \mathcal{S}$, we have $f_1 \equiv g_1$ and hence $f \equiv g$.

Case 3. Let $\infty \in S$ and $0 \notin S$. Then $0 \in S^*$. This case can be dealt exactly in the same way as Case 2 of this theorem.

Case 4. Let $\{0, \infty\} \subset S$. Then $\{0, \infty\} \subset S^*$. Since $E_f(\infty) = E_{f_1}(0)$, $E_f(0) = E_{f_1}(\infty)$ and $E_g(\infty) = E_{g_1}(0)$, $E_g(0) = E_{g_1}(\infty)$, this case can also be resorted similarly like Case 2 of this theorem. So we omit the detail.

($\Longleftarrow$) For the converse part assuming $b_i = \frac{1}{a_i}$, the proof can be carried out in the same line of proof as done in the first part. Hence the theorem. $\square$

So, we have obtained the answer of Question 1 in affirmative. Now we concentrate on the discussion behind Question 1 which actually demands the answer of the following question.

Question 2. *Does $S^* = S$ hold, when $S \in \mathcal{S}$?*

In this article, we prove that the answer to this question is negative. On the way of providing the answer of this question we develop the following results.

For $S \subseteq \overline{\mathbb{C}}$ and $k_1 \in \mathbb{C} \setminus \{0\}, k_2 \in \mathbb{C}$ let us define

$$k_1 S + k_2 = \{k_1 a_i + k_2 : a_i \in S\}.$$

We prove the following theorem.

Theorem 2. *Let $k_1 \in \mathbb{C} \setminus \{0\}, k_2 \in \mathbb{C}$. Then, $S \in \mathcal{S} \iff k_1 S + k_2 \in \mathcal{S}$.*

Proof. ($\Longrightarrow$) Let $S \in \mathcal{S}$. Then we need to show that $k_1 S + k_2 \in \mathcal{S}$. Let f and g be two meromorphic functions such that $E_f(k_1 S + k_2) = E_g(k_1 S + k_2)$. Now we split this part of the proof in two cases.

Case 1. Let $\infty \notin S$. Since $E_f(k_1 S + k_2) = E_g(k_1 S + k_2)$, we get $\bigcup_{a_i \in S} E_f(k_1 a_i + k_2) = \bigcup_{a_i \in S} E_g(k_1 a_i + k_2)$.

Now for $f_1(z) = \frac{f(z) - k_2}{k_1}$ we get $E_f(k_1 a_i + k_2) = \{(z,p) : f(z) - k_1 a_i - k_2 = 0\} = \{(z,p) : k_1 f_1(z) + k_2 - k_1 a_i - k_2 = 0\} = \{(z,p) : f_1(z) - a_i = 0\} = E_{f_1}(a_i)$.

Similarly, for the meromorphic functions g and g_1, where $g_1(z) = \frac{g(z)-k_2}{k_1}$, we have $E_g(k_1a_i + k_2) = E_{g_1}(a_i)$. Hence,

$$\bigcup_{a_i\in S} E_{f_1}(a_i) = \bigcup_{a_i\in S} E_f(k_1a_i + k_2) = \bigcup_{a_i\in S} E_g(k_1a_i + k_2) = \bigcup_{a_i\in S} E_{g_1}(a_i).$$

Since $S \in \mathcal{S}$, $f_1 \equiv g_1$ and hence $f \equiv g$. Therefore $k_1S + k_2 \in \mathcal{S}$.
Case 2. Let $\infty \in S$. Then for the meromorphic function $f_1(z) = \frac{f(z)-k_2}{k_1}$, we always have $E_{f_1}(\infty) = E_f(\infty)$. Similarly for the meromorphic function $g_1(z) = \frac{g(z)-k_2}{k_1}$, we always have $E_{g_1}(\infty) = E_g(\infty)$. Hence proceeding similarly for all other points of S like in Case 1 of this theorem we shall obtain

$$\bigcup_{a_i\in S} E_{f_1}(a_i) = \bigcup_{a_i\in S\setminus\{\infty\}} E_f(k_1a_i+k_2)\bigcup E_f(\infty) = \bigcup_{a_i\in S\setminus\{\infty\}} E_g(k_1a_i+k_2)\bigcup E_g(\infty) = \bigcup_{a_i\in S} E_{g_1}(a_i).$$

Therefore, $f_1 \equiv g_1$, i.e. $f \equiv g$.
($\Longleftarrow$) The converse part follows immediately as an application of the first part, so we omit it. $\square$

Taking into account Theorem 2 with $k_1 = k,\ k_2 = 0$ we obtain the following statement.

Proposition 1. $S \in \mathcal{S} \Longleftrightarrow kS \in \mathcal{S}$.

Define a relation $\mathcal{R}$ on $\mathcal{S}$ by

$$\mathcal{R} = \{(S_1, S_2) : S_1 = kS_2,\ k \in \mathbb{C} \setminus \{0\}\}.$$

The following statement is an elementary consequence of Proposition 1.

Proposition 2. *The relation $\mathcal{R}$ is an equivalence.*

Since $\mathcal{R}$ is an equivalence relation, we have a partition of $\mathcal{S}$ into different equivalence classes.

Now we ask a more general question than Question 2.

Question 3. *Does $S^* \in [S]$ hold?*

In the following theorem we obtain the answer of above Question 3 as well as the answer of Question 2.

Theorem 3. *If $S \in \mathcal{S}$ then $S^* \notin [S]$.*

Proof. On the contrary suppose that $S^* \in [S]$. Then $\exists\ kS \in [S]$ such that $kS = S^*$. Now we prove the theorem considering the following cases.
Case 1. Let $0, \infty \notin S$. Then from the definition of kS and S^*, we have $kS = \{ka_i : a_i \in S\}$ and $S^* = \{\frac{1}{a_i} : a_i \in S\}$. Clearly for every $ka_i \in kS$, there exist $\frac{1}{a_j} \in S^*$ such that $ka_i = \frac{1}{a_j}$ and viceversa. Therefore for those a_i, a_j we have $a_ia_j = \frac{1}{k}$. Now let f be a meromorphic function, then $\frac{1}{kf}$ is so. We have

$$E_{\left(\frac{1}{kf}\right)}(a_i) = \left\{(z,p) : \tfrac{1}{kf(z)} - a_i = 0\right\} = \left\{(z,p) : \tfrac{a_ia_j}{f(z)} - a_i = 0\right\} =$$
$$= \left\{(z,p) : \tfrac{a_i}{f(z)}\left[a_j - f(z)\right] = 0\right\} = \left\{(z,p) : f(z) - a_j = 0\right\} = E_f(a_j).$$

Therefore $\bigcup_{a_i\in S} E_{\left(\frac{1}{kf}\right)}(a_i) = \bigcup_{a_i\in S} E_f(a_i)$. Since $S \in \mathcal{S}$, so $\frac{1}{kf} = f$, which is a contradiction to the fact that f is non-constant. Hence $S^* \notin [S]$.

Case 2. Let $0 \in S$ and $\infty \notin S$. Then from the definition of kS and S^*, we have that $\infty \notin kS$ and $\infty \in S^*$. Hence $S^* \neq kS$, a contradiction.

Case 3. Let $\infty \in S$ and $0 \notin S$. Then from the definition of kS and S^*, we have that $0 \notin kS$ and $0 \in S^*$. Hence $S^* \neq kS$, a contradiction.

Case 4. Let $0, \infty \in S$. Then from the definition of kS and S^*, we have $\{0, \infty\} \subset kS \cap S^*$. Now proceeding in the same fashion as done in Case 1 of this theorem for other points of kS and S^*, we would obtain $\bigcup_{a_i\in S\setminus\{0,\infty\}} E_{\left(\frac{1}{kf}\right)}(a_i) = \bigcup_{a_i\in S\setminus\{0,\infty\}} E_f(a_i)$. Also $E_{\left(\frac{1}{kf}\right)}(\infty) = E_f(0)$ and $E_{\left(\frac{1}{kf}\right)}(0) = E_f(\infty)$ always hold for any meromorphic function f. Hence we obtain

$$\bigcup_{a_i\in S} E_{\left(\frac{1}{kf}\right)}(a_i) = \bigcup_{a_i\in S} E_f(a_i).$$

Since $S \in \mathcal{S}$, again we arrive at a contradiction as f is non-constant. Hence $S^* \notin [S]$. $\square$

Remark 1. *Since $S^* \notin [S]$, clearly $S^* \neq S$, which settles Question 2. Hence the elements of an $S \in \mathcal{S}$ can not be reciprocal in nature.*

Now taking Proposition 1 into account, one can ask the parallel question to Question 2 for kS as follows.

Question 4. *Is $kS = S$, when $S \in \mathcal{S}$ and $k \in \mathbb{C} \setminus \{0, 1\}$?*

With respect to Question 4, we prove the following statement.

Proposition 3. *Let $S \in \mathcal{S}$. Then $kS \neq S$, for all $k \in \mathbb{C} \setminus \{0, 1\}$.*

Proof. We assume that $kS = S$ for some $k \in \mathbb{C} \setminus \{0, 1\}$. Now we prove the proposition under the following two cases.

Case 1. Let $0, \infty \notin S$. Then for every $a_j \in kS$ there exist $a_i \in S$ such that $ka_i = a_j$. Let f be a meromorphic function. Then kf is so. Now

$$E_{(kf)}(a_j) = E_{kf}(ka_i) = \{(z, p) : kf(z) - ka_i = 0\} =$$
$$= \{(z, p) : k(f(z) - a_i) = 0\} = \{(z, p) : f(z) - a_i = 0\} = E_f(a_i).$$

So clearly $\bigcup_{a_i\in S} E_{(kf)}(a_i) = \bigcup_{a_i\in S} E_f(a_i)$. Since $S \in \mathcal{S}$, so $kf = f$, which contradicts that f is non-constant.

Case 2. Let $0 \in S$ or $\infty \in S$ or both of them belong to S. Since $E_{(kf)}(0) = E_f(0)$ and $E_{(kf)}(\infty) = E_f(\infty)$, so this case can be dealt in a similar manner like Case 1 of this theorem. $\square$

We know that every Möbius transformation of $\overline{\mathbb{C}}$ takes the circles and straight lines onto circles or straight lines. Also cross ratios remain invariant under any Möbius transformation. Now in view of Theorem 1 and Theorem 2, we explore another beautiful property of Möbius transformation as follows.

Proposition 4. *If $S \in \mathcal{S}$ and h is a Möbius transformation of $\overline{\mathbb{C}}$, then $h(S) \in \mathcal{S}$.*

Proof. Let $S \in \mathcal{S}$ and $h(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ be a Möbius transformation defined in the extended plane $\overline{\mathbb{C}}$. Now clearly for $\gamma \neq 0$ we have $h(S) = \frac{\alpha}{\gamma} + \frac{\beta\gamma-\alpha\delta}{\gamma}(\gamma S + \delta)^*$. For $\gamma = 0$ we have $h(S) = \frac{\alpha}{\delta}S + \frac{\beta}{\delta}$. Since $S \in \mathcal{S}$, so applying Theorem 2 and Theorem 1 for $h(S)$ we obtain that $h(S) \in \mathcal{S}$. $\square$

Now in view of Theorem 3 and Proposition 3, one may naturally enquire about the fact whether all sets $S \in \mathcal{S}$ disjoint or they may have some common points. We have the following statement.

Proposition 5. *There exist two sets $S_1 \in \mathcal{S}$ and $S_2 \in \mathcal{S}$ such that $S_1 \cap S_2 \neq \varnothing$.*

Proof. Let $S_1 \in [S]$. Then we claim that there exists a set $S_2 \in \mathcal{S}$, $S_2 \notin [S]$ such that $S_1 \cap S_2 \neq \varnothing$. Let $T \in [T]$, where $[T] \neq [S]$. Consider $a \in S_1 \setminus \{0\}$ and $b \in T \setminus \{0\}$. Then $kb \in kT \in [T]$ for any $k \in \mathbb{C} \setminus \{0\}$. So let us choose $k = \frac{a}{b}$. Hence $kb = a \in kT = S_2$ (say). Therefore, $a \in S_1 \cap S_2$, i.e., $S_1 \cap S_2 \neq \varnothing$. □

Pertinent to Proposition 5, we formulate the following questions.

Question 5. *Do any two $S_1,\ S_2 \in \mathcal{S}$ intersect each other?*

Question 6. *Do there exist two sets $S_1 \in \mathcal{S}$, $S_2 \in \mathcal{S}$ such that $S_1 \cap S_2 = \varnothing$?*

Following theorem shows that the answer of Question 6 is affirmative.

Theorem 4. *There exist two sets $S_1 \in \mathcal{S}$, $S_2 \in \mathcal{S}$ such that $S_1 \cap S_2 = \varnothing$.*

Proof. Let $S_1 = \{b_1, b_2, \ldots, b_n\} \in \mathcal{S}$ be such that $b_i \neq 0, \infty$ for all $i \in \{1, 2, \ldots, n\}$. In view of Proposition 1, $kS_1 \in \mathcal{S}$, where $k \in \mathbb{C}\setminus\{0\}$. Now we can arrange the elements of S_1 in non-decreasing moduli. In that case, let b_j and b_l are the elements in S_1 so that $|b_j| \leq |b_i| \leq |b_l|$ for all $i \in \{1, 2, \ldots, n\}$. Let $k = \frac{r}{|b_j|}$, where $r > |b_l|$ is a real number. Clearly each b_i is of the form $b_i = |b_i|e^{i\theta}$. Then $kb_i = \frac{r}{|b_j|}|b_i|e^{i\theta}$ for all $i \in \{1, 2, \ldots, n\}$. Hence $|kb_i| = \frac{r}{|b_j|}|b_i| > |b_l|$ for all $i \in \{1, 2, \ldots, n\}$. That is by multiplying this suitable k with each element of S_1 we have just amplified the distance of these elements from the origin so that the distance of each kb_i from the origin is larger than the largest distant element of S_1 from the origin. Therefore $b_i \notin kS_1$ for all $i \in \{1, 2, \ldots, n\}$ which implies $S_1 \cap kS_1 = \varnothing$. Denote kS_1 by S_2. Then we write $S_1 \cap S_2 = \varnothing$. This proves the theorem. □

From Theorem 4, we can conclude that arbitrary intersection of URSM's may not be a URSM. After the completion of Theorem 4, one would naturally ask the following questions regarding the union of sets from $\mathcal{S}$.

Question 7. 1^0. *Do the union of URSM's again become a URSM?*
2^0. *Is the union of all URSM's dense in $\mathbb{C}$ or $\overline{\mathbb{C}}$?*
3^0. *Do the union of all URSM's cover $\mathbb{C}$ or $\overline{\mathbb{C}}$?*

In this connection, first of all we would like to pay attention towards the fact whether 0 belongs to any $S \in \mathcal{S}$ or not. Below we provide an example of a set $S \in \mathcal{S}$ which contains 0.

Example 1. Let

$$P(z) = \frac{z^{13}}{13} + \frac{z^{12}}{6} + \frac{6z^{11}}{11} + z^{10} + \frac{5z^9}{3} + \frac{5}{2}z^8 + \frac{20}{7}z^7 + \frac{10}{3}z^6 + 3z^5 + \frac{5z^4}{2} + 2z^3 + z^2 + z.$$

Observe that $z = 0$ is a zero of $P(z)$ and $P'(z) = (z+1)^2(z-i)^5(z+i)^5$. Now consider $S = \{z : P(z) = 0\}$. So clearly: *i)* $P(-1) \neq P(i) \neq P(-i)$, *ii)* none of $-1, i, -i$ is arithmetic mean of the other two, *iii)* $\frac{P(-1)}{P(i)} \neq \frac{P(i)}{P(-i)} \neq \frac{P(-i)}{P(-1)}$. Then according to Theorem 5.2 and Theorem 1.4 of [6], $S \in \mathcal{S}$.

Now we consider the following example showing that there exist $S \in \mathcal{S}$ containing ∞.

Example 2. Consider the following polynomial

$$Q(z) = z^{12} + z^{11} + 2z^{10} + \frac{5z^9}{2} + 3z^8 + \frac{10z^7}{3} + \frac{20z^6}{7} + \frac{5z^5}{2} + \frac{5z^4}{3} + z^3 + \frac{6z^2}{11} + \frac{z}{6} + \frac{1}{13}.$$

Let $T = \{z : Q(z) = 0\} \bigcup \{\infty\}$. Note that all the zeros of $Q(z)$ are nothing but the reciprocals of the non-zero zeros of $P(z)$ in Example 1. Hence in view of Theorem 1 and Example 1, $T \in \mathcal{S}$.

Remark 2. Though from the very definition of $S \in \mathcal{S}$, it is a subset of $\overline{\mathbb{C}}$ but a set $S \in \mathcal{S}$ containing ∞ is for the first time being exemplified in the literature by the above example.

Theorem 5. *$\bigcup_{S \in \mathcal{S}} S = \overline{\mathbb{C}}$.*

Proof. Let $S_1 \in \mathcal{S}$ be such that $0, \infty \notin S_1$. Clearly $kS_1 \in \mathcal{S}$. Since $kS_1 \subseteq \overline{\mathbb{C}} \setminus \{0, \infty\}$, so $\bigcup_{kS_1 \in [S_1]} kS_1 \subseteq \overline{\mathbb{C}} \setminus \{0, \infty\}$. Now let $z \in \overline{\mathbb{C}} \setminus \{0, \infty\}$ be arbitrary. Then for any $a \in S_1$, there exist $\frac{z}{a} = k \in \mathbb{C}$ such that $z = ka \in kS_1 \subseteq \bigcup_{kS_1 \in [S_1]} kS_1$. That is, $\bigcup_{kS_1 \in [S_1]} kS_1 \supseteq \overline{\mathbb{C}} \setminus \{0, \infty\}$. Therefore, $\bigcup_{kS_1 \in [S_1]} kS_1 = \overline{\mathbb{C}} \setminus \{0, \infty\}$. Now consider $S_2 \in \mathcal{S}$, $S_3 \in \mathcal{S}$ such that $\{0, \infty\} \subset S_2 \bigcup S_3$. Then $\bigcup_{kS_1 \in [S_1]} kS_1 \bigcup S_2 \bigcup S_3 = \overline{\mathbb{C}}$. $\square$

The following statement immediately follows from Theorem 5.

Corollary 1. *Every point $a \in \overline{\mathbb{C}}$ is an element of some set $S \in \mathcal{S}$.*

Hence Theorem 5 or Corollary 1 fully answers Questions 7.2^0 and 7.3^0 together. Now with respect to Question 7.1^0, we have the following counterexamples that an arbitrary union of URSM's may not be a URSM.

Example 3. Consider the functions $f(z) = \frac{1}{z}$ and $g(z) = -\dfrac{1}{z}$.
Then $E_f(\overline{\mathbb{C}}) = E_g(\overline{\mathbb{C}})$ but $f \not\equiv g$. Hence $\overline{\mathbb{C}} \notin \mathcal{S}$.

Example 4. Consider the functions $f(z) = z$ and $g(z) = -z$.
Then $E_f(\mathbb{C}) = E_g(\mathbb{C})$ but $f \not\equiv g$. Therefore $\mathbb{C} \notin \mathcal{S}$.

Example 5. Consider the functions $f(z) = e^z$ and $g(z) = e^{-z}$.
Then $E_f(\mathbb{C} \setminus \{0\}) = E_g(\mathbb{C} \setminus \{0\})$ but $f \not\equiv g$. Hence $\mathbb{C} \setminus \{0\} \notin \mathcal{S}$.

Example 6. Consider the functions $f(z) = \operatorname{tg} z$ and $g(z) = \operatorname{ctg} z$.
Then $E_f(\mathbb{C} \setminus \{i, -i\}) = E_g(\mathbb{C} \setminus \{i, -i\})$ but $f \not\equiv g$. Hence $\mathbb{C} \setminus \{i, -i\} \notin \mathcal{S}$.

Example 7. Consider the functions $f(z) = e^z + a$ and $g(z) = -e^z + a$.
Then $E_f(\mathbb{C} \setminus \{a\}) = E_g(\mathbb{C} \setminus \{a\})$ but $f \not\equiv g$. Hence for any given complex number a, $\mathbb{C} \setminus \{a\} \notin \mathcal{S}$.

Example 8. Consider the functions $f(z) = \frac{ae^z - b}{e^z - 1}$ and $g(z) = \frac{be^z - a}{1 - e^z}$.
Then $E_f(\mathbb{C} \setminus \{a, b\}) = E_g(\mathbb{C} \setminus \{a, b\})$ but $f \not\equiv g$. Hence for any two arbitrary complex number a, b, one has $\mathbb{C} \setminus \{a, b\} \notin \mathcal{S}$.

So, after the answers of Questions 7, it is clear that every element of the extended complex plane participates in forming at least one $S \in \mathcal{S}$. Mainly Theorem 5 and Corollary 1 proves that the elements of a URSM are distributed all over the extended plane. But they are distributed over the extended plane with a very special arrangement as follows from the other theorems proved above.

Now we focus our research on the distribution of URSM's over the extended plane. In this connection, we prove the following theorem.

Before stating the theorem we need to recall some basic definitions of the metric space $(\mathbb{C}, d)$, $d(a,b) = |a-b|$ for $a, b \in \mathbb{C}$.

We also recall the usual notion of diameter of a non-empty set A in $\mathbb{C}$ as

$$\delta(A) = \sup\{d(a,b)\colon a, b \in A\}.$$

Theorem 6. *For any given positive real number r there exists a set $S \in \mathcal{S}$ of diameter r. In fact, there exists uncountably many sets $S \in \mathcal{S}$ of diameter $\delta(S) = r$.*

Proof. Let $r > 0$ be an arbitrary real number. Let $S = \{a_1, a_2, \ldots, a_n\}$ be a URSM of diameter p and $\infty \notin S$. The existence of such URSM is already given by Example 1. Now $\delta(S) = p = \sup\limits_{a_i, a_j \in S} d(a_i, a_j) = \max\limits_{a_i, a_j \in S} d(a_i, a_j)$, as S is finite. Then suppose $\max\limits_{a_i, a_j \in S} d(a_i, a_j) = |a_l - a_m|$ for some $l, m \in \{1, 2, \ldots, n\}$. That is $|a_l - a_m| \geq |a_i - a_j|$ for all $i, j \in \{1, 2, \ldots, n\}$. Hence $|ka_l - ka_m| \geq |ka_i - ka_j|$ for all $i, j \in \{1, 2, \ldots, n\}$, where $k \in \mathbb{R}^+$. Therefore for the set kS; $\delta(kS) = \max\limits_{ka_i, ka_j \in kS} d(ka_i, ka_j) = |ka_l - ka_m| = k|a_l - a_m| = kp$. Since this is true for any $k \in \mathbb{R}^+$, let us choose $k = \frac{r}{p}$. Then $\frac{r}{p}S$ is of diameter r. Now $\frac{r}{p} \in \mathbb{R}^+ \subseteq \mathbb{C} \setminus \{0\}$. So in view of Proposition 1, $\frac{r}{p}S \in \mathcal{S}$. Since r was chosen arbitrarily, so the existence the set $S \in \mathcal{S}$ of diameter r for any $r \in \mathbb{R}^+$ is proved.

Now let $\theta_1, \theta_2, \ldots, \theta_n$ be the principal arguments of the elements $\frac{r}{p}a_1, \frac{r}{p}a_2, \ldots, \frac{r}{p}a_n$, respectively. Choose $\theta \in \big(0, \min\big\{|\theta_i - \theta_j|\colon i, j \in \{1, 2, \ldots, n\}, \theta_i \neq \theta_j\big\}\big)$. Then according to Proposition 1, $e^{i\theta}\frac{r}{p}S \in \mathcal{S}$. Observe that the diameter of $e^{i\theta}\frac{r}{p}S$ is also r. We can always have uncountably many such θ's. □

3. Results on unique range set for meromorphic functions ignoring multiplicity. All results and remarks except Examples 1, 2 and Remark 2 obtained in the above section for URSM's are also true for URSM-IM's. But in this section we shall prove some other results regarding URSM-IM's which may or may not be true for all URSM's.

On this occasion, we would like to recall the fact that till date a lot of the sets $S \in \mathcal{S}$ and $S \in \mathcal{S}_I$ have been obtained in the literature [3–5, 10, 11, 15, 16] and eventually all these sets S are finite. So natural query arises:

Question 8. *Can we have a set $S \in \mathcal{S}$ or $S \in \mathcal{S}_I$ containing infinitely many elements?*

In the next two statements, we answer Question 8 and also find a necessary and sufficient condition for $S \notin \mathcal{S}_I$.

Proposition 6. *Let $S \subseteq \overline{\mathbb{C}}$. Then $S \notin \mathcal{S}_I \Longleftrightarrow \overline{\mathbb{C}} \setminus S \notin \mathcal{S}_I$.*

Proof. $(\Longrightarrow)$ Suppose $S \notin \mathcal{S}_I$. Then there exist meromorphic functions f and g such that $\overline{E}_f(S) = \overline{E}_g(S)$ and $f \not\equiv g$. Since f and g are defined in $\mathbb{C}$, for this f and g we have $\overline{E}_f(\overline{\mathbb{C}} \setminus S) = \overline{E}_f(\overline{\mathbb{C}}) \setminus \overline{E}_f(S) = \mathbb{C} \setminus \overline{E}_g(S) = \overline{E}_g(\overline{\mathbb{C}}) \setminus \overline{E}_g(S) = \overline{E}_g(\overline{\mathbb{C}} \setminus S)$. Hence $\overline{\mathbb{C}} \setminus S \notin \mathcal{S}_I$.
$(\Longleftarrow)$ For the converse let us assume $\overline{\mathbb{C}} \setminus S \notin \mathcal{S}_I$. Then according to the above part $S = \overline{\mathbb{C}} \setminus (\overline{\mathbb{C}} \setminus S) \notin \mathcal{S}_I$. □

The next statement immediately follows from the above theorem.

Corollary 2. *Let $S \subseteq \overline{\mathbb{C}}$. Then $S \in \mathcal{S}_I \Longleftrightarrow \overline{\mathbb{C}} \setminus S \in \mathcal{S}_I$.*

Remark 3. Since $\mathcal{S}_I \subset \mathcal{S}$, considering any finite URSM-IM, from Corollary 2 one can easily obtain a set $S_1 \in \mathcal{S}_I$ (as well as $S_1 \in \mathcal{S}$) having uncountable number of elements.

Remark 4. Corollary 2 also confirms that union of two $S_1, S_2 \in \mathcal{S}$ (or $S_1, S_2 \in \mathcal{S}_I$) is not always a URSM (or URSM-IM, respectively). For example if $S_1 \in \mathcal{S}_I$, then $S_1 \in \mathcal{S}$ and $S_2 = \overline{\mathbb{C}} \setminus S_1 \in \mathcal{S}$. But $S_1 \bigcup S_2 = \overline{\mathbb{C}} \notin \mathcal{S}$ (as well $S_1 \bigcup S_2 \notin \mathcal{S}_I$).

From Corollary 2 we also observe that for two sets $S_1, S_2 \in \mathcal{S}_I$, if $S_1 \cap S_2 = \varnothing$ then $S_2 \bigcup (\overline{\mathbb{C}} \setminus \{S_1\}) = \overline{\mathbb{C}} \setminus \{S_1\} \in \mathcal{S}$ (as well $S_2 \cap (\overline{\mathbb{C}} \setminus \{S_1\}) = S_2 \in \mathcal{S}$). Hence keeping all the results and remarks of this section and the above section for union and intersection of URSM's (URSM-IM's) in mind we conjecture the following.

Conjecture 1. *Union or intersection of two $S_1, S_2 \in \mathcal{S}$ ($S_1, S_2 \in \mathcal{S}_I$) is again a set from $\mathcal{S}$ (a set from $\mathcal{S}_I$) if and only if one of them is contained in the other.*

Now we prove another result which may lead us to obtain the smallest possible URSM-IM provided the answer of the following question is affirmative.

Question 9. *Is $S \bigcup \{a\} \in \mathcal{S}_{\mathcal{I}}$, where $S \in \mathcal{S}_I$ and $a \notin S$?*

With respect to Question 9, we obtain the following theorem.

Theorem 7. *Let $S \in \mathcal{S}_I$. Then there exists $a \in \overline{\mathbb{C}} \setminus S$, the set $S \bigcup \{a\}$ is not a set from $\mathcal{S}_I$.*

Proof. On the contrary suppose that $S \bigcup \{a\} \in \mathcal{S}_I$ for all a. Now consider a finite set $S \in \mathcal{S}_I$ of n elements. In particular, let $S = \{a_1, a_2, \ldots, a_n\}$. Since $S \in \mathcal{S}_I$, according to Corollary 2, $\overline{\mathbb{C}} \setminus S \in \mathcal{S}_I$. So from our assumption, $(\overline{\mathbb{C}} \setminus S) \bigcup \{a_1\} \in \mathcal{S}_I$, where $a_1 \in S$. Using the same logic, we get that $(\overline{\mathbb{C}} \setminus S) \bigcup \{a_1\} \bigcup \{a_2\} \in \mathcal{S}_I$. Proceeding in this way we would get that $(\overline{\mathbb{C}} \setminus S) \bigcup S = \overline{\mathbb{C}} \in \mathcal{S}_I$, which is a contradiction. Hence $S \bigcup \{a\}$ is not always a URSM-IM. □

From Theorem 2, we know that for a set $S \in \mathcal{S}_I$, $\overline{\mathbb{C}} \setminus S \in \mathcal{S}_I$. Hence, for any set $S \in \mathcal{S}_I$, if $(\overline{\mathbb{C}} \setminus S) \bigcup \{a\} \in \mathcal{S}_I$ and this happens for a set $S \in \mathcal{S}_I$ of cardinality 17, then at instant we would be able to find out a set $S_1 \in \mathcal{S}_I$ of cardinality 16 and so on. Therefore the following questions are natural under this situation.

Question 10. $\mathbf{1^0}$. *Can the answer of Question 9 be affirmative for at least a single set $S \in \mathcal{S}_I$?*
$\mathbf{2^0}$. *What can be the relation between any two meromorphic functions sharing $S \bigcup \{a\}$, where $S \in \mathcal{S}_I$ and $a \notin S$?*
$\mathbf{3^0}$. *Under which condition any two meromorphic functions sharing a set S will share $S \bigcup \{a\}$, where $a \notin S$?*

The following example to show the existence of a set $S \in \mathcal{S}_I$ such that $\infty \in S$.

Example 9. Let

$$P(z)=\frac{z^{19}}{19}-\frac{z^{18}}{9}+\frac{10z^{17}}{17}-z^{16}+\frac{12z^{15}}{5}-4z^{14}+\frac{84z^{13}}{13}-\frac{28z^{12}}{3}+\frac{126z^{11}}{11}-14z^{10}+$$
$$+14z^9-14z^8+12z^7-\frac{28z^6}{3}+\frac{36z^5}{5}-4z^4+3z^3-z^2+z.$$

Observe that $z=0$ is a zero of $P(z)$ and $P^{'}(z)=(z-1)^2(z-i)^8(z+i)^8$. Now consider $S=\{z:P(z)=0\}$. So clearly: $i)$ $P(1)\neq P(i)\neq P(-i)$, $ii)$ None of $1,i,-i$ is arithmetic mean of the other two, $iii)$ $\frac{P(1)}{P(i)}\neq\frac{P(i)}{P(-i)}\neq\frac{P(-i)}{P(1)}$. Then according to Theorem 5.2 and Theorem 1.4 of [6], $S\in\mathcal{S}_I$.

Now consider the polynomial

$$Q(z)=z^{18}-z^{17}+3z^{16}-4z^{15}+\frac{36z^{14}}{5}-\frac{28z^{13}}{3}+12z^{12}-14z^{11}+14z^{10}-14z^9+$$
$$+\frac{126z^8}{11}-\frac{28z^7}{3}+\frac{84z^6}{13}-4z^5+\frac{12z^4}{5}-z^3+\frac{10z^2}{17}-\frac{z}{9}+\frac{1}{19}.$$

Note that all the zeros of $Q(z)$ are nothing but the reciprocals of the non-zero zeros of $P(z)$. Let us denote $T=\{z:Q(z)=0\}\bigcup\{\infty\}$. Hence $T=S^{*}$. Therefore in view of analog of Theorem 1 for the class $\mathcal{S}_I$ see, the first sentence at the beginning of Section 3, $T\in\mathcal{S}_I$.

Now we would like to conclude this section with a short discussion. In the previous section observing the Examples 3–8 carefully, it seems that if $S\notin\mathcal{S}$ then $\overline{\mathbb{C}}\setminus S$ would not be in $\mathcal{S}$ too. In this section, we have proved this result for the clas of the sets $\mathcal{S}_I$ and more generally we have proved this to be a necessary and sufficient condition for a set S to be $\mathcal{S}_I$. Hence for $S\in\mathcal{S}\cap\mathcal{S}_I$ our presumption is true. In view of these observations, we conjecture the following.

Conjecture 1. *Let $S\subseteq\overline{\mathbb{C}}$. Then, $S\in\mathcal{S}\Longleftrightarrow\overline{\mathbb{C}}\setminus S\in\mathcal{S}$.*

Note that if Conjecture 1 is proved to be true, then all the results and discussions of this section would be improved up to the class $\mathcal{S}$.

4. On analogues for entire functions. For two arbitrary non-constant entire functions f, g and $S\subseteq\mathbb{C}$ if $E_f(S)=E_g(S)$ implies $f\equiv g$, then S is called a unique range set for entire functions or URSE in short [5] (we write $S\in\mathcal{S}E$). Similarly if $\overline{E}_f(S)=\overline{E}_g(S)$ implies $f\equiv g$, then S is called a unique range set for entire functions ignoring multiplicities or URSE-IM in short [4] and we write $S\in\mathcal{S}E_I$. Like URSM and URSM-IM there is a similar literature of URSE and URSE-IM [14, see section 10.5.3 to 10.6.2]. However, now we discuss the analogues of all the results proved in Section 2 and Section 3 with respect to $\mathcal{S}E$ and $\mathcal{S}E_I$, respectively.

Note that all the results proved in Section 2 and Section 3 except Theorems 1 and 3 and Proposition 4 are also true for the classes of the sets $\mathcal{S}E$ and $\mathcal{S}E_I$. For the results of Theorem 1, Theorem 3 and Proposition 4 to be true or false in case of the classes of the sets $\mathcal{S}E$ and $\mathcal{S}E_I$, we need proofs. But for other results we just need to replace $\overline{\mathbb{C}}$ by $\mathbb{C}$ as every set $S\in\mathcal{S}E$ or $S\in\mathcal{S}E_I$ by definition is a subset of $\mathbb{C}$. For example, analogues to Theorem 5 and Theorem 2 for $\mathcal{S}E$ and $\mathcal{S}E_I$ are as follows.

Theorem 8. $\bigcup_{S\in\mathcal{S}E}S=\bigcup_{S\in\mathcal{S}E_I}S=\mathbb{C}$.

Theorem 9. *Let $S \subset \mathbb{C}$. Then, $S \in \mathcal{S}E_I \iff \mathbb{C} \setminus S \in \mathcal{S}E_I$.*

Proof. The proof is obvious as for any non-constant entire function f we always have $\overline{E}_f(\mathbb{C}) = \mathbb{C}$. $\square$

Remark 5. On this occasion, we would like to recall that in case of the classs $\mathcal{S}E$ ($\mathcal{S}E_I$), there are examples of finite sets $S \in \mathcal{S}E$ ($S \in \mathcal{S}E_I$) (as well a infinite set $S \in \mathcal{S}E$). But there is only one example of infinite set $S \in \mathcal{S}E$ [8] which is as follows $\{z : e^z + z = 0\}$. Obviously this is a countable set. So it is still unfold whether there can be any URSE having uncountable number of elements. Observe that, Theorem 9 confirms the existence of a set $S \in \mathcal{S}E$ (as well a set $S \in \mathcal{S}E_I$) having uncountable number of elements.

Since Theorem 1, Theorem 3 and Proposition 4 are uncertain to be true for the class $\mathcal{S}E$ or $\mathcal{S}E_I$, so one would naturally raise the following question for further investigations.

Question 11. *Can Theorem 1, 3 and Proposition 4 be proved for the class $\mathcal{S}E$ or $\mathcal{S}E_I$?*

5. Some applications.

Application 1 (Simple proofs of existence of some unique range set). Note that as an application of Theorem 1, one can make a very small and simple proof of the result of [16] using the result of [15] and also the result of [4] using the result of [17].

In [15] and [16], respectively, H. X. Yi proved that the set of zeros of each of the polynomials $z^n + cz^m + d$ and $z^n + az^{n-m} + b$ are URSM's, where the polynomials have only simple zeros, $n \geq 2m + 9$, $m \geq 2$ and $\gcd(m, n) = 1$. Now choose $a, b \in \mathbb{C} \setminus \{0\}$ in such a way that the polynomial $P(z) = z^n + \frac{a}{b} z^m + \frac{1}{b}$ has only simple zeros. Then according to Yi's result in [15], $S = \{z : P(z) = 0\} \in \mathcal{S}$ for $n \geq 2m + 9$, $m \geq 2$ and $\gcd(m, n) = 1$.

Observe that the zeros of $Q(z) = z^n + az^{n-m} + b$ are nothing but the reciprocals of the zeros of $P(z)$. Hence $T = \{z : Q(z) = 0\} = S^*$. Therefore according to Theorem 1, $T \in \mathcal{S}$.

Let $n \geq 17$ and $a, b \in \mathbb{C} \setminus \{0\}$ be such that $ab^{n-2} \neq 2$, $\mathbf{U}_Y := \mathbf{U}(P_{a,b,n})$ be the set of zeros of polynomial

$$P_{a,b,n}(z) = az^n - n(n-1)z^2 + 2n(n-2)bz - (n-1)(n-2)b^2.$$

In [17] H. X. Yi proved that $\mathbf{U}_Y \in \mathcal{S}_I$. S. Bartels [4] proved that the set $\mathbf{U}_B := \mathbf{U}(Q_{c,n})$ of zeros of polynomial

$$Q_{c,n}(z) = (n-1)(n-2)z^n - 2n(n-2)z^{n-1} + n(n-2)z^{n-2} - 2c$$

is also a URSM-IM, i.e. $\mathbf{U}_B \in \mathcal{S}_I$ for $n \geq 17$, where $c \neq 0, 1$.

Now for $b = 1$ and $a = 2c$ we get $P_{2c,1,n}(z) = 2cz^n - n(n-1)z^2 + 2n(n-2)z - (n-1)(n-2)$, where $c \neq 0, 1$. Observe that the zeros of $Q_{c,n}(z)$ are nothing but the reciprocals of the zeros of $P_{2c,1,n}(z)$. Denote $V = \mathbf{U}(Q_{c,n})$ and $U = \mathbf{U}(P_{2,1,n})$. Hence $V = U^*$. Therefore according to Theorem 1, $V \in \mathcal{S}_I$ for $n \geq 17$.

Moreover, in [3] A. Banerjee and S. Mallick proved that the set of zeros of the polynomial $R(z) = z^n + az^{n-m} + bz^{n-2m} + c$; is a URSM for $n \geq 2m + 9$, where $a, b, c \in \mathbb{C} \setminus \{0\}$ be such that $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$ and the polynomial has only simple zeros. Observe that as an application of Theorem 1 one can easily prove that the set of zeros of the polynomial $S(z) = cz^n + bz^{2m} + az^m + 1$ is a URSM under the same condition as taken for $R(z)$ and the most interesting fact in this case is that we do not need any more detailed proof of this result.

Application 2. Note that for the following sets $S = -S$: real line, imaginary line, set of integers, set of rational numbers, cross halves, i.e. 2nd quadrant $\bigcup$ 4th quadrant, 1st quadrant $\bigcup$ 3rd quadrant, upper half $\bigcup$ lower half, right half $\bigcup$ left half. So these sets are not URSM's according to Proposition 3.

Now the following questions are natural under this situation.

Question 12. $\mathbf{1^0}$. *Is each quadrant of the complex plane a URSM?*
$\mathbf{2^0}$. *Is the upper half or the lower half of the complex plane a URSM?*
$\mathbf{3^0}$. *Is the right half or the left half of the complex plane a URSM?*

The answers of Questions 12 are also negative as can be obtained by simple application of Proposition 3.

Application 3. Let us denote the first, second, third and fourth quadrant of the complex plane by C_1, C_2, C_3 and C_4, respectively. Also denote the upper half, lower half, right half and left half by $\mathcal{UPH}$, $\mathcal{LOH}$, $\mathcal{RIH}$ and $\mathcal{LEH}$, respectively. Then clearly $C_i = 2C_i$ for all $i = 1, 2, 3, 4$ and $\mathcal{UPH} = 2\ \mathcal{UPH}$, $\mathcal{LOH} = 2\ \mathcal{LOH}$, $\mathcal{RIH} = 2\mathcal{RIH}$ and $\mathcal{LEH} = 2\mathcal{LEH}$. Hence according to Proosition 3, none of C_1, C_2, C_3, C_4, $\mathcal{UPH}$, $\mathcal{LOH}$ can be URSM.

Not only the infinite sets, as an above application of Proposition 3, we can prove a finite set like $\mathcal{U}_0 = \{z : z^n = 1\}$ is not a URSM as $\omega \cdot \mathcal{U}_0 = \mathcal{U}_0$, where $\omega \in \mathcal{U}_0$.

Application 4. It is obvious that $\mathcal{U}_0 \subset \mathcal{C} := \{z \colon |z| = 1\}$. Also note that real and imaginary lines are not URSM's. So natural inquisition gives rise to the following question.

Question 13. *What can we say about any circle or any line in* $\mathbb{C}$ *with respect to URSM's?*

As an application of Theorem 2, we find the answer of Question 13 as follows.

Let $r\mathcal{C} = \{re^{i\theta} : 0 \le \theta < 2\pi\}$ be a circle of radius r centered at the origin. Now consider $e^{i\psi}$, where $0 < \psi < 2\pi$. Then $e^{i\psi}r\mathcal{C} = \{re^{i(\theta+\psi)} : 0 \le \theta \le 2\pi, 0 < \psi < 2\pi\} = r\mathcal{C}$, i.e. $e^{i\psi}r\mathcal{C}$ is nothing but a circle of radius r centered at the origin. Since $e^{i\psi} \ne 1$, so $r\mathcal{C} \notin \mathcal{S}$ according to Proposition 3. Hence for any $\alpha \in \mathbb{C}$, $r\mathcal{C} + \alpha \notin \mathcal{S}$ according to Theorem 2. Obviously $r\mathcal{C} + \alpha$ is nothing but a circle of radius r centered at α. Since $\alpha \in \mathbb{C}$ is arbitrary, so we can conclude that no circle in $\mathbb{C}$ is a URSM.

Let $\mathcal{L}$ denotes an arbitrary line in $\mathbb{C}$ passing through the origin. Clearly $\mathcal{L} \notin \mathcal{S}$ as $\mathcal{L} = -\mathcal{L}$. Now every line parallel to $\mathcal{L}$ is nothing but $\{z + c : z \in \mathcal{L}\ \text{ and }\ c \in \mathbb{R}\}$. Clearly $\{z + c : z \in \mathcal{L}\ \text{ and }\ c \in \mathbb{R}\} = \mathcal{L} + c$. Since $\mathcal{L} \notin \mathcal{S}$, so $\mathcal{L} + c \notin \mathcal{S}$. Since every line in $\mathbb{C}$ not passing through the origin is parallel to a line passing through the origin. Hence no line in $\mathbb{C}$ is a URSM.

Applying our results, we can also similarly prove that $A = \{z : (z - a)^m - b = 0\} \notin \mathcal{S}$ (see H. Fujimoto, [6, p. 1183]).

Application 5. In Examples 1 and 2 we have seen that given 0 or ∞ we can find URSM containing 0 or ∞. Now let $c \in \mathbb{C} \setminus \{0\}$ be arbitrary. Denote (see Application 1) $\mathbf{U}_Y = \mathbf{U}(P_{a,b,n}) = \{c_1, c_2, \ldots, c_n\}$ for $n \ge 11$. Clearly $c_i \ne 0$ for all $i \in \{1, 2, \ldots, n\}$. Then as an application of Proposition 1 we find that for each $i \in \{1, 2, \ldots, n\}$ the set $c \in \frac{c}{c_i}\mathbf{U} \in \mathcal{S}$. Thus for every element in $\mathbb{C}$ there exists many URSM's containing the element.

Hence the following question becomes inevitable in this situation.

Question 14. *For each finite set* $B = \{b_j \colon 2 \le j \le n\} \subset \mathbb{C}$, $n \in \mathbb{N}$, *is it possible to find a set* $S \in \mathcal{S}$ *such that* $B \subset S$?

The following theorem contains a complete answer to this question.

Theorem 10. *Let* $B = \{b_j \colon 2 \le j \le n\} \subset \mathbb{C}$ *be any given set of pairwise distinct numbers. Then there exists a set* $S \in \mathcal{S}$ *such that* $B \subset S$.

Proof. **1**. $n = 2$. Let $B = \{c, d\} \subset \mathbb{C}$ be arbitrary. We at first prove that there exists a set $S \in \mathcal{S}$ containing c, d. Let $S = \{a_1, a_2, a_3, \ldots, a_n\} \in \mathcal{S}$ be such that $a_i \neq 0$, $\forall\ i \in \{1, 2, 3, \ldots, n\}$. We already have many examples of such sets as given by H. X. Yi [16], G. Frank and M. Reinders [5], S. Mallick [10] and many others. Now consider the following function $h(z) = \frac{(c-d)z+da_1-ca_2}{a_1-a_2}$, where $a_1, a_2 \in S$. Obviously, h is a bi-linear transformation as $a_1 \neq a_2$ and $c \neq d$. Observe that $h(a_1) = c$ and $h(a_2) = d$.

Now $h(S) = \frac{(c-d)}{(a_1-a_2)}S + \frac{da_1-ca_2}{a_1-a_2}$. Since $S \in \mathcal{S}$ and $h(z)$ is a Möbius transformation, so according to Proposition 4 we get that $h(S) \in \mathcal{S}$. Therefore, $\frac{(c-d)}{(a_1-a_2)}S + \frac{da_1-ca_2}{a_1-a_2} \in \mathcal{S}$ and is the set containing these arbitrarily given c and d.

2. $n = 3$. Let $B = \{a, b, c\} \subset \mathbb{C}$ be arbitrary. We now prove that there exists a set $S \in \mathcal{S}$ containing a, b, c. Let $S = \{a_1, a_2, a_3, \ldots, a_n\} \in \mathcal{S}$ be such that $a_i \neq 0$, $\forall\ i \in \{1, 2, 3, \ldots, n\}$. Now consider the following Möbius transformation $h(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$, where $\alpha = y_2z_1 - y_1z_2, \gamma = x_1z_2 - x_2z_1, \delta = x_2y_1 - x_1y_2$ and $\beta = -\alpha a_1 + \gamma a a_1 + \delta a$ with $x_1 = a_1 - a_2, x_2 = a_2 - a_3, x_3 = a_3 - a_1$, $y_1 = a_2b - a_1a, y_2 = a_3c - a_2b, y_3 = a_1a - a_3c$, $z_1 = b - a, z_2 = c - b, z_3 = a - c$. Observe that $h(a_1) = a$, $h(a_2) = b$, $h(a_3) = c$.

Now clearly for $\gamma \neq 0$ we have $h(S) = \frac{\alpha}{\gamma} + \frac{\beta\gamma-\alpha\delta}{\gamma}(\gamma S + \delta)^*$. For $\gamma = 0$ we have $h(S) = \dfrac{\alpha}{\delta}S + \dfrac{\beta}{\delta}$. Hence according to Proposition 4 we obtain that $h(S) \in \mathcal{S}$ is the required set containing a, b, c.

3. $n \geq 4$. Let us consider the set $S = \{z : 120z^{17} - 255z^{16} + 136z^{15} + 1 = 0\}$. Obviously, $S \in \mathcal{S}_I$ according to S. Bartels [4], where $n = 17$.

The elements of S are all distributed in all quadrants.

If $b_i \in S$, $\forall i \in \{1, 2, \ldots, n\}$, then we are done.

If $b_i \notin S$ for at least one $i \in \{1, 2, \ldots, n\}$, then we find the the maximum of $|b_i|$ for all $i \in \{1, 2, \ldots, n\}$. Suppose $\max |b_i| = l$. Further suppose $\min\{|z| : z \in S\} = p$ and $k > \frac{l}{p}$. Then obviously $\min\{|kz| : kz \in kS\} > l$. Hence kS does not contain b_i, $\forall i \in \{1, 2, \ldots, n\}$. Since $S \in \mathcal{S}_I$, so is kS and so is $\overline{\mathbb{C}} \setminus kS$, according to Proposition 1 and Theorem 6, respectively. Since $\mathcal{S}_I \subset \mathcal{S}$, hence $b_i \in \overline{\mathbb{C}} \setminus kS \in \mathcal{S}$ $(\forall i \in \{1, 2, \ldots, n\})$. □

Remark 6. For the cases $n = 2$ and $n = 3$ in Theorem 10, we found a finite set $S \in \mathcal{S}$ containing the given points but for the case $n \geq 4$ we have not always found that rather we have found an infinite set $S \in \mathcal{S}$ containing the given points in some cases.

One can also frame an alternate version of Theorem 10 as the following assertion which identifies the Fundamental Theorem of Algebra to a more specific region.

Corollary 3. *All zeros of a polynomial in* $\mathbb{C}$ *lie in some set* $S \in \mathcal{S}$.

Since $\mathbb{C}$ and $\overline{\mathbb{C}}$ are not URSM's, so for any uncountable subset of $\mathbb{C}$ it is not always possible to find out a URSM that contains the given uncountable set. Though there are some uncountable sets which are themselves URSM's as clarified in Remark 3.

However, for arbitrary countable subset of $\mathbb{C}$ we will always have URSM's containing the set. We prove this result by the following theorem.

Theorem 11. *Let* $\mathcal{T}$ *be a countable subset of* $\mathbb{C}$. *Then there exis an* $S \in \mathcal{S}$ *containing* $\mathcal{T}$.

Proof. Consider a set $S \in \mathcal{S}_I$ such that no two of its elements have same arguments. For our convenience we may work with the set $S \in \mathcal{S}_I$ used in the proof of Theorem 10 ($n \geq 4$ case).

Now consider the collection $\{k_iS\}$ for all $k_i \in \mathbb{R}^+\setminus(0,1]$. Obviously, this is an uncountable collection and $k_lS \cap k_mS = \varnothing$ for distinct $k_l, k_m \in \mathbb{R}^+ \setminus (0,1]$. Note that for $\mathcal{T}$, we must have at least one $k_p \in \mathbb{R}^+ \setminus (0,1]$ such that $k_pS \cap \mathcal{T} = \varnothing$. Otherwise, $\mathcal{T}$ will be uncountable as $k_lS \cap k_mS = \varnothing$ and $\{k_iS\}$ is an uncountable collection. Since $S \in \mathcal{S}_I$, so according to Proposition 1, $k_pS \in \mathcal{S}_I$. k_pS does not intersect $\mathcal{T}$ which in view of Theorem 6 implies that $\mathcal{T} \subset \overline{\mathbb{C}} \setminus k_pS \in \mathcal{S}_I$. □

Observe that, we can frame an alternate version of Theorem 11 as the following statement.

Corollary 4. *All zeros or poles of a meromorphic function lie in a URSM.*

Remark 7. Note that Corollary 3 and Theorem 11 are also true for finite or countable subsets of $\overline{\mathbb{C}}$. In this case, if R is the given subset containing ∞, then we shall find infinite URSM containing $R \setminus \{\infty\}$ as found in the proof of Corollary 3 and Theorem 11 and that URSM automatically contains R.

Application 6. Consider the polynomial $Q_{c,n}(z)$ (see Application 1) for $c = 2$ and $n = 11$. We get $Q_{2,11}(z) = 90z^{11} - 198z^{10} + 110z^9 - 4$. Let $\mathbf{W} = \mathbf{U}(Q_{2,11})$ and it's diameter be p. Now suppose we want a set $S \in \mathcal{S}$ of diameter $\delta(S) = 10$. Then according to Theorem 6, $S = \frac{10}{p}\mathbf{W} \in \mathcal{S}$ has the diameter 10. Hence using Theorem 6, we can find $S_1 \in \mathcal{S}$ of any diameter we wish.

Remark 8. Since $\mathcal{S}_I \subset \mathcal{S}$, so we can say that all the applications from Application 2 to 6 also hold good for the class $\mathcal{S}_I$.

Application 7. Let $S \in \mathcal{S}$ be a finite set containing ∞. Let f and g be any two distinct non-constant entire functions. Then $E_f(S) \neq E_g(S)$. Since for any two non-constant entire functions $E_f(\infty) = E_g(\infty) = \varnothing$, so $E_f(S \setminus \{\infty\}) \neq E_g(S \setminus \{\infty\})$. This is true for any two distinct non-constant entire functions. Hence $S \setminus \{\infty\} \in \mathcal{S}E$. Similar conclusion can be drawn for URSE-IM with respect to URSM-IM.

Application 8. In view of Theorem 3, we conclude that every multiplicative subgroup of $\mathbb{C}$ under usual multiplication of the complex numbers can not be a URSM or URSM-IM. Also in view of Proposition 3 any additive subgroup of $\mathbb{C}$ can not be a URSM or URSM-IM.

Alternatively, we can say that URSM's or URSM-IM's can never be a subgroup of $\mathbb{C}$ under usual addition or multiplication.

In case of URSE or URSE-IM we can say that they can never be a subgroup of $\mathbb{C}$ under usual addition.

Application 9. In view of Pproposition 3 for a $S \in \mathcal{S}$, clearly $k_1S \neq k_2S$ where $k_1, k_2 \in \mathbb{C}\setminus\{0\}$ and $k_1 \neq k_2$. Though S is not a subgroup of $\mathbb{C}$ under ususal addition or multiplication but if we define the operation $(*_1)$ on $[S]$ by $k_1S *_1 k_2S = k_1k_2S$, where $k_1, k_2 \in \mathbb{C} \setminus \{0\}$, then as an application of Proposition 1, we get that $([S], *_1)$ is an abelian group.

6. A few more open problems. Now we pose the following questions for further investigations.

Question 15. 1^0. *Can all the elements of a set $S \in \mathcal{S}$ belong to a line?*
2^0. *Can all the elements of a set $S \in \mathcal{S}$ belong to a circle?*
3^0. *Under which condition $S \bigcup \{a\} \in \mathcal{S}_I$, where $S \notin \mathcal{S}_I$ and $a \notin S$?*

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