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ON THE VALUE DISTRIBUTION OF A DIFFERENTIAL MONOMIAL AND SOME NORMALITY CRITERIA

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The aim of this paper is to study the zero distribution of the differential polynomial
$$af^{q_0}(f')^{q_1}...(f^{(k)})^{q_k}-\varphi,$$
where f is a transcendental meromorphic function and $a=a(z)(\not\equiv 0,\infty)$ and $\varphi(\not\equiv 0,\infty)$ are small functions of f. Moreover, using this value distribution result, we prove the following normality criterion for family of analytic functions:
Let $\mathscr{F}$ be a family of analytic functions on a domain D and let $k\geq 1$, $q_0\geq 2$, $q_i\geq 0$ $(i=1,2,\ldots,k-1)$, $q_k\geq 1$ be positive integers. If for each $f\in\mathscr{F}$: i. f has only zeros of multiplicity at least k, ii.
$$f^{q_0}(f')^{q_1}\ldots(f^{(k)})^{q_k}\neq 1,$$
then $\mathscr{F}$ is normal on domain D.

1. Introduction. The topic of this article has its origin in Hayman's ([3]) result that *if f is a transcendental meromorphic function and $n\geq 3$, then f^nf' assumes all finite values except possibly zero infinitely often.*

Later this result was complemented by E. Mues ([8]) (for $n=2$) and H. Y. Chen and M. L. Fang ([1]) (for $n=1$). Using Bloch's principle and Mues's result ([8]), in 1989, X. C. Pang ([9]) gave an analogous theorem for meromorphic functions in the unit disc (or bounded domain) in terms of normality of a family of meromorphic functions as follows:

Theorem A. ([9]) *Let $\mathscr{F}$ be a family of meromorphic function on a domain D. If each $f\in\mathscr{F}$ satisfies $f^2f'\neq 1$, then $\mathscr{F}$ is normal on domain D.*

The result of Mues was qualitative result. In 1992, Q. Zhang ([16]) gave the quantitative version of Mues's result as follows:

Theorem B. *For a transcendental meromorphic function f, the following inequality holds*
$$T(r,f)\leq 6N\Big(r,\frac{1}{f^2f'-1}\Big)+S(r,f).$$

In this direction, X. Huang and Y. Gu ([4]) further extended the Zhang's result ([16]) by replacing f' by $f^{(k)}$, $(k\in\mathbb{N})$.

Theorem C. ([4]) *Let f be a transcendental meromorphic function and k be a positive integer. Then*
$$T(r,f)\leq 6N\Big(r,\frac{1}{f^2f^{(k)}-1}\Big)+S(r,f).$$

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Moreover, in the same paper, C. X. Huang and Y. Gu ([4]) proved the following normality criterion for family of meromorphic functions:

Theorem D. ([4]) *Let $\mathscr{F}$ be a family of meromorphic functions on a domain D and let k be a positive integer. If for each $f \in \mathscr{F}$, f has only zeros of multiplicity at least k and $f^2 f^{(k)} \neq 1$, then $\mathscr{F}$ is normal on domain D.*

To study the value distribution of a differential polynomial in more general settings, in 2003, I. Lahiri and S. Dewan ([6]) proved the following theorem:

Theorem E. *Let f be a transcendental meromorphic function and $\alpha = \alpha(z)(\not\equiv 0, \infty)$ be a small function of f. If $\psi = \alpha(f)^n(f^{(k)})^p$, where $n(\geq 0)$ $p(\geq 1)$, $k(\geq 1)$ are integers, then for any small function $a = a(z)(\not\equiv 0, \infty)$ of ψ,*

$$(p+n)T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + pN_k(r,0;f) + \overline{N}(r,a;\psi) + S(r,f),$$

where $N_k(r,0;f)$ the counting function of zeros of f, where a zero of f with multiplicity q is counted q times if $q \leq k$, and is counted k times if $q > k$.

In this direction, a lot of investigations were made (e.g., ([12]), ([13]), ([14]), ([15])). Moreover, one can go through the Steinmetz' book, Nevanlinna theory, normal families, and algebraic differential equations ([11]) for the generalizations the Hayman result (Chapter 3, Section 3.2.).

Moreover, Theorem 4.12 of the same book ([11]) gave the following normality criterion:

Theorem F. ([11]) *Let $k \geq 1$ and $n \geq 1$ be integers, and $\mathscr{F}$ be a family of analytic functions f on some domain D, with zeros having multiplicity at least $k \geq 1$ and such that $f^n f^{(k)}$ omits some fixed value $a \neq 0$. Then $\mathscr{F}$ is normal on the domain D.*

The aim of this paper is to study the zero distribution of the differential polynomial

$$a(z)(f)^{q_0}(f')^{q_1}...(f^{(k)})^{q_k},$$

where $a(z)(\not\equiv 0, \infty)$ is a small function of f. Moreover, using this value distribution result, we give some normality criterion for family of analytic functions.

2. Main Results. Let f be a transcendental meromorphic function and $a(z)$ be a small function of f. Also, let $q_0, q_1, ..., q_k \in \mathbb{N} \cup \{0\}$. Let us define

$$M[f] \;:=\; a(z)(f)^{q_0}(f')^{q_1}...(f^{(k)})^{q_k}. \tag{1}$$

Also, we define $\mu := q_0 + q_1 + ... + q_k$ and $\mu_* := q_1 + 2q_2 + ... + kq_k$.

Theorem 1. *Let $f(z)$ be a transcendental meromorphic function and $\varphi(z)(\not\equiv 0, \infty)$ be a small function of $f(z)$. If $q_0 \geq 0$, $q_k \geq 1$, then*

$$\mu T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \sum_{i=1}^{k} q_i N_i(r,0;f) + \overline{N}(r,\varphi;M[f]) + S(r,f).$$

Remark 1. Clearly Theorem 1 extends Theorem E.

Theorem 2. *Let $f(z)$ be a transcendental meromorphic function and $\varphi(z)(\not\equiv 0, \infty)$ be a small function of $f(z)$ such that φ and f has no common zero. Moreover, we assume that $\frac{1}{a(z)}$ and f has no common zero. If every pole of $f(z)$ has multiplicity at least $l(\geq 1)$, $q_0 > 1 + \frac{1}{l}$ and $q_k \geq 1$, then*

$$T(r,f) \leq \frac{1}{q_0 - 1 - \frac{1}{l}} N\Big(r, \frac{1}{M[f]-\varphi}\Big) + S(r,f).$$

Corollary 1. *Let $f(z)$ be a transcendental entire function and $\varphi(z)(\not\equiv 0,\infty)$ be a small function of $f(z)$ such that φ and f has no common zero. Moreover, we assume that $\frac{1}{a(z)}$ and f has no common zero. If $q_0>1$ and $q_k\geq 1$, then*

$$T(r,f)\leq \frac{1}{q_0-1}N\Big(r,\frac{1}{M[f]-\varphi}\Big)+S(r,f).$$

Corollary 2. *Let $f(z)$ be a transcendental entire (resp. meromorphic function such that every pole of $f(z)$ has multiplicity at least $l(\geq 1)$) and $\varphi(z)(\not\equiv 0,\infty)$ be a small function of $f(z)$ such that φ and f has no common zero. Moreover, we assume that $\frac{1}{a(z)}$ and f has no common zero. If $q_0>1$ (resp. $1+\frac{1}{l}$) and $q_k\geq 1$, then $M[f]-\varphi$ has infinitely many zeros.*

Morevoer, as an application of corollary 2, we prove a normality criterion for a family of analytic functions.

Theorem 3. *Let $\mathscr{F}$ be a family of analytic functions on a domain D and let $k(\geq 1)$, $q_0(\geq 2)$, $q_i(\geq 0)$ $(i=1,2,\ldots,k-1)$, $q_k(\geq 1)$ be positive integers. If for each $f\in\mathscr{F}$: i. f has only zeros of multiplicity at least k, ii. $f^{q_0}(f')^{q_1}\ldots(f^{(k)})^{q_k}\neq 1$, then $\mathscr{F}$ is normal on domain D.*

3. Lemmas.

Lemma 1. *For a non-constant meromorphic function g, we obtain*

$$N\Big(r,\frac{g'}{g}\Big)-N\Big(r,\frac{g}{g'}\Big)=\overline{N}(r,g)+N\Big(r,\frac{1}{g}\Big)-N\Big(r,\frac{1}{g'}\Big).$$

Proof. The proof is same as the formula (12) of ([5]). □

Lemma 2. *Let f be a transcendental meromorphic function and $M[f]$ be a differential polynomial defined in (1), then*

$$T\big(r,M[f]\big)=O(T(r,f))\ \text{ and }\ S\big(r,M[f]\big)=S(r,f).$$

Proof. The proof is similar to the proof of the Lemma 2.4 of ([7]). □

Lemma 3. *Let f be a transcendental meromorphic function and $M[f]$ be a differential polynomial defined in (1) with $q_0\geq 1$, then $M[f]$ must be non-constant.*

Proof. Here $\left(\frac{1}{f}\right)^{\mu}=a(z)\left(\frac{f'}{f}\right)^{q_1}\left(\frac{f''}{f}\right)^{q_2}\ldots\left(\frac{f^{(k)}}{f}\right)^{q_k}\frac{1}{M[f]}$. Thus by the first fundamental theorem and lemma of logarithmic derivative, we have

$$\mu T(r,f)\leq\sum_{i=1}^{k}q_iN\left(r,\infty;\frac{f^{(i)}}{f}\right)+T(r,M[f])+S(r,f)\leq$$

$$\leq\sum_{i=1}^{k}iq_i\left(\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right)+T(r,M[f])+S(r,f)\leq\sum_{i=1}^{k}iq_i\Big(N(r,0;M[f])+$$

$$+N(r,\infty;M[f])\Big)+T(r,M[f])+S(r,f)\leq(2\mu_*+1)T(r,M[f])+S(r,f), \tag{2}$$

Since f is a transcendental meromorphic function, thus $M[f]$ must be non-constant. This completes the proof. □

Lemma 4. ([10]) *Let $\mathscr{F}$ be a family of meromorphic functions on the unit disc Δ such that all zeros of functions in $\mathscr{F}$ have multiplicity at least k. Let α be a real number satisfying $0 \le \alpha < k$. Then $\mathscr{F}$ is not normal in any neighbourhood of $z_0 \in \Delta$ if and only if there exist (i) points $z_n \in \Delta$, $z_n \to z_0$, (ii) positive numbers ρ_n, $\rho_n \to 0$ and (iii) functions $f_n \in \mathscr{F}$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \to g(\zeta)$ spherically uniformly on compact subsets of $\mathbb{C}$, where g is a non-constant meromorphic function.*

4. Proof of the Theorems.

Proof of Theorem 1. Since $\frac{1}{f^\mu} = \frac{M[f]}{f^\mu}\frac{1}{M[f]}$, so by the first fundamental theorem and lemma of logarithmic derivative, we have

$$\mu T(r,f) = N\Big(r, \frac{1}{f^\mu}\Big) + m\Big(r, \frac{1}{f^\mu}\Big) + O(1) \le N(r,0;f^\mu) + m\Big(r, \frac{1}{M[f]}\Big) + S(r,f) \le$$
$$\le N(r,0;f^\mu) + T(r,M[f]) - N(r,0;M[f]) + S(r,f). \tag{3}$$

Now, by Nevanlinna's three small functions theorem ([2], pp. 47), we have

$$T(r,M[f]) \;\le\; \overline{N}(r,0;M[f]) + \overline{N}(r,\infty;M[f]) + \overline{N}(r,\varphi;M[f]) + S(r,M[f]). \tag{4}$$

Let z_0 be a zero of f with multiplicity $q(\ge 1)$.
Case-I If $q \le k$, then z_0 is a zero of $M[f]$ of order at least $qq_0 + (q-1)q_1 + (q-2)q_2 + \ldots + 2q_{q-2} + q_{q-1} + t$ (where $t = 0$ if $a(z)$ has no zero or pole at z_0; $t = s$ if $a(z)$ has zero of order s at z_0, and $t = -s$ if $a(z)$ has pole of order s at z_0). Now

$$\mu q + 1 - (qq_0 + (q-1)q_1 + (q-2)q_2 + \ldots + 2q_{q-2} + q_{q-1}) - t =$$
$$= 1 + \{q_1 + 2q_2 + \ldots + (q-2)q_{q-2} + (q-1)q_{q-1}\} + (qq_q + qq_{q+1} + \ldots + qq_k) - t.$$

Case-II If $q \ge k+1$, then z_0 is a zero of $M[f]$ of order $q\mu - \mu_* + t$ (where $t = 0$ if $a(z)$ has no zero or pole at z_0; $t = s$ if $a(z)$ has zero of order s at z_0, and $t = -s$ if $a(z)$ has pole of order s at z_0). Now

$$\mu q + 1 - (q\mu - \mu_*) - t = 1 + q_1 + 2q_2 + \ldots + kq_k - t.$$

Thus from the above discussion, we have

$$N(r,0;f^\mu) + \overline{N}(r,0;M[f]) - N(r,0;M[f]) \le \overline{N}(r,0;f) + \sum_{i=1}^{k} q_i N_i(r,0;f) + S(r,f). \tag{5}$$

Combining (3),(4) and (5), we have

$$\mu T(r,f) \le N(r,0;f^\mu) + T(r,M[f]) - N(r,0;M[f]) + S(r,f) \le$$
$$\le N(r,0;f^\mu) + \overline{N}(r,0;M[f]) + \overline{N}(r,\infty;M[f]) + \overline{N}(r,\varphi;M[f]) - N(r,0;M[f]) + S(r,f) \le$$
$$\le \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \sum_{i=1}^{k} q_i N_i(r,0;f) + \overline{N}(r,\varphi;M[f]) + S(r,f). \tag{6}$$

This completes the proof. □

Proof of Theorem 2. Let us define $b = b(z) =: \frac{1}{\varphi(z)}$. Now by Lemma 3, it is clear that $b(z)M[f]$ is non-constant. Again

$$\frac{1}{f^\mu} = \frac{bM[f]}{f^\mu} - \frac{(bM[f])'}{f^\mu} \cdot \frac{(bM[f]-1)}{(bM[f])'}.$$

Thus in view of Lemmas 1 and 2, the first fundamental theorem and lemma of logarithmic derivative, we have

$$\begin{aligned}\mu m\left(r,\frac{1}{f}\right) &\le m\left(r,\frac{bM[f]}{f^{\mu}}\right)+m\left(r,\frac{(bM[f])'}{f^{\mu}}\right)+m\left(r,\frac{bM[f]-1}{(bM[f])'}\right)+O(1)\le\\ &\le 2m\left(r,\frac{bM[f]}{f^{\mu}}\right)+m\left(r,\frac{(bM[f])'}{bM[f]}\right)+m\left(r,\frac{bM[f]-1}{(bM[f])'}\right)+O(1)\le\\ &\le T\left(r,\frac{(bM[f])'}{bM[f]-1}\right)-N\left(r,\frac{bM[f]-1}{(bM[f])'}\right)+S(r,f)\le\\ &\le \overline{N}(r,\infty;f)+N\left(r,\frac{1}{bM[f]-1}\right)-N\left(r,0;(bM[f])'\right)+S(r,f)\le\\ &\le \frac{1}{l}N(r,\infty;f)+N\left(r,\frac{1}{M[f]-\varphi}\right)-(q_0-1)N\left(r,0;f\right)+S(r,f).\end{aligned}\tag{7}$$

Now, using the first fundamental theorem and (7), we obtain

$$(\mu-q_0+1)m\left(r,\frac{1}{f}\right)+(q_0-1)T(r,f)\le N\left(r,\frac{1}{M[f]-\varphi(z)}\right)+\frac{1}{l}N(r,\infty;f)+S(r,f).\tag{8}$$

As $q_0 > 1+\frac{1}{l}$, then from (8), we have

$$T(r,f)\le \frac{1}{q_0-1-\frac{1}{l}}N\left(r,\frac{1}{M[f]-\varphi(z)}\right)+S(r,f).$$

This completes the proof. □

Proof of Theorem 3. Since normality is a local property, we may assume that $D=\Delta$. If possible, suppose that $\mathscr{F}$ is not normal on Δ, then by Lemma 4, there exist $\{f_n\}\subset\mathscr{F}$, $z_n\in\Delta$ and positive numbers ρ_n with $\rho_n\to 0$ such that

$$g_n(\zeta)=\rho_n^{-\alpha}f_n(z_n+\rho_n\zeta)\to g(\zeta)$$

locally, uniformly in spherical metric, where we choose $\alpha=\frac{\mu_*}{\mu}$. Now, by Lemma 4, $g(\zeta)$ is a non-constant meromorphic function, moreover, by Hurwitz's theorem, all zeros of $g(\zeta)$ are of multiplicity at least k. Next, we define

$$H_n(\zeta)=(g_n(\zeta))^{q_0}(g_n^{'}(\zeta))^{q_1}\ldots(g_n^{(k)}(\zeta))^{q_k},\quad H(\zeta)=(g(\zeta))^{q_0}(g^{'}(\zeta))^{q_1}\ldots(g^{(k)}(\zeta))^{q_k}.$$

Thus $H(\zeta)\not\equiv 0$, otherwise, $g(\zeta)$ will become a polynomial of degree at most $k-1$, which is impossible. Also

$$\begin{aligned}H_n(\zeta)&=\rho_n^{\mu_*-\alpha\mu}(f_n(z_n+\rho_n\zeta))^{q_0}(f_n^{'}(z_n+\rho_n\zeta))^{q_1}\ldots(f_n^{(k)}(z_n+\rho_n\zeta))^{q_k}=\\ &=(f_n(z_n+\rho_n\zeta))^{q_0}(f_n^{'}(z_n+\rho_n\zeta))^{q_1}\ldots(f_n^{(k)}(z_n+\rho_n\zeta))^{q_k}\ \to H(\zeta)\end{aligned}$$

locally, uniformly in spherical metric. Since, $H_n(\zeta)\neq 1$, thus by the Hurwitz's Theorem, $H(\zeta)\neq 1$. Thus by Corollary 2, $g(\zeta)$ must be non-constant rational function, otherwise, $H(\zeta)-1$ has infinitely many solution, which is not possible.

Since $\mathscr{F}$ is a family of analytic functions, so $g_n(\zeta)$ is analytic. Since, $g_n(\zeta)\to g(\zeta)$ locally, uniformly in spherical metric, so either $g(\zeta)\equiv\infty$, or, $g(\zeta)$ is an analytic function. But, since $g(\zeta)$ is non-constant, so, $g(\zeta)$ must be a polynomial, say, $g(\zeta)=c_0+c_1\zeta+\ldots+c_l\zeta^l$.

If $l \geq k$, then $H(\zeta)$ becomes a non-constant polynomial, which contradicts that $H(\zeta) \neq 1$. Thus $l < k$, which, in view of Hurwitz's Theorem, contradicts our assumptions on zeros of $f \in \mathscr{F}$. Thus our assumption is wrong. So $\mathscr{F}$ is normal. This completes the proof. □

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