C. Belabbaci

SPECTRAL RADIUS OF S-ESSENTIAL SPECTRA

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In this paper, we study the spectral radius of some S-essential spectra of a bounded linear operator defined on a Banach space. More precisely, via the concept of measure of noncompactness, we show that for any two bounded linear operators T and S with S non zero and non compact operator the spectral radius of the S-Gustafson, S-Weidmann, S-Kato and S-Wolf essential spectra are given by the following inequalities

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e,S}(T) \leq \frac{\alpha(T)}{\beta(S)}, \tag{1}$$

where $\alpha(.)$ stands for the Kuratowski measure of noncompactness and $\beta(.)$ is defined in [11]. In the particular case when the index of the operator S is equal to zero, we prove the last inequalities for the spectral radius of the S-Schechter essential spectrum. Also, we prove that the spectral radius of the S-Jeribi essential spectrum satisfies inequalities (2) when the Banach space X has no reflexive infinite dimensional subspace and the index of the operator S is equal to zero (the S-Jeribi essential spectrum, introduced in [7] as a generalisation of the Jeribi essential spectrum).

1. Introduction. The spectral theory of operators pencils $(\lambda S - T)$ (operator-valued functions of a complex argument) play a crucial role in many branches of mathematical physics see, for example, [2, 5, 4, 12].

The purpose of this paper is to study the spectral radius of some S-essential spectra of a bounded linear operator on a Banach space X. More precisely, we give a localization of the spectral radius of the S-essential spectra of an operator T via the concept of measure of noncompactness when S is a non compact and non zero bounded linear operator and T is any bounded linear operator. This work continues research begun in [7], here we aim to prove that the spectral radius of the S-Gustafson, S-Weidmann, S-Kato and S-Wolf essential spectrum is given by the following inequalities:

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e,S}(T) \leq \frac{\alpha(T)}{\beta(S)}, \tag{2}$$

where $\alpha(.)$ stands for the Kuratowski measure of noncompactness and $\beta(.)$ is defined in the next section.

When the index of the operator S is equal to zero, then the inequalities (2) holds for the S-Schechter essential spectrum. Also, we prove the above inequalities (2) for the S-Jeribi essential spectrum, introduced in [7] as a generalisation of the Jeribi essential spectrum, in the particular case when the Banach space X has no reflexive infinite dimensional subspace.

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The paper is organised as follows. Section 2 contains an overview of the necessary background and Section 3 contains the main results.

2. Preliminaries. The notion of measure of non-compactness have been successfully applied in many different domains in mathematics; in fixed point theory, differential equations, functional equations, integral and integro-differential equations, etc. (see for example [14, 13, 3, 8]). In order to recall this notion denote by X a Banach space, M_X the set of all nonempty and bounded subsets of X. The Kuratowski measure of noncompactness of a set A in M_X, denoted $\alpha(A)$, is defined by

$$\alpha(A) = \inf\left\{\varepsilon > 0\colon \quad A \subset \bigcup_{i=1}^{n} S_i, \quad S_i \subset X, \quad \operatorname{diam} S_i \leq \varepsilon, \quad i = 1, ..., n\right\}.$$

Denote by $\mathcal{L}(X)$ the space of all bounded linear operators of X into X and let $T \in \mathcal{L}(X)$. The Kuratowski measure of noncompactness of an operator T is defined as follows

$$\alpha(T) = \inf\{k \geq 0\colon \alpha(T(A)) \leq k\,\alpha(A) \text{ for any set } A \in M_X\},$$

it can be equivalently defined as

$$\alpha(T) = \sup\left\{\frac{\alpha(T(A))}{\alpha(A)}; A \in M_X, \alpha(A) > 0\right\}.$$

We define the following nonnegative quantity, $\beta(T)$, as (see [11])

$$\beta(T) = \inf\left\{\frac{\alpha(T(A))}{\alpha(A)}; A \in M_X, \alpha(A) > 0\right\}.$$

In what follows, we give some fundamental properties of α, β already given in [11].

Proposition 1. *Let T, S be in $\mathcal{L}(X)$. Then the following claims hold:*

1. $\alpha(\lambda T) = |\lambda|\,\alpha(T)$ *and* $\beta(\lambda T) = |\lambda|\,\beta(T)$ *for all* $\lambda \in \mathbb{C}$.
2. $|\alpha(T) - \alpha(S)| \leqslant \alpha(T + S) \leqslant \alpha(T) + \alpha(S)$.
3. $\beta(T) - \alpha(S) \leqslant \beta(T + S) \leqslant \beta(T) + \alpha(S)$.
4. $\alpha(T \circ S) \leqslant \alpha(T)\alpha(S)$ *and* $\beta(T \circ S) \geqslant \beta(T)\beta(S)$.
5. $\alpha(T) = 0$ *if and only if* T *is compact.*

Recall now the following important operators. Let $T \in \mathcal{L}(X)$, denote by $R(T)$ the range of T and $\ker(T)$ the null space of T. The nullity of T, $n(T)$, is defined as the dimension of $\ker(T)$ and the deficiency of T, $d(T)$, is defined as the codimension of the range $R(T)$ in X. An operator $T \in \mathcal{L}(X)$ is called upper semi Fredholm operator if $R(T)$ is closed and $n(T)$ is finite, it is called lower semi Fredholm operator if $d(T)$ is finite. The set of all upper (respectively lower) semi Fredholm operators is denoted respectively by $\Phi_+(X)$ and $\Phi_-(X)$. $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ is called the set of Fredholm operators and $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$ is called the set of semi Fredholm operators. The index of an operator $T \in \Phi_\pm(X)$ is defined as $\operatorname{ind}(T) = n(T) - d(T)$. The following properties are well known.

Proposition 2. *Let $T \in \mathcal{L}(X)$ and T^* be the adjoint operator of T. Then the following claims hold:*

1. *$\beta(T)>0$ if and only if $T\in\Phi_+(X)$;*
2. *$\beta(T^*)>0$ if and only if $T\in\Phi_-(X)$.*

Let T, $S\in\mathcal{L}(X)$ with S non zero. We define the S-resolvent set of T by

$$\rho_S(T)=\{\lambda\colon\ \lambda S-T \text{ has a bounded inverse }\}$$

and the S-spectrum of T by $\sigma_S(T)=\mathbb{C}\backslash\rho_S(T)$. There are several definitions of the S-essential spectrum of a bounded linear operator on a Banach space, in this paper we are interested with the following definitions:

$$\begin{array}{ll}
\text{S-Wolf} & \sigma_{e_{1,S}}(T)=\{\lambda\in\mathbb{C}\colon (\lambda S-T)\notin\Phi(X)\};\\
\text{S-Gustafson} & \sigma_{e_{2,S}}(T)=\{\lambda\in\mathbb{C}\colon (\lambda S-T)\notin\Phi_+(X)\};\\
\text{S-Weidmann} & \sigma_{e_{3,S}}(T)=\{\lambda\in\mathbb{C}\colon (\lambda S-T)\notin\Phi_-(X)\};\\
\text{S-Kato} & \sigma_{e_{4,S}}(T)=\{\lambda\in\mathbb{C}\colon (\lambda S-T)\notin\Phi_\pm(X)\};\\
\text{S-Schechter} & \sigma_{e_{5,S}}(T)=\mathbb{C}/\Phi_0(T);\\
 & \Phi_0(T)=\{\lambda\colon (\lambda S-T)\in\Phi(X)\text{with } \operatorname{ind}(\lambda S-T)=0\};\\
\text{S-Jeribi} & \sigma_{e_{j,S}}(T)=\bigcap\limits_{W\in\mathcal{W}_*(X)}\sigma_S(T+W);
\end{array}$$

where $\mathcal{W}_*(X)$ stands for each one of the sets of weakly compact operators $\mathcal{W}(X)$ and strictly singular operators $\mathcal{S}(X)$. The following inclusion $\sigma_{e_{j,S}}(T)\subset\sigma_{e_{5,S}}(T)$ is always satisfies since the set $\mathcal{W}_*(X)$ contains the set of compact operators $\mathcal{K}(X)$. In general, we have the following inclusions:

$$\sigma_{e_{4,S}}(T)=\sigma_{e_{2,S}}(T)\cap\sigma_{e_{3,S}}(T)\subseteq\sigma_{e_{1,S}}(T)\subseteq\sigma_{e_{5,S}}(T).$$

Note that if S is the identity operator, we recover the usual definition of essential spectra of a bounded linear operator T; $\sigma_{e_i}(T)$ for $i=1,\dots,5$ and $\sigma_j(T)$. The interested reader may find further results on the essential spectra and S-essential spectra in [10], [9]. When the S-essential spectrum is a non empty set, we define the S-essential spectral radius of T as follows

$$r_{e_i,S}(T)=\sup\{|\lambda|:\lambda\in\sigma_{e_i,S}(T)\};\qquad i=1,\dots,5,j.$$

3. Main results. In this section we present our main results. We give a localization for the spectral radius of some S-essential spectra. In the particular, the S-Gustafson, S-Wolf, S-Weidmann, S-Kato, S-Schechter and S-Jeribi essential spectrum.

We begin by the spectral radius of the S-Gustafson essential spectrum. We have the following main theorem.

Theorem 1. *Let T, S be two bounded operators on X, with S nonzero and noncompact. Then the radius of the S-Gustafson essential spectrum is given by*

$$\frac{\beta(T)}{\alpha(S)}\le r_{e_2,S}(T)\le\frac{\alpha(T)}{\beta(S)}.\tag{3}$$

Proof. Let us consider $\lambda\in\mathbb{C}$ such that $|\lambda|>\frac{\alpha(T)}{\beta(S)}$. Then we have $\beta(\lambda S)-\alpha(T)>0$. Using respectively assertion (1) of Proposition 1 and assertion (2) of Proposition 2, we obtain $(\lambda S-T)\in\Phi_+(X)$, i.e. λ is not in the S-Gustafson essential spectrum $\sigma_{e_2,S}(T)$. That is, if λ belongs to $\sigma_{e_2,S}(T)$ then $|\lambda|\le\frac{\alpha(T)}{\beta(S)}$. Hence

$$r_{e_2,S}(T)\le\frac{\alpha(T)}{\beta(S)}.\tag{4}$$

To prove the last inequality, let $\lambda \in \mathbb{C}$ such that $|\lambda| < \frac{\beta(T)}{\alpha(S)}$. Then we have $\beta(T) - \alpha(\lambda S) > 0$. Using assertion (1) of Proposition 1 and assertion (2) of Proposition 2 respectively, we get $(\lambda S - T) \in \Phi_+(X)$, i.e. $\lambda \notin \sigma_{e_2,S}(T)$. Thus

$$\lambda \in \sigma_{e_2,S}(T) \Rightarrow |\lambda| \geq \frac{\beta(T)}{\alpha(S)}.$$

So, we have

$$r_{e_2,S}(T) \geq \frac{\beta(T)}{\alpha(S)} \tag{5}$$

From the above inequalities (4) and (5), we deduce that

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e_2,S}(T) \leq \frac{\alpha(T)}{\beta(S)}.$$

□

We shall prove Theorem 1 for the S-Weidmann essential spectrum, but first we need to prove the following lemma:

Lemma 1. *Let T, S be two bounded operators on X, with S nonzero and noncompact. Then we have*

$$\frac{\alpha(T)}{\beta(S)} = \frac{\alpha(T^*)}{\beta(S^*)},$$

where T^*, S^* are the adjoint operator of T and S, respectively.

Proof. The following inequalities are well known (see [10])

$$\frac{1}{2}\alpha(T) \leq \alpha(T^*) \leq \alpha(T). \tag{6}$$

For any bounded set A in X with A non relatively compact set, we have

$$\frac{1}{2}\frac{\alpha(S(A))}{\alpha(A)} \leq \frac{\alpha(S^*(A))}{\alpha(A)} \leq \frac{\alpha(S(A))}{\alpha(A)}.$$

This implies that

$$\inf\left\{\frac{\alpha(S^*(A))}{\alpha(A)}, \alpha(A) > 0\right\} \leq \frac{1}{2}\inf\left\{\frac{\alpha(S(A))}{\alpha(A)}, \alpha(A) > 0\right\}$$

and

$$\inf\left\{\frac{\alpha(S(A))}{\alpha(A)}, \alpha(A) > 0\right\} \leq \inf\left\{\frac{\alpha(S^*(A))}{\alpha(A)}, \alpha(A) > 0\right\}.$$

By definition of β, we get $\beta(S^*) \leq \frac{1}{2}\beta(S)$ and $\beta(S) \leq \beta(S^*)$ i.e

$$\frac{2}{\beta(S)} \leq \frac{1}{\beta(S^*)} \leq \frac{1}{\beta(S)}. \tag{7}$$

Timing the inequalities (7) and (6) we get the results. □

The following main theorem holds.

Theorem 2. *Let T, S be two bounded operators on X, with S non zero and non compact. Then the spectral radius of the S-essential spectra is given by*

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e_i,S}(T) \leq \frac{\alpha(T)}{\beta(S)} \quad \text{for } i = 3, 4, 1. \tag{8}$$

Proof. First, we prove the inequalities (8) for the S-Weidmann essential spectrum. Let $\lambda \in \mathbb{C}$ such that $|\lambda| < \frac{\beta(T^*)}{\alpha(S^*)}$, then $\beta(T^*) - \alpha(\lambda S^*) > 0$. From proposition (2), this implies that $\beta((\lambda S - T)^*) > 0$. Using properties of β, we see that $(\lambda S - T) \in \Phi_-(X)$. i.e $\lambda \notin \sigma_{e_3,S}(T)$. Therefore, if $\lambda \in \sigma_{e_3,S}(T)$ then $|\lambda| \geq \frac{\alpha(T^*)}{\beta(S^*)}$. By the use of Lemma (1) we conclude that

$$r_{e_3,S}(T) \geqslant \frac{\alpha(T)}{\beta(S)}. \tag{9}$$

In order to prove the other inequality, let us take $\lambda \in \mathbb{C}$ such that $|\lambda| > \frac{\alpha(T^*)}{\beta(S^*)}$. Then $\beta(\lambda S^*) > \alpha(T^*)$. From assertion (1) of proposition (1) and assertion (2) of proposition (2) respectively, we get $(\lambda S - T) \in \Phi_-(X)$. This means that $\lambda \notin \sigma_{e_3,S}(T)$. Consequently, by the use of Lemme (1) we get the following implication

$$\lambda \in \sigma_{e_3,S}(T) \Rightarrow |\lambda| \leqslant \frac{\alpha(T)}{\beta(S)} \tag{10}$$

Hence

$$r_{e_3,S}(T) \leqslant \frac{\alpha(T)}{\beta(S)}. \tag{11}$$

From the two inequalities (9) and (11), we get a localization for the S-Weidmann essential spectrum

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e_3,S}(T) \leq \frac{\alpha(T)}{\beta(S)}.$$

Now, we prove the above inequalities for the S-Wolf essential spectrum. From the inclusion $\sigma_{e_2,S}(T) \subset \sigma_{e_1,S}(T)$ we have $r_{e_2,S}(T) \leq r_{e_1,S}(T)$. So the following inequality holds

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e_1,S}(T). \tag{12}$$

For the last inequality, let us suppose $\lambda \in \mathbb{C}$ such that $|\lambda| > \frac{\alpha(T)}{\beta(S)}$. Then using Lemme (1), we have $|\lambda| > \frac{\alpha(T)}{\beta(S)} = \frac{\alpha(T^*)}{\beta(S^*)}$. i.e $|\lambda|\,\beta(S) > \alpha(T)$ and $|\lambda|\,\beta(S^*) > \alpha(T^*)$. Using properties of β we get $\beta(\lambda S - T) > 0$ and $\beta(\lambda S^* - T^*) > 0$. This implies that $(\lambda S - T) \in \Phi_+(X) \cap \Phi_-(X) = \Phi(X)$. Hence $\lambda \notin \sigma_{e_1,S}(T)$. Therefore

$$\lambda \in \sigma_{e_1,S}(T) \Rightarrow |\lambda| \leq \frac{\alpha(T)}{\beta(S)}$$

Consequently we have the following inequality

$$r_{e_1,S}(T) \leq \frac{\alpha(T)}{\beta(S)}. \tag{13}$$

From the two inequalities (13) and (12), we get the results.

In what follows, we prove the above localization for the S-Kato essential spectrum. From the equality $\sigma_{e_4,S}(T) = \sigma_{e_2,S}(T) \cap \sigma_{e_3,S}(T)$, we infer that $r_{e_4,S}(T) \leqslant \frac{\alpha(T)}{\beta(S)}$. Now it suffices to show that $r_{e_4,S}(T) \geq \frac{\beta(T)}{\alpha(S)}$. Suppose that $|\lambda| < \frac{\beta(T)}{\alpha(S)}$, then using properties of β we see that $(\lambda S - T) \in \Phi_+(X)$. Consequently $\lambda \notin \sigma_{e_4,S}(T)$. We deduce that, if λ belongs to $\sigma_{e_4,S}(T)$ then $|\lambda| \geq \frac{\alpha(T)}{\beta(S)}$. Hence $r_{e_4,S}(T) \geq \frac{\alpha(T)}{\beta(S)}$. □

In the following theorem, we prove formulae (2) for the S-Schechter essential spectrum.

Theorem 3. *Let T, S be two bounded operators on a Banach space X, with S non zero and non compact. Suppose that ind $S = 0$. Then the spectral radius of the S-Schechter essential spectrum is given by*

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e_5,S}(T) \leq \frac{\alpha(T)}{\beta(S)}. \tag{14}$$

Proof. It follows immediately from the inclusion $\sigma_{e_{1,S}}(T) \subset \sigma_{e_{5,S}}(T)$ that $\frac{\beta(T)}{\alpha(S)} \leqslant r_{e_5,S}(T)$. In order to prove the last inequality, let us take $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant \frac{\alpha(T)}{\beta(S)}$. Using [1, Theorem 2.2], we obtain that $(\lambda S - T) \in \Phi_+(X)$ and $\operatorname{ind}(\lambda S - T) = \operatorname{ind}(\lambda S)$. According to the hypothesis ensuring ind $S = 0$, it follows that $(\lambda S - T) \in \Phi(X)$ with $\operatorname{ind}(\lambda S - T) = 0$. Hence $\lambda \in \sigma_{e_{1,S}}(T)$ implies that $|\lambda| \leq \frac{\alpha(T)}{\beta(S)}$. We deduce that $r_{e_5,S}(T) \leq \frac{\alpha(T)}{\beta(S)}$. □

We end this section by the spectral radius of the S-Jeribi essential spectrum. In the definition of the S-Jeribi essential spectrum, we restrict K belonging to $\mathcal{W}(X)$ only since X is a Banach space. The other main results is the following theorems:

Theorem 4. *Let X be a Banach space which has no reflexive infinite dimensional subspaces and $T \in \mathcal{L}(X)$. Then we have*

$$\sigma_{e_{i,S}}(T) \subset \sigma_{j,S}(T), \quad i = 1, \dots, 4.$$

Proof. Reasoning in the same way as the proof of [6, Theorem 3.3]. □

The following main theorem gives a localization for the spectral radius of the S-Jeribi essential spectrum in the particular case when the Banach space X has no reflexive infinite dimensional subspaces.

Theorem 5. *Let X be a Banach space which has no reflexive infinite dimensional subspaces, $T, S \in \mathcal{L}(X)$ and ind $S = 0$. Then the spectral radius of the S-Jeribi essential spectrum satisfies formulae (2).*

Proof. From the inclusion $\sigma_{e_{j,S}}(T) \subset \sigma_{e_{5,S}}(T)$, we infer that $r_{e_{j,S}}(T) \leq \frac{\alpha(T)}{\beta(S)}$. The last inequality is satisfies by using Theorem 4. □

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Laboratory of Pure and Applied Mathematics
Department of mathematics, University of Laghouat, Algeria
c.belabbaci@lagh-univ.dz

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