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POINTWISE ESTIMATES FOR THE DERIVATIVE OF ALGEBRAIC POLYNOMIALS

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We give a sufficient condition on coefficients a_k of an algebraic polynomial $P(z)=\sum\limits_{k=0}^{n} a_k z^k$, $a_n\neq 0$, such that the pointwise Bernstein inequality $|P'(z)|\le n|P(z)|$ is true for all $z,\ |z|\le 1$.

1. Introduction and main result. Let P be an algebraic polynomial with complex coefficients, and let $z_1, z_2, \dots, z_m$ be distinct zeros of P with multiplicities $r_1, r_2, \dots, r_m$, respectively, enumerated in ascending order of their moduli $|z_1|\le|z_2|\le\dots\le|z_m|$; $\sum_{k=1}^m r_k=\deg P$. Here and in what follows, we assume that $\sum_{k=1}^0=0$.

Consider the real part of the logarithmic derivative of P

$$\operatorname{Re}\frac{zP'(z)}{P(z)}=\operatorname{Re}\sum_{k=1}^{m}\frac{r_k z}{z-z_k}=\frac{n}{2}+\frac{1}{2}\sum_{k=1}^{m} r_k\frac{|z|^2-|z_k|^2}{|z-z_k|^2}, \tag{1}$$

where $n=\deg P$. Since $|\operatorname{Re} w|\le|w|,\ w\in\mathbb{C}$, for all $z\in\mathbb{C}\setminus\{z_1,\dots,z_m\}$ we obtain

$$\Big|\frac{n}{2}+\frac{1}{2}\sum_{k=1}^{m} r_k\frac{|z|^2-|z_k|^2}{|z-z_k|^2}\Big|\le\Big|\frac{zP'(z)}{P(z)}\Big|. \tag{2}$$

Denote $\mathbb{D}:=\{z\in\mathbb{D}: |z|<1\}$, $\mathbb{T}:=\{z\in\mathbb{C}: |z|=1\}$. Assume that $z_k\notin\mathbb{T}$, $k\in\{1,\dots,m\}$. By the Cauchy theorem, $\sum_{k=1}^j r_k=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{P'(z)}{P(z)}dz$, therefore

$$\sum_{k=1}^{j} r_k\le\frac{1}{2\pi}\int_0^{2\pi}\Big|\frac{P'(e^{i\theta})}{P(e^{i\theta})}\Big|d\theta\le\max\Big\{\Big|\frac{P'(z)}{P(z)}\Big|\colon z\in\mathbb{T}\Big\}, \tag{3}$$

where $j\le m$ is the non-negative integer such that $|z_j|<1<|z_{j+1}|$.

From (2), (3) and the following Bernstein inequality

$$\max\{|P'(z)|\colon z\in\mathbb{T}\}\le n\max\{|P(z)|\colon z\in\mathbb{T}\}, \tag{4}$$

we readily conclude that for any algebraic polynomial P, $\deg P=n$, having all its zeros in $\mathbb{D}$, the following inequalities hold $\frac{n}{1+|z_m|}\le\min\Big\{\Big|\frac{P'(z)}{P(z)}\Big|\colon z\in\mathbb{T}\Big\}\le n\le\max\Big\{\Big|\frac{P'(z)}{P(z)}\Big|\colon z\in\mathbb{T}\Big\}$.

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The first inequality was observed by Govil [1], the second one is the consequence of (4) and the third one is the consequence of (3). All these results are sharp. The equalities are attained for the polynomial $P(z) = a_n(z-c)^n$ and suitable $c \in \mathbb{D}$.

Assume that all zeros $z_1, \dots, z_m$ of P lie in the domain $\mathbb{U} := \{z \in \mathbb{C} : |z| \geq 1\}$. Then it follows from (1) that $\operatorname{Re}\frac{zP'(z)}{P(z)} \leq n/2$ for all $z \in \overline{\mathbb{D}} \setminus \{z_1, \dots, z_m\}$. Aziz [2] noted that this gives (see also Lemma 1 below), $|zP'(z)| \leq |nP(z) - zP'(z)|$ for all $z \in \overline{\mathbb{D}}$.

It is easy to see that if P is a polynomial of degree n having all its zeros in $\mathbb{U}_2 := \{z \in \mathbb{C} : |z| \geq 2\}$, then $\max\left\{\left|\frac{zP'(z)}{P(z)}\right| : z \in \overline{\mathbb{D}}\right\} \leq \sum_{k=1}^{m} \frac{r_k}{|z_k|-1} \leq n$. This is equivalent to $|zP'(z)| \leq n|P(z)|$ for all $z \in \overline{\mathbb{D}}$. We will call the last relation the *pointwise Bernstein inequality*. Combining *Aziz's inequality* $|zP'(z)| \leq |nP(z) - zP'(z)|$ and the pointwise Bernstein inequality, in the case $\{z_1, \dots, z_m\} \in \mathbb{U}_2$ we obtain that for all $z \in \overline{\mathbb{D}}$

$$|zP'(z)| \leq \min\{|nP(z) - zP'(z)|, n|P(z)|\}. \tag{5}$$

In this note we give the sufficient condition on coefficients of the polynomial P such that the pointwise Bernstein inequality is true for all $z \in \overline{\mathbb{D}}$. As we will see, our condition implies (5) and does not require that all zeros of P lie in $\mathbb{U}_2$.

For further information about the estimates of derivative and the logarithmic derivative of polynomials we refer to [3–6] and references therein.

Our main result is the following theorem.

Theorem 1. *Let $n \in \mathbb{Z}_+$ and $\{k_\nu\}_{\nu=0}^n$, $0 \leq k_0 < k_1 < \ldots < k_n$, be positive integers and let $P(z) = \sum_{\nu=0}^n a_\nu z^{k_\nu}$ be an algebraic polynomial of degree k_n with coefficients $\{a_\nu\}_{\nu=0}^n \in \mathbb{C} \setminus \{0\}$. If*

$$\min_{t \in \overline{\mathbb{D}}} \operatorname{Re} \sum_{j=0}^{n-\nu} \frac{a_{j+\nu}}{a_\nu} t^{k_{j+\nu} - k_\nu} \geq \frac{1}{2}, \quad \nu = 0, 1, \dots, n, \tag{6}$$

then the following assertions hold:
(i) The polynomial P has no zeros in $\overline{\mathbb{D}}$, provided $k_0 = 0$, and has no zeros in $\overline{\mathbb{D}} \setminus \{0\}$ for $k_0 > 0$.
(ii) For all $z \in \overline{\mathbb{D}}$

$$|zP'(z)| \leq k_n |P(z)|. \tag{7}$$

For $z \in \mathbb{D}$ the equality in this inequality is attained only in the case $n = 0$, that is for $P(z) = a_0 z^{k_0}$, $k_0 > 0$.
(iii) for $k_0 = 0$ and for all $n \geq 1$, $z \in \mathbb{D}$ we have $|P'(z)| < k_n|P(z)|$.

Remark 1. Let P be chosen as in Theorem 1. Then we have the implication $(ii) \Rightarrow (i)$.

This is a consequence of the Riemann theorem on removable singularities applied to the function $z \mapsto \frac{zP'(z)}{P(z)} = \sum_{k=1}^{m} \frac{r_k z}{z - z_k}$.

Corollary 1. *Let P be as in Theorem 1 with $a_0 \geq a_1 \geq \ldots \geq a_n > 0$, $n \in \mathbb{N}$ and $k_0 = 0$. If*

$$0 \leq \Delta^2(a_\nu) := \begin{cases} a_{\nu+2} - 2a_{\nu+1} + a_\nu, & \text{if } \nu = 0, 1, \dots, n-2, \\ a_{n-1} - 2a_n, & \text{if } \nu = n-1, \\ a_n, & \text{if } \nu = n, \end{cases}$$

then $|zP'(z)| \leq \min\left\{|k_n P(z) - zP'(z)|, k_n|P(z)|\right\}$ for all $z \in \overline{\mathbb{D}}$.

We denote $\lambda_{k,\nu} = \begin{cases} \frac{a_{k+\nu}}{a_\nu}, \ 0 \le k \le n-\nu, \\ 0, \ k = n-\nu+1. \end{cases}$ For each $\nu \in \{0,1,\dots,n\}$ the sequence $\{\lambda_{k,\nu}\}_{k=0}^{n-\nu+1}$ is non-negative, monotonically non-increasing and convex, i.e. $\lambda_{0,\nu} \ge \lambda_{1,\nu} \ge \dots \ge \lambda_{n-\nu,\nu} > \lambda_{n-\nu+1,\nu} = 0$ and $\Delta^2(\lambda_{k,\nu}) \ge 0$ for $k = 0,1,\dots,n-\nu+1$. Thus by the Fejér Theorem (see [3, p.310]) the trigonometric polynomials $\frac{\lambda_{0,\nu}}{2} + \sum_{k=1}^{n-\nu} \lambda_{k,\nu} \cos kx$, $\nu = 0,1,\dots,n$, are non-negative for all $x \in \mathbb{R}$. This is equivalent to the condition (6).

Example 1. Let $n \in \mathbb{N} \setminus \{1\}$ and $P(z) = \sum_{k=0}^n (n+1-k) z^k$. Then for $t = \mathrm{e}^{\mathrm{i}x}$, $x \in \mathbb{R}$, we have

$$\frac{1}{2} + \operatorname{Re} \sum_{k=1}^{n-\nu} \frac{n+1-(k+\nu)}{n+1-\nu} t^k = \frac{1}{2} + \sum_{k=1}^{n-\nu} \big(1 - \frac{k}{n+1-\nu}\big) \cos kx = F_{n-\nu+1}(x) \ge 0,$$

for all $x \in \mathbb{R}$, $\nu \in \{0,1,\dots,n\}$, where F_k is the Fejér kernel (see [3, p.311, p.313]).

Therefore, combining Aziz's inequlity and (7), we obtain

$$\Big|\sum_{k=1}^{n} (n+1-k) k z^k\Big| \le \min\left\{\Big|\sum_{k=0}^{n-1} (n+1-k)(n-k) z^k\Big|, n\Big|\sum_{k=0}^{n} (n+1-k) z^k\Big|\right\}.$$

By the Eneström–Kakeya's Theorem (see [4, p.255]) with refinement given by Anderson, Saff and Varga [7, Corollary 2], zeros of P satisfy $|z_k| < 2$, $k \in \{1,\dots,n\}$.

2. Lemmas. For the proof of Theorem 1 we require the following lemmas.

Lemma 1. *Let P and Q be functions defined on a compact set $K \subset \mathbb{C}$, $\mathcal{Z}(Q) := \{z \in \mathbb{C} : Q(z) = 0\}$ and $K \setminus \mathcal{Z}(Q) \ne \varnothing$. In order that $|P(z) - Q(z)| \le |P(z)|$ for all $z \in K$ it is necessary and sufficient that* $\inf\{\operatorname{Re} \frac{P(z)}{Q(z)} : z \in K \setminus \mathcal{Z}(Q)\} \ge \frac{1}{2}$.

Proof. The assertion readily follows from the obvious identity $|w|^2 - |w-1|^2 = 2\operatorname{Re} w - 1$, for $w = \frac{P(z)}{Q(z)}$ with $z \in K \setminus \mathcal{Z}(Q)$. □

Lemma 2. *Let $P(z) = \sum_{j=0}^n a_j z^j$, $n \in \mathbb{N}$, and $a_n \ne 0$. Then for all $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$ we have*

$$\left|z \frac{P(z) - P(w)}{z - w}\right| \le A(z,w) \max\Big\{\Big|P(z) - \sum_{j=0}^{k} a_j z^j\Big| : k \in \{0,\dots,n-1\}\Big\},$$

$A(z,w) = \begin{cases} \frac{|z|^n - |w|^n}{|z|^{n-1}(|z|-|w|)}, & \text{if } |z| \ne |w|, \\ n, & \text{if } |z| = |w|. \end{cases}$ *The result is best possible and the equality holds for the polynomial $P(z) = a_0 + a_n z^n$ in the case* $\arg z = \arg w$.

Proof. Fix $z \in \mathbb{C} \setminus \{0\}$. Summation by parts yields

$$P(w) = P(z) \left(\frac{w}{z}\right)^n + \left(1 - \frac{w}{z}\right) \sum_{k=1}^{n} \left(\sum_{j=0}^{k-1} a_j z^j\right) \left(\frac{w}{z}\right)^{k-1}.$$

This gives $z \frac{P(z)-P(w)}{z-w} = \sum_{k=0}^{n-1} \left(P(z) - \sum_{j=0}^{k} a_j z^j\right) \left(\frac{w}{z}\right)^k$. From this equality it follows the assertion of the lemma. □

3. Proof of Theorem 1. Denote

$$\rho_k(P)(z) := \sum_{j=k}^{k_n} c_j z^j, \ k = 0,1,\dots,k_n, \quad c_j = \begin{cases} 0, & \text{if } j \notin \{k_\nu\}_{\nu=0}^n, \\ a_j, & \text{if } j \in \{k_\nu\}_{\nu=0}^n. \end{cases}$$

(i) By Lemma 1 the condition (6) is equivalent to

$$|P(z)| \geq |\rho_{k_0}(P)(z)| \geq \ldots \geq |\rho_{k_n}(P)(z)| = \left|a_n z^{k_n}\right| \quad \forall z \in \overline{\mathbb{D}}. \tag{8}$$

This gives that $P(z) \neq 0$ for $z \in \overline{\mathbb{D}} \setminus \{0\}$. If $k_0 = 0$ then in addition $P(0) = a_{k_0} \neq 0$.
(ii) It follows from (8) that the sequence $\{|\rho_{k_\nu}(P)(z)|\}_{\nu=0}^n$ is non-increasing. Since $\rho_j(P) = \rho_{k_\nu}(P)$ for $k_{\nu-1} < j \leq k_\nu$, $\nu = 0, 1, \ldots, n$, where $k_{-1} = -1$, we conclude that the sequence $\{|\rho_j(P)(z)|\}_{j=0}^{k_n}$ is also non-increasing. Therefore, by Lemma 2 we get

$$\left| z\frac{P(z) - P(zt)}{1 - t} \right| \leq k_n |\rho_{n_0}(P)(z)| \leq k_n |P(z)|$$

for all $t \in \mathbb{T}$. In particularly, for $t = 1$ we obtain (7).

Now assume that the equality in (7) is attained at some $z \in \mathbb{D}$. Then by part (i) of Theorem 1, the function $F(t) := \frac{tP'(t)}{k_n P(t)} = \frac{k_0}{k_n} + \frac{(k_1 - k_0)a_1}{k_n a_0} t^{k_1 - k_0} + \ldots$ is holomorphic in $\mathbb{D}$, $|F(t)| \leq 1$ for all $t \in \mathbb{D}$ and $|F(z)| = 1$. Therefore, by the maximum modulus principle $F(t) = c$ for all $t \in \mathbb{D}$ with $|c| = 1$. But $F(0) = k_0/k_n$. So, $c = k_0/k_n = 1$. This is equivalent to $n = 0$ and $P(t) = \mathrm{e}^{\mathrm{M}} t^{k_0}$ for some $M \in \mathbb{C}$.
(iii) Let $k_0 = 0$. In view of proved properties of the function F, we have that $F(0) = 0$. Therefore, by the Schwarz Lemma we get $|F(t)| \leq |t|$ for all $t \in \mathbb{D}$. Moreover, if $|F(z)| = |z|$ for some $z \in \mathbb{D} \setminus \{0\}$, then $F(t) = ct$ for some $c \in \mathbb{C}$ with $|c| = 1$. It follows that $c = F'(t) = \frac{k_1^2 a_1}{k_n a_0} t^{k_1 - 1} + \cdots, \quad t \in \mathbb{D}$. Hence, it is necessary that $k_1 = 1$ and $|a_1| = k_n |a_0|$. However, under condition (6),

$$|a_1/a_0| = \frac{1}{2\pi} \Big| \int_0^{2\pi} e^{i(k_1 - k_0)\theta} \mathrm{Re}\Big(1 + 2\sum_{j=1}^n \frac{a_j}{a_0} e^{i(k_j - k_0)\theta}\Big) d\theta \Big| \leq$$
$$\leq \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\Big(1 + 2\sum_{j=1}^n \frac{a_j}{a_0} e^{i(k_j - k_0)\theta}\Big) d\theta = 1.$$

Thus, $k_n = 1$ or equivalently, $n = 1$. But for $n = 1$ the condition (6) implies $|a_0| \geq 2|a_1|$. This is a contradiction. Hence, $|F(t)| < |t|$ for all $t \in \mathbb{D}$.

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