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FINITE M/M/1 RETRIAL MODELS WITH CHANGEABLE SERVICE RATE

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The article deals with M/M/1 -type retrial queueing system with finite orbit. It is supposed that service rate depends on the loading of the system. The explicit formulae for ergodic distribution of the number of customers in the system are obtained. The theoretical results are illustrated by numerical examples.

1. Introduction. In ordinary queueing systems, an arriving customer who finds a free server receives the service immediately and leaves the system after the service completion. However, if the arriving customer finds the server busy upon the arrival, it either joins the waiting line and waits for service according to some queueing discipline or leaves the system forever. Retrial queues differ from the ordinary ones by the phenomenon that the arriving customer (i.e., *primary customer* or *primary call*) who finds the server busy upon arrival leaves only the service area but not the entire system. It becomes a *repeated customer* (or *unsatisfied customer* or *repeated call*) and repeats its attempt to get service after some time called *retrial time*. All unsatisfied customers form an *orbit* (or *pool*) and if the sequential attempt of an unsatisfied customer to get service fails, it returns to the orbit and the process repeats. As for the repeated customer, the usual supposition enables the customers to act independently both on each other and in relation to the primary customers. The retrial time follows an exponential distribution. It was stressed by many specialists in queueing theory (see [1], [2], [3], [4]) that the retrial phenomenon should be considered in solving a number of practically important problems arising in computer and communication networks, telephone systems, airport operation, etc. It explains, the interest related to such systems that has considerably increased in recent years, and excelent bibliography on this subject one can find in [1], [2].

2. Mathematical model and preliminaries. This article deals with the retrial queue, where customers arrive following a Poisson process with rate λ. The retrial time distribution is exponential with parameter ν, and hence, any unsatisfied customer produces a Poisson process of repeated calls with intensity ν. The service time distribution function is exponential for both primary and repeated calls. The parameter of the distribution is defined as follows. If at the instant when a customer (primary or repeated) gets the service, the system has j, $j \geq 1$ customers (j is the number of customers in the orbit $+1$), then the parameter of the service time is equal to μ_j.

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Traditionally, the study of any queueing systems starts with a description of its evolution by the Markovian process. In the case where service intensity is stable, (i.e. $\mu_i = \mu = const,$), that description looks as follows [3]. Let $\xi(t)$ stand for the number of customers in the orbit at time t, and $\eta(t)$ define the state of the server at that time: if the server is free, then $\eta(t) = 0$, and $\eta(t) = 1$ otherwise. The two component process $(\eta(t), \xi(t))$ is a homogeneous Markov process with phase space $S = \{(i, j),\ i = 0, 1,\ j = 0, 1, 2, ...,\}$.

In the case where service intensity depends on the number of customers in the system the process $(\eta(t), \xi(t))$, in contrast to [3], is not Markovian. But we can make it Markovian prescribing another sense to the component $\eta(t)$.

Let, as above, $\xi(t)$ denote the number of customers in the orbit at the time t. If at the time t the server is busy and the service rate is μ_i, $i \geq 1$, then we say that the server is in the phase i. If at the time t the server is free, we say that the server is in the phase 0. Let $\eta(t) \in \{0, 1, 2, ...,\}$ denote the phase of the server at time t.

It is clear that if $\eta(t) = 0$, and $\xi(t) = i \geq 0$, then there are i customers in the system at the time t, they all are in the orbit and the server is free. If $\eta(t) \geq 1$ and $\xi(t) = i \geq 0$, then there are $i + 1$ customers in the system at the time t: (i.e., one customer in service and i customers in the orbit). The process $(\eta(t), \xi(t))$ is homogeneous Markov process with state space $\{(i, j), 0 \leq j, 0 \leq i \leq j + 1$.

We say the process $(\eta(t), \xi(t))$ is ergodic if the following limits exist:

$$\pi_{ij} = \lim_{t\to\infty} \mathbf{P}\{\eta(t) = i, \xi(t) = j/\eta(0) = k, \xi(t) = l)\} > 0,$$

for all $i, j, k, l \geq 0$, and $\sum_{ij} \pi_{ij} = 1$.

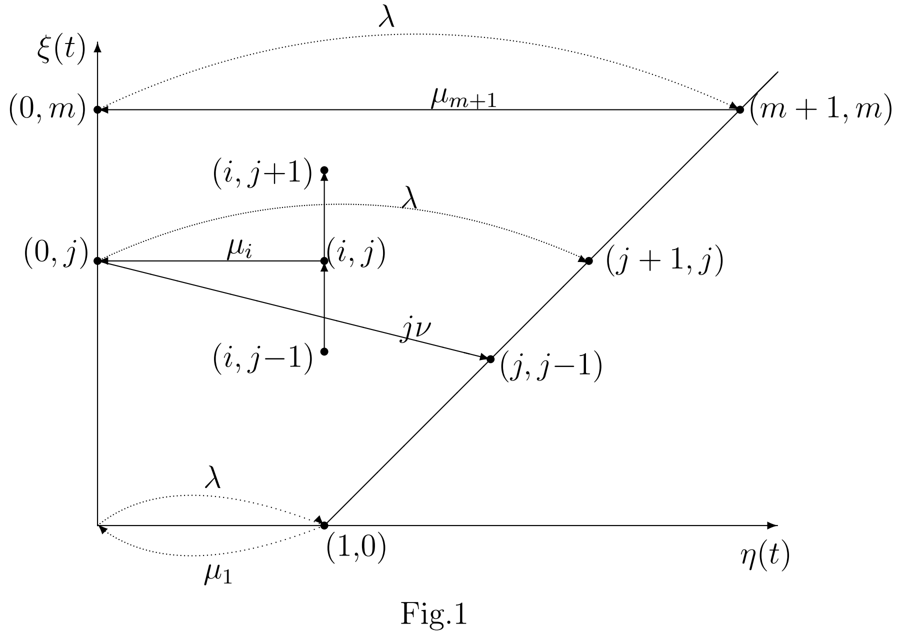

Fig.1

In this article we deal with a system with finite orbit, i.e. the number of waiting places in the orbit is finite, say, $m < \infty$. It means that the customer, that finds $m+1$ customers in system at the moment of arriving, is lost. The Markov process $(\eta(t), \xi(t))$ corresponding to such a system is always ergodic and its state space has the form $E = \{(i, j), 0 \leq j \leq m,\ 0 \leq i \leq j + 1$. Figure 1 presents the graph and transition rates of the process $(\eta(t), \xi(t))$.

3. Formulae for ergodic distribution. We put

$$\varkappa=-\lambda/\nu,\quad \beta_i=\lambda/(\lambda+\mu_i),\quad a(i,l)=\frac{\lambda+\mu_i}{\mu_i}\Big[\beta_i^{l+1}\prod_{k=0}^{l-1}(i+k)-\varkappa^l\Big].$$

For the sequence $a(i,k), i,k\geq 1$ we define the "convolution" in the following way

$$a^{*n}(i,l)=\sum_{k=1}^{l-n+1}a(i,k)a^{*(n-1)}(i+k,l-k),\qquad a^{*1}(i,l)=a(i,l),\quad 1\leq n\leq l$$

and let

$$H(i,l)=\sum_{k=1}^{l}a^{*k}(i,l),\quad l\geq 0,\quad H(i,0)=1,\quad A_i=-\nu\sum_{k=1}^{i}\varkappa^kH(k,i-k).$$

Theorem 1. *Let $m<\infty$ be the number of waiting spaces in the orbit. The ergodic distribution of the process $(\eta(t),\xi(t))$ can be presented as follows*

$$\pi_{ij}=\pi_{00}\frac{\beta_i^{j-i+1}A_i}{(i-1)!\mu_i},\quad 0\leq i-1\leq j\leq m-1,$$

$$\pi_{0j}=\pi_{00}\frac{\lambda}{j\nu}\sum_{k=1}^{j}\frac{\beta_k^{j-k}A_k}{(k-1)!\mu_k},\quad 1\leq j\leq m,\qquad \pi_{im}=\pi_{00}\frac{\lambda\beta_i^{m-i}A_i}{(i-1)!\mu_i^2},\quad 1\leq i\leq m,$$

$$\pi_{00}=\Big(1+\sum_{k=1}^{m}d(k,m)A_k\Big)^{-1},\ \pi_{m+1m}=\frac{\pi_{00}\lambda^2}{m\nu\mu_{m+1}}\sum_{k=1}^{m}\frac{\beta_k^{m-k}A_k}{(k-1)!\mu_k},$$

where

$$d(k,m)=\frac{1}{(k-1)!\mu_k}\Big[\frac{\lambda+\mu_k}{\mu_k}+\frac{\lambda^2\beta_k^{m-k}}{m\nu\mu_{m+1}}+\frac{\lambda}{\nu}\sum_{l=0}^{m-k}\frac{\beta_k^l}{l+k}\Big].$$

Proof. The following method (see [5]) of obtaining equilibrium equations for the system in question is applied.

Let us consider an ergodic Markov process $\xi(t)$ with discrete state space S, transition rate a_{ij}, $i,j\in S$ and let π_i stand for its ergodic distribution. Let $S=S_1\cup S_2$, $S_1,S_2\neq\varnothing$ and $S_1\cap S_2=\varnothing$. The quantity $\sum_{S_1\to S_2}a_{ij}\pi_i$, where $\sum_{S_1\to S_2}$ means that summation is carried out over those pairs (i,j), $i\in S_1$, $j\in S_2$ for which one-step transition from i to j is possible, is called *flew of probability from S_1 to S_2.* Then we have

$$\sum_{S_1\to S_2}a_{ij}\pi_i=\sum_{S_2\to S_1}a_{ij}\pi_i. \tag{1}$$

Using (1) and Fig.1 we have the set of equilibrium equations for the probabilities π_{ij}

$$\begin{cases} j\nu\pi_{0j} &= \lambda\sum_{k=1}^{j}\pi_{kj-1}, & 1\leq j\leq m,\\ \pi_{ij}(\lambda+\mu_i) &= \lambda\pi_{ij-1}, & 1\leq i\leq j\leq m-1,\\ \pi_{jj-1}(\lambda+\mu_j) &= \lambda\pi_{0j-1}+j\nu\pi_{0j}, & 1\leq j\leq m,\end{cases} \tag{2}$$

For the states $(0,0),(1,0)$ and $\{(i,m): 0 \le i \le m+1\}$ (see Fig. 1) we have

$$\begin{cases} \pi_{00}\lambda &= \mu_1\pi_{10}, \\ \pi_{im}\mu_i &= \pi_{im-1}\lambda, \quad 1 \le i \le m, \\ \pi_{m+1m}\mu_{m+1} &= \pi_{0m}\lambda, \end{cases}$$

which gives

$$\pi_{10} = \frac{\lambda}{\mu_1}\pi_{00}, \quad \pi_{m+1m} = \frac{\lambda}{\mu_{m+1}}\pi_{0m}, \quad \pi_{im} = \frac{\lambda}{\mu_i}\pi_{im-1}, \ 1 \le i \le m. \tag{3}$$

From the second equation in (2) we get

$$\pi_{ij} = \beta_i\pi_{ij-1} = \beta_i^{j-i+1}\pi_{ii-1} = \beta_i^{j-i+1}\Big(\frac{\lambda}{\lambda+\mu_i}\pi_{0i-1} + \frac{i\nu}{\lambda+\mu_i}\pi_{0i}\Big) =$$
$$= \beta_i^{j-i+2}\Big(\pi_{0i-1} + \frac{i\nu\pi_{0i}}{\lambda}\Big) = \beta_i^{j-i+2}\Big(\pi_{0i-1} - \frac{i\pi_{0i}}{\varkappa}\Big), \qquad 1 \le i \le j \le m-1.$$

From this and the third equation in (2) rewritten in the form

$$\pi_{jj-1} = \beta_j\big(\pi_{0j-1} - \frac{j\pi_{0j}}{\varkappa}\big), \quad 1 \le j \le m$$

we conclude that

$$\pi_{ij} = \beta_i^{j-i+2}\Big(\pi_{0i-1} - \frac{i\pi_{0i}}{\varkappa}\Big), \qquad 0 \le i-1 \le j \le m-1. \tag{4}$$

Using (4) in the first equation of the system (2) we have

$$j\nu\pi_{0j} = \lambda\sum_{k=1}^{j}\beta_k^{j-k+1}\Big(\pi_{0k-1} - \frac{k\pi_{0k}}{\varkappa}\Big). \tag{5}$$

If we set

$$\rho_i = (i-1)!\beta_i\mu_i\big(\pi_{0i-1} - \frac{i\pi_{0i}}{\varkappa}\big), \tag{6}$$

then from (4), (5) it follows

$$\pi_{ij} = \frac{\beta_i^{j-i}}{(i-1)!\mu_i}\rho_i, \qquad 0 \le i-1 \le j \le m, \tag{7}$$

$$j\nu\pi_{0j} = \lambda\sum_{k=1}^{j}\frac{\beta_k^{j-k}}{(k-1)!\mu_k}\rho_k, \qquad 1 \le j \le m. \tag{8}$$

From (6) we have the requrent equation for π_{0i}

$$\pi_{0i} = \frac{\varkappa}{i}\pi_{0i-1} - \frac{\varkappa}{i!\beta_i\mu_i}\rho_i,$$

which gives

$$\pi_{0i} = -\frac{1}{i!}\sum_{k=1}^{i}\frac{\varkappa^{i-k+1}}{\mu_k\beta_k}\rho_k + \pi_{00}\frac{\varkappa^i}{i!}.$$

Using this in (8) leads to

$$\rho_j = \sum_{k=1}^{j-1} \frac{\lambda+\mu_k}{\mu_k}\Big[\beta_k^{j-k+1}\prod_{l=0}^{j-k-1}(k+l) - \varkappa^{j-k}\Big]\rho_k + \pi_{00}\lambda\varkappa^{j-1}, \qquad 1\le j\le m$$

or

$$\rho_j = \sum_{k=1}^{j-1} a(k,j-k)\rho_k + \pi_{00}\lambda\varkappa^{j-1}, \qquad 1\le j\le m. \tag{9}$$

In [6] it was shown that the solution of equation (9) can be presented in the form

$$\rho_j = \lambda\pi_{00}\sum_{k=1}^{j}\varkappa^{k-1}H(k,j-k) = \pi_{00}A_j.$$

From this and (3), (7), (8) all the formulae for p_{ij}, except for π_{00}, from the theorem follow. To find π_{00} we use the normalizing condition

$$\sum_{i=1}^{m+1}\sum_{j=i-1}^{m}\pi_{ij} + \sum_{j=0}^{m}\pi_{0j} = 1,$$

which completes the proof. □

Let $\hat\pi_{0j} = \lim_{t\to\infty}\mathbf{P}\{\eta(t)=0,\xi(t)=j\}$, and $\hat\pi_{1j} = \lim_{t\to\infty}\mathbf{P}\{\eta(t)\ge 1,\xi(t)=j\} = \sum_{i=1}^{j+1}\pi_{ij}$. These probabilities describe the state of the server and the number of the customers in the orbit at steady-state regime. The next result follows from Theorem 1.

Corollary 1. *We have*

$$\hat\pi_{0j} = \hat\pi_{00}\frac{\lambda}{j\nu}\sum_{k=1}^{j}\frac{\beta_k^{j-k}A_k}{(k-1)!\mu_k}, \qquad 1\le j\le m,$$

$$\hat\pi_{1j} = \begin{cases} \hat\pi_{00}\sum_{k=1}^{j+1}\frac{\beta_k^{j-k+1}A_k}{(k-1)!\mu_k}, & 1\le j\le m-1,\\ \frac{\hat\pi_{00}\lambda}{m\nu\mu_{m+1}}\sum_{k=1}^{m}\frac{\beta_k^{m-k}A_k(m\nu\mu_{m+1}+\lambda\mu_k)}{(k-1)!\mu_k^2}, & j=m,\end{cases}$$

and

$$\hat\pi_{00} = \Big(1+\sum_{k=1}^{m}d(k,m)A_k\Big)^{-1}.$$

4. Optimization problem and numerical examples. In this section, as an example of applying obtained results, we consider the problem of optimizing the profit of the functioning system from the previous section. For a system with finite waiting places the next aspects are rather essential. First of all, we would like to avoid the situation, where $\xi(t)$-the number of customers in the orbit, is close to m, since it leads to increasing the number of lost customers. The situation, where $\xi(t)$ is close to 0 is also undesirable, because the server can be in the idle state too often (i.e., the system is underloaded). We also want that the probability of rejection from the service of a primary customer upon arrival should be small. Having a possibility to change the service rate μ_i, the manager of the system can satisfy these aspects by increasing μ_i if $\xi(t)$ is approaching to m and decreasing it if $\xi(t)$ is approaching to 0. Such a scheme was considered in [7], where it was assumed that the service rate equals $\mu_l I\{\xi(t) \le T\} + \mu_h I\{\xi(t) \ge T+1\}$. It is the so-called threshold strategy. In this paper, in addition to the threshold strategy, we consider the system with the *proportional changes of service rate* (or proportional strategy). But some notation first. So, let

α_i stands for the profit from the service a customer if the system is in the phase i;

c_1, c_2 stand for a fine for a lost customer and the service rejection, respectively.

As an objective function we take

$$F = \sum_{i=1}^{m+1} \alpha_i \mu_i \sum_{j=i-1}^{m} \pi_{ij} - c_1 \hat{\pi}_{1m} - c_2 \sum_{j=0}^{m-1} \hat{\pi}_{1j}.$$

Example 1 (threshold strategy). Let us consider the system with the following parameters: $m = 20, \lambda = 2, \nu = 1$, $c_1 = 2$, $c_2 = 1$, and

$$\mu_i = \begin{cases} 1.5 & \text{if} \quad 0 \le i \le 4, \\ 2 & \text{if} \quad 5 < i \le h, \\ 2.5 & \text{if} \quad h < i \le 21, \end{cases} \qquad \alpha_i = \begin{cases} 0.5 & \text{if} \quad 0 \le i \le 4, \\ 1 & \text{if} \quad 5 < i \le h, \\ 1.5 & \text{if} \quad h < i \le 21. \end{cases}$$

The value of the threshold $h = h^*$ we find as a solution of the following problem

$$F(h^*) = \sup_{4 \le h \le 21} F(h) = \sup_{4 \le h \le 21} \Big[\sum_{i=1}^{m} \alpha_i \mu_i \sum_{j=i-1}^{m} \pi_{ij} - c_1 \hat{\pi}_{1m} - c_2 \sum_{j=0}^{m-1} \hat{\pi}_{1j}\Big]. \tag{10}$$

Using the program Mathematica 11.3 and Corollary 1 we have the values of the functional $F(h)$ for $4 \le h \le 21$.

h	4	5	6	7	8	9	10	11	12	13	**14**
$F(h)$	0.6846	0.7073	0.7272	0.7439	0.7578	0.7692	0.7785	0.7858	0.7913	0.7950	**0.7968**

h	15	16	17	18	19	20	21
$F(h)$	0.7967	0.7946	0.7900	0.7824	0.7712	0.7562	0.7546

It follows that $h^* = 14$ is the solution of the problem (10) and $F(h^*) = 0.7968$.

Example 2 (proportional strategy). In this example we consider the system with same parameters as in the previous one: $m = 20, \lambda = 2, \nu = 1, c_1 = 1, c_2 = 2$ and we suppose that both the service rate and the value of α_i follow the proportional strategy:

$$\mu_i(\beta) = \frac{\beta(i-21)}{400} + 2.6, \qquad \alpha_i = 2 + \frac{i}{20}. \tag{11}$$

Now the value of $\beta = \beta^*$ we find as the solution of the following problem

$$F(\beta^*) = \sup_{0 \le \beta \le 50} F(\beta) = \sup_{0 \le \beta \le 50} \Big[\sum_{i=1}^{m} \alpha_i \mu_i \sum_{j=i-1}^{m} \pi_{ij} - c_1 \hat{\pi}_{1m} - c_2 \sum_{j=0}^{m-1} \hat{\pi}_{1j}\Big], \tag{12}$$

where μ_i, α_i are from (11). Using the program Mathematica 11.3 we have the solution of the problem (12): $\beta^* = 26$ (hence $\mu_i = 0.065i + 1.235$) and $F(\beta^*) = 3.454$.

For the system with the optimal value of parameter μ_i we find the steady-state distribution from Corollary 1. Using the program Mathematica 11.3 we have

j	0	1	2	3	4	5	6	7	8	9	10
$\hat{\pi}(0,j)$	0.00006	0.00018	0.0004	0.00075	0.00127	0.00198	0.0029	0.00399	0.00523	0.00652	0.00779
$\hat{\pi}(1,j)$	0.00009	0.0004	0.00112	0.00253	0.00495	0.00869	0.01398	0.02091	0.02936	0.03894	0.04905

j	11	12	13	14	15	16	17	18	19	20
$\hat{\pi}(0,j)$	0.00892	0.00982	0.01042	0.01067	0.01056	0.01012	0.0094	0.00846	0.00741	0.00630
$\hat{\pi}(1,j)$	0.05892	0.06772	0.07467	0.07920	0.08094	0.07986	0.07618	0.07037	0.06302	0.05598

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