
In this paper, we study the spectral radius of some S-essential spectra of a bounded linear operator defined on a Banach space. More precisely, via the concept of measure of noncompactness, we show that for any two bounded linear operators $T$ and $S$ with $S$ non zero and non compact operator the spectral radius of the S-Gustafson, S-Weidmann, S-Kato and S-Wolf essential spectra are given by the following inequalities

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e,S}(T) \leq \frac{\alpha(T)}{\beta(S)},$$

(1)

where $\alpha(.)$ stands for the Kuratowski measure of noncompactness and $\beta(.)$ is defined in [11]. In the particular case when the index of the operator $S$ is equal to zero, we prove the last inequalities for the spectral radius of the S-Schechter essential spectrum. Also, we prove that the spectral radius of the S-Jeribi essential spectrum satisfies inequalities (2) when the Banach space $X$ has no reflexive infinite dimensional subspace and the index of the operator $S$ is equal to zero (the S-Jeribi essential spectrum, introduced in [7] as a generalisation of the Jeribi essential spectrum).

1. Introduction. The spectral theory of operators pencil $(\lambda S - T)$ (operator-valued functions of a complex argument) play a crucial role in many branches of mathematical physics see, for example, [2, 5, 4, 12].

The purpose of this paper is to study the spectral radius of some S-essential spectra of a bounded linear operator on a Banach space $X$. More precisely, we give a localization of the spectral radius of the S-essential spectra of an operator $T$ via the concept of measure of noncompactness when $S$ is a non compact and non zero bounded linear operator and $T$ is any bounded linear operator. This work continues research begun in [7], here we aim to prove that the spectral radius of the S-Gustafson, S-Weidmann, S-Kato and S-Wolf essential spectrum is given by the following inequalities:

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e,S}(T) \leq \frac{\alpha(T)}{\beta(S)},$$

(2)

where $\alpha(.)$ stands for the Kuratowski measure of noncompactness and $\beta(.)$ is defined in the next section.

When the index of the operator $S$ is equal to zero, then the inequalities (2) holds for the S-Schechter essential spectrum. Also, we prove the above inequalities (2) for the S-Jeribi essential spectrum, introduced in [7] as a generalisation of the Jeribi essential spectrum, in the particular case when the Banach space $X$ has no reflexive infinite dimensional subspace.

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The paper is organised as follows. Section 2 contains an overview of the necessary background and Section 3 contains the main results.

2. Preliminaries. The notion of measure of non-compactness have been successfully applied in many different domains in mathematics; in fixed point theory, differential equations, functional equations, integral and integro-differential equations, etc. (see for example [14, 13, 3, 8]). In order to recall this notion denote by $X$ a Banach space, $M_X$ the set of all nonempty and bounded subsets of $X$. The Kuratowski measure of noncompactness of a set $A$ in $M_X$, denoted $\alpha(A)$, is defined by

$$\alpha(A) = \inf \left\{ \varepsilon > 0 : A \subset \bigcup_{i=1}^{n} S_i, \quad S_i \subset X, \quad \text{diam} S_i \leq \varepsilon, \quad i = 1, \ldots, n \right\}.$$ 

Denote by $\mathcal{L}(X)$ the space of all bounded linear operators of $X$ into $X$ and let $T \in \mathcal{L}(X)$. The Kuratowski measure of noncompactness of an operator $T$ is defined as follows

$$\alpha(T) = \inf \{ k \geq 0 : \alpha(T(A)) \leq k \alpha(A) \text{ for any set } A \in M_X \},$$

it can be equivalently defined as

$$\alpha(T) = \sup \left\{ \frac{\alpha(T(A))}{\alpha(A)} ; A \in M_X, \alpha(A) > 0 \right\}.$$

We define the following nonnegative quantity, $\beta(T)$, as (see [11])

$$\beta(T) = \inf \left\{ \frac{\alpha(T(A))}{\alpha(A)} ; A \in M_X, \alpha(A) > 0 \right\}.$$ 

In what follows, we give some fundamental properties of $\alpha$, $\beta$ already given in [11].

**Proposition 1.** Let $T$, $S$ be in $\mathcal{L}(X)$. Then the following claims hold:

1. $\alpha(\lambda T) = |\lambda| \alpha(T)$ and $\beta(\lambda T) = |\lambda| \beta(T)$ for all $\lambda \in \mathbb{C}$.
2. $|\alpha(T) - \alpha(S)| \leq \alpha(T + S) \leq \alpha(T) + \alpha(S)$.
3. $\beta(T) - \alpha(S) \leq \beta(T + S) \leq \beta(T) + \alpha(S)$.
4. $\alpha(T \circ S) \leq \alpha(T) \alpha(S)$ and $\beta(T \circ S) \geq \beta(T) \beta(S)$.
5. $\alpha(T) = 0$ if and only if $T$ is compact.

Recall now the following important operators. Let $T \in \mathcal{L}(X)$, denote by $R(T)$ the range of $T$ and $\ker(T)$ the null space of $T$. The nullity of $T$, $n(T)$, is defined as the dimension of $\ker(T)$ and the deficiency of $T$, $d(T)$, is defined as the codimension of the range $R(T)$ in $X$. An operator $T \in \mathcal{L}(X)$ is called upper semi Fredholm operator if $R(T)$ is closed and $n(T)$ is finite, it is called lower semi Fredholm operator if $d(T)$ is finite. The set of all upper (respectively lower) semi Fredholm operators is denoted respectively by $\Phi_+(X)$ and $\Phi_-(X)$. $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ is called the set of Fredholm operators and $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$ is called the set of semi Fredholm operators. The index of an operator $T \in \Phi_\pm(X)$ is defined as $\text{ind}(T) = n(T) - d(T)$. The following properties are well known.

**Proposition 2.** Let $T \in \mathcal{L}(X)$ and $T^*$ be the adjoint operator of $T$. Then the following claims hold: 
1. $\beta(T) > 0$ if and only if $T \in \Phi_+(X)$;

2. $\beta(T^*) > 0$ if and only if $T \in \Phi_-(X)$. 

Let $T, S \in \mathcal{L}(X)$ with $S$ non zero. We define the S-resolvent set of $T$ by 

$$\rho_S(T) = \{ \lambda : \lambda S - T \text{ has a bounded inverse} \}$$

and the S-spectrum of $T$ by $\sigma_S(T) = \mathbb{C} \setminus \rho_S(T)$. There are several definitions of the S-essential spectrum of a bounded linear operator on a Banach space, in this paper we are interested with the following definitions:

- **S-Wolf**
  $$\sigma_{e_1,S}(T) = \{ \lambda \in \mathbb{C} : (\lambda S - T) \notin \Phi(X) \};$$

- **S-Gustafson**
  $$\sigma_{e_2,S}(T) = \{ \lambda \in \mathbb{C} : (\lambda S - T) \notin \Phi_+(X) \};$$

- **S-Weidmann**
  $$\sigma_{e_3,S}(T) = \{ \lambda \in \mathbb{C} : (\lambda S - T) \notin \Phi_-(X) \};$$

- **S-Kato**
  $$\sigma_{e_4,S}(T) = \{ \lambda \in \mathbb{C} : (\lambda S - T) \notin \Phi_\pm(X) \};$$

- **S-Schechter**
  $$\sigma_{e_5,S}(T) = \mathbb{C}/\Phi_0(T);$$

- **S-Jeribi**
  $$\sigma_{e_j,S}(T) = \bigcap_{W \in \mathcal{W}_i(X)} \sigma_S(T + W);$$

where $\mathcal{W}_i(X)$ stands for each one of the sets of weakly compact operators $W(X)$ and strictly singular operators $S(X)$. The following inclusion $\sigma_{e_1,S}(T) \subset \sigma_{e_5,S}(T)$ is always satisfies since the set $\mathcal{W}_i(X)$ contains the set of compact operators $K(X)$. In general, we have the following inclusions:

$$\sigma_{e_4,S}(T) = \sigma_{e_2,S}(T) \cap \sigma_{e_3,S}(T) \subseteq \sigma_{e_1,S}(T) \subseteq \sigma_{e_5,S}(T).$$

Note that if $S$ is the identity operator, we recover the usual definition of essential spectrum of a bounded linear operator $T$; $\sigma_e(T)$ for $i = 1, \ldots, 5$ and $\sigma_j(T)$. The interested reader may find further results on the essential spectra and S-essential spectra in [10], [9]. When the S-essential spectrum is a non empty set, we define the S-essential spectral radius of $T$ as follows

$$r_{e_i,S}(T) = \sup \{ |\lambda| : \lambda \in \sigma_{e_i,S}(T) \}; \quad i = 1, \ldots, 5, j.$$

### 3. Main results

In this section we present our main results. We give a localization for the spectral radius of some S-essential spectra. In the particular, the S-Gustafson, S-Wolf, S-Weidmann, S-Kato, S-Schechter and S-Jeribi essential spectrum.

We begin by the spectral radius of the S-Gustafson essential spectrum. We have the following main theorem.

**Theorem 1.** Let $T, S$ be two bounded operators on $X$, with $S$ nonzero and noncompact. Then the radius of the S-Gustafson essential spectrum is given by

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e_{2,S}}(T) \leq \frac{\alpha(T)}{\beta(S)}.$$  \hspace{1cm} (3)

**Proof.** Let us consider $\lambda \in \mathbb{C}$ such that $|\lambda| > \frac{\alpha(T)}{\beta(S)}$. Then we have $\beta(\lambda S) - \alpha(T) > 0$. Using respectively assertion (1) of Proposition 1 and assertion (2) of Proposition 2, we obtain $(\lambda S - T) \in \Phi_+(X)$, i.e. $\lambda$ is not in the S-Gustafson essential spectrum $\sigma_{e_{2,S}}(T)$. That is, if $\lambda$ belongs to $\sigma_{e_2,S}(T)$ then $|\lambda| \leq \frac{\alpha(T)}{\beta(S)}$. Hence

$$r_{e_{2,S}}(T) \leq \frac{\alpha(T)}{\beta(S)}.$$  \hspace{1cm} (4)
To prove the last inequality, let \( \lambda \in \mathbb{C} \) such that \( |\lambda| < \frac{\beta(T)}{\alpha(S)} \). Then we have \( \beta(T) - \alpha(\lambda S) > 0 \). Using assertion (1) of Proposition 1 and assertion (2) of Proposition 2 respectively, we get \( (\lambda S - T) \in \Phi_+(X) \), i.e. \( \lambda \notin \sigma_{e_2}(T) \). Thus

\[
\lambda \in \sigma_{e_2}(T) \Rightarrow |\lambda| \geq \frac{\beta(T)}{\alpha(S)}.
\]

So, we have

\[
r_{e_2}(T) \geq \frac{\beta(T)}{\alpha(S)} \tag{5}
\]

From the above inequalities (4) and (5), we deduce that

\[
\frac{\beta(T)}{\alpha(S)} \leq r_{e_2}(T) \leq \frac{\alpha(T)}{\beta(S)}.
\]

We shall prove Theorem 1 for the S-Weidmann essential spectrum, but first we need to prove the following lemma:

**Lemma 1.** Let \( T, S \) be two bounded operators on \( X \), with \( S \) nonzero and noncompact. Then we have

\[
\frac{\alpha(T)}{\beta(S)} = \frac{\alpha(T^*)}{\beta(S^*)},
\]

where \( T^*, S^* \) are the adjoint operator of \( T \) and \( S \), respectively.

**Proof.** The following inequalities are well known (see [10])

\[
\frac{1}{2} \alpha(T) \leq \alpha(T^*) \leq \alpha(T).
\]

For any bounded set \( A \) in \( X \) with \( A \) non relatively compact set, we have

\[
\frac{1}{2} \frac{\alpha(S(A))}{\alpha(A)} \leq \frac{\alpha(S^*(A))}{\alpha(A)} \leq \frac{\alpha(S(A))}{\alpha(A)}.
\]

This implies that

\[
\inf \left\{ \frac{\alpha(S^*(A))}{\alpha(A)}, \alpha(A) > 0 \right\} \leq \frac{1}{2} \inf \left\{ \frac{\alpha(S(A))}{\alpha(A)}, \alpha(A) > 0 \right\}
\]

and

\[
\inf \left\{ \frac{\alpha(S(A))}{\alpha(A)}, \alpha(A) > 0 \right\} \leq \inf \left\{ \frac{\alpha(S^*(A))}{\alpha(A)}, \alpha(A) > 0 \right\}.
\]

By definition of \( \beta \), we get \( \beta(S^*) \leq \frac{1}{2} \beta(S) \) and \( \beta(S) \leq \beta(S^*) \) i.e

\[
\frac{2}{\beta(S)} \leq \frac{1}{\beta(S^*)} \leq \frac{1}{\beta(S)} \tag{7}
\]

Timing the inequalities (7) and (6) we get the results. The following main theorem holds.
Theorem 2. Let $T, S$ be two bounded operators on $X$, with $S$ non zero and non compact. Then the spectral radius of the $S$-essential spectrum is given by

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e_i,S}(T) \leq \frac{\alpha(T)}{\beta(S)} \quad \text{for } i = 3, 4, 1.$$  \hfill (8)

Proof. First, we prove the inequalities (8) for the $S$-Weidmann essential spectrum. Let $\lambda \in \mathbb{C}$ such that $|\lambda| < \frac{\beta(T^*)}{\alpha(S^*)}$, then $\beta(T^*) - \alpha(\lambda S^*) > 0$. From proposition (2), this implies that $\beta((\lambda S - T)^*) > 0$. Using properties of $\beta$, we see that $(\lambda S - T) \in \Phi_-(X)$, i.e. $\lambda \notin \sigma_{e_3,S}(T)$. Therefore, if $\lambda \in \sigma_{e_3,S}(T)$ then $|\lambda| \geq \frac{\alpha(T)}{\beta(S)}$. By the use of Lemma (1) we conclude that

$$r_{e_3,S}(T) \geq \frac{\alpha(T)}{\beta(S)}.$$  \hfill (9)

In order to prove the other inequality, let us take $\lambda \in \mathbb{C}$ such that $|\lambda| > \frac{\alpha(T^*)}{\beta(S^*)}$. Then $\beta(\lambda S^*) > \alpha(T^*)$. From assertion (1) of proposition (1) and assertion (2) of proposition (2) respectively, we get $(\lambda S - T) \in \Phi_-(X)$. This means that $\lambda \notin \sigma_{e_3,S}(T)$. Consequently, by the use of Lemme (1) we get the following implication

$$\lambda \in \sigma_{e_3,S}(T) \Rightarrow |\lambda| \leq \frac{\alpha(T)}{\beta(S)}.$$  \hfill (10)

Hence

$$r_{e_3,S}(T) \leq \frac{\alpha(T)}{\beta(S)}.$$  \hfill (11)

From the two inequalities (9) and (11), we get a localization for the $S$-Weidmann essential spectrum

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e_3,S}(T) \leq \frac{\alpha(T)}{\beta(S)}.$$  \hfill (12)

Now, we prove the above inequalities for the $S$-Wolf essential spectrum. From the inclusion $\sigma_{e_2,S}(T) \subset \sigma_{e_1,S}(T)$ we have $r_{e_2,S}(T) \leq r_{e_1,S}(T)$. So the following inequality holds

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e_1,S}(T).$$  \hfill (13)

For the last inequality, let us suppose $\lambda \in \mathbb{C}$ such that $|\lambda| > \frac{\alpha(T)}{\beta(S)}$. Then using Lemme (1), we have $|\lambda| > \frac{\alpha(T)}{\beta(S)} = \frac{\alpha(T^*)}{\beta(S^*)}$, i.e. $|\lambda| |\beta(S)| > \alpha(T)$ and $|\lambda| |\beta(S^*)| > \alpha(T^*)$. Using properties of $\beta$ we get $\beta(\lambda S - T) > 0$ and $\beta(\lambda S^* - T^*) > 0$. This implies that $(\lambda S - T) \in \Phi_+(X) \cap \Phi_-(X) = \Phi(X)$. Hence $\lambda \notin \sigma_{e_1,S}(T)$. Therefore

$$\lambda \in \sigma_{e_1,S}(T) \Rightarrow |\lambda| \leq \frac{\alpha(T)}{\beta(S)}.$$  

Consequently we have the following inequality

$$r_{e_1,S}(T) \leq \frac{\alpha(T)}{\beta(S)}.$$  \hfill (13)

From the two inequalities (13) and (12), we get the results.
Theorem 4. Let $T, S$ be two bounded operators on a Banach space $X$, with $S$ non zero and non compact. Suppose that $\text{ind} S = 0$. Then the spectral radius of the S-Schechter essential spectrum is given by

$$\frac{\beta(T)}{\alpha(S)} \leq r_{e_5,S}(T) \leq \frac{\alpha(T)}{\beta(S)}. \quad (14)$$

Proof. It follows immediately from the inclusion $\sigma_{e_1,S}(T) \subset \sigma_{e_5,S}(T)$ that $\frac{\beta(T)}{\alpha(S)} \leq r_{e_5,S}(T)$. In order to prove the last inequality, let us take $\lambda \in \mathbb{C}$ such that $|\lambda| \geq \frac{\alpha(T)}{\beta(S)}$. Using [1, Theorem 2.2], we obtain that $(\lambda S - T) \in \Phi_+(X)$ and $\text{ind}(\lambda S - T) = \text{ind}(\lambda S)$. According to the hypothesis ensuring $\text{ind} S = 0$, it follows that $(\lambda S - T) \in \Phi(X)$ with $\text{ind}(\lambda S - T) = 0$. Hence $\lambda \in \sigma_{e_1,S}(T)$ implies that $|\lambda| \leq \frac{\alpha(T)}{\beta(S)}$. We deduce that $r_{e_5,S}(T) \leq \frac{\alpha(T)}{\beta(S)}$. \qed

In the following theorem, we prove formulae (2) for the S-Schechter essential spectrum.

**Theorem 3.** Let $T, S$ be two bounded operators on a Banach space $X$, with $S$ non zero and non compact. Suppose that $\text{ind} S = 0$. Then the spectral radius of the S-Jeribi essential spectrum is given by

$$|\lambda| = \frac{\alpha(T)}{\beta(S)} \quad \text{and} \quad \text{ind} \left( T \right) = \lambda S.$$ 

Proof. Reasoning in the same way as the proof of [6, Theorem 3.3]. \qed

We end this section by the spectral radius of the S-Jeribi essential spectrum. In the definition of the S-Jeribi essential spectrum, we restrict $K$ belonging to $\mathcal{W}(X)$ only since $X$ is a Banach space. The other main results is the following theorems:

**Theorem 4.** Let $X$ be a Banach space which has no reflexive infinite dimensional subspaces and $T \in \mathcal{L}(X)$. Then we have

$$\sigma_{e_i,S}(T) \subset \sigma_{j,S}(T), \quad i = 1, \ldots, 4.$$ 

Proof. Reasoning in the same way as the proof of [6, Theorem 3.3]. \qed

The following main theorem gives a localization for the spectral radius of the S-Jeribi essential spectrum in the particular case when the Banach space $X$ has no reflexive infinite dimensional subspaces.

**Theorem 5.** Let $X$ be a Banach space which has no reflexive infinite dimensional subspaces, $T, S \in \mathcal{L}(X)$ and $\text{ind} S = 0$. Then the spectral radius of the S-Jeribi essential spectrum satisfies formulae (2).

Proof. From the inclusion $\sigma_{e_1,S}(T) \subset \sigma_{e_5,S}(T)$, we infer that $r_{e_1,S}(T) \leq \frac{\alpha(T)}{\beta(S)}$. The last inequality is satisfies by using Theorem 4. \qed

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