FACTORISATION OF ORTHOGONAL PROJECTORS

1. Introduction. Let \( H := L_2(\mathbb{R}) \) be a Hilbert space with the standard norm \( \| \cdot \| \) and the inner product \( (\cdot | \cdot) \), and let \( \mathcal{B} := \mathcal{B}(H) \) be the Banach algebra of all linear continuous operators in \( H \). Let us fix in the algebra \( \mathcal{B} \) the continuous chain of orthoprojectors \( \mathcal{E} := \{E(\xi)\}_{\xi \in \mathbb{R}} \), where \( E(\xi) \) is the multiplication operator on the characteristic function of the interval \( (-\infty, \xi) \).

An operator \( A \in \mathcal{B} \) is called an upper-triangular operator with respect to the chain \( \mathcal{E} \) if for every \( E \in \mathcal{E} \) the subspace \( E \perp H \) is an invariant subspace of \( A \), i.e.,
\[
E \perp AE = 0, \quad E \in \mathcal{E} \quad (E^\perp := I - E).
\]
Similarly, an operator \( A \in \mathcal{B} \) is called a lower-triangular operator with respect to the chain \( \mathcal{E} \) if for every \( E \in \mathcal{E} \) the subspace \( E \perp H \) is an invariant subspace of \( A \), i.e.,
\[
EAE^\perp = 0, \quad E \in \mathcal{E}.
\]

We set
\[
\mathcal{B}^+ := \{ B \in \mathcal{B} : \forall E \in \mathcal{E} \quad E^\perp BE = 0 \},
\]
\[
\mathcal{B}^- := \{ B \in \mathcal{B} : \forall E \in \mathcal{E} \quad EBE^\perp = 0 \}.
\]

\( \mathcal{B}^+ \) and \( \mathcal{B}^- \) are closed subalgebras in the algebra \( \mathcal{B} \). It is easy to see that if \( A \in \mathcal{B}^+ \), then the adjoint operator \( A^* \) belongs to the algebra \( \mathcal{B}^- \).

**Definition 1.** We say that an operator \( A \in \mathcal{B} \) admits UL-factorisation if there exist \( A_+ \in \mathcal{B}^+, A_- \in \mathcal{B}^- \) such that \( A = A_+A_- \).
Definition 1 is wider than usually accepted in the mathematical literature since it does not assume invertibility of $A$ (see, e.g., [1]). We only know the paper [2], where the factorisation problem was studied for nonnegative non-invertible operators.

In the case when $A \in \mathcal{B}$ is a nonnegative self-adjoint operator, we consider a special factorisation.

**Definition 2.** Let $A \in \mathcal{B}$ and $A \geq 0$. We say that $A$ admits a special factorisation if there exists $A_+ \in \mathcal{B}^+$ such that $A = A_+A_+^*$.

In the present paper, we study the following problem.

**Problem 1.** Does every orthogonal projector $P \in \mathcal{B}$ with $\dim \ker P < \infty$ admit a special factorisation in the form $P = VV^*$, where $V$ is an isometric operator in $\mathcal{B}^+$?

It follows from the results of Larson [3] that not every uniformly positive operator $A \in \mathcal{B}$ admits a factorisation $A = BB^*$, where $B \in \mathcal{B}^+$. In the case when an operator is non-invertible the problem of its special factorisation is much more difficult. In the mentioned work [2], a special factorisation of an orthogonal projector $P$ with $\dim \ker P < \infty$ was considered in the Hilbert space $L_2(0,1)$ with chain of orthoprojectors $\{\tilde{E}(\xi) \mid \xi \in [0,1]\}$, where $\tilde{E}(\xi)$ is the multiplication operator on the characteristic function of the interval $[0,\xi)$. In [2], it was proved that an orthogonal projector $P$ with $\dim \ker P < \infty$ admits a special factorisation if the additional condition

$$\forall \xi \in [0,1] \quad \dim \tilde{E}(\xi) \ker P = \dim \ker P$$

holds.

The main result of this paper gives an explicit formula for an isometric operator $V \in \mathcal{B}^+$ such that $VV^* = P$ for an orthogonal projector $P \in \mathcal{B}$ with $\dim \ker P < \infty$.

2. **Factorisation of an orthogonal projector.** Denote by $G$ the Hilbert space $\mathbb{C}^n$ with the standard inner product

$$(x \mid y)_{G} := \sum_{j=1}^{n} x_j \bar{y}_j, \quad x = (x_j)_{j=1}^{n}, \quad y = (y_j)_{j=1}^{n}.$$

Let $P$ be an orthogonal projector in $\mathcal{B}$ with $\dim \ker P = n \in \mathbb{N}$, and let $(\varphi_j)_{j=1}^{n}$ be an orthonormal basis in the space $\ker P$.

Let us consider the function $\Phi: \mathbb{R} \to G'$ ($G'$ is the dual space to $G$) that is defined by the formula

$$\Phi(t)c = \sum_{j=1}^{n} c_j \varphi_j(t), \quad t \in \mathbb{R}, \quad c \in (c_j)_{j=1}^{n} \subseteq G.$$

It is easy to see that $\Phi \in L_2(\mathbb{R}, G')$. Denote by $\Phi^*(t)$ the operator that is adjoint to $\Phi(t)$. It acts from $\mathbb{C}$ into $G$ by the formula

$$\Phi^*(t)c = c(\varphi_1(t), \ldots, \varphi_n(t)), \quad t \in \mathbb{R}, \quad c \in \mathbb{C}.$$

We also define the function

$$A(x) := \int_{-\infty}^{x} \Phi^*(t)\Phi(t)dt, \quad x \in \mathbb{R}, \quad (1)$$
which plays an important role in the study. Clearly, this function is absolutely continuous, and $A'(x) = \Phi^*(x)\Phi(x)$ for almost every $x \in \mathbb{R}$. Moreover, $A^*(x) = A(x)$ for all $x \in \mathbb{R}$, and
\[
A(x_1) \leq A(x_2) \quad \text{for} \quad x_1 \leq x_2. \tag{2}
\]
Let us consider the subspaces
\[
F(x) := \ker A(x), \quad G(x) := \text{ran } A(x), \quad x \in \mathbb{R}.
\]
Since $G(x) \oplus F(x) = G$ for all $x \in \mathbb{R}$, it follows by (2) that
\[
F(x_1) \supset F(x_2), \quad G(x_1) \subset G(x_2) \quad \text{for} \quad x_1 \leq x_2.
\]
It is easy to check that the function
\[
\rho(x) := \dim F(x), \quad x \in \mathbb{R},
\]
is nonincreasing, left-continuous, and piecewise-constant. We denote by $(\xi_k)_{k=1}^m$ a strictly increasing sequence of all points of discontinuity of the function $\rho$, and let
\[
\Delta_k := \begin{cases} 
(-\infty, \xi_1), & \text{if } s = 1; \\
(\xi_{s-1}, \xi_s), & \text{if } 1 < s \leq m; \\
(\xi_m, +\infty), & \text{if } s = m + 1.
\end{cases}
\]
Put $F_k := F(\xi_k), \ G_k := G(\xi_k), \ k = 1, \ldots, m$. It follows from the above that
\[
F(x) = F_k, \quad G(x) = G_k, \quad x \in \Delta_k, \quad k = 1, \ldots, m,
\]
and, moreover, $F(x) = \{0\} =: F_{m+1}, \ G(x) = G =: G_{m+1}, \ x \in \Delta_{m+1}$. Thus the equalities
\[
\text{ran } A(x) = G_k, \quad x \in \Delta_k, \quad k = 1, \ldots, m + 1, \tag{3}
\]
hold.

**Lemma 1.** For an arbitrary $x \in \Delta_k$ $(k = 1, \ldots, m + 1)$ the operator
\[
A_k(x) := A(x)|_{G_k}
\]
is invertible in $G_k$. Moreover, the function $x \mapsto A_k^{-1}(x)$ is continuous on $\Delta_k$.

**Proof.** Let $x \in \Delta_k$. Since the operator $A(x)$ is self-adjoint, it follows from (3) that $A$ maps $G_k$ onto itself. Hence the operator $A_k(x)$ is invertible. As already mentioned, the function $x \mapsto A(x)$ is continuous. Thus the function $\Delta_k \ni x \mapsto A_k(x)$ is continuous, too. Taking into account that the operators $A_k(x), \ x \in \Delta_k$, are invertible, we obtain that the functions $x \mapsto A_k^{-1}(x)$ are continuous on $\Delta_k$. □

**Definition 3.** Denote by $A^\flat: \mathbb{R} \to B(G)$ the function acting by the formula
\[
A^\flat(x) := A_k^{-1}(x) \oplus O_k, \quad x \in \Delta_k, \quad k = 1, \ldots, m + 1, \tag{4}
\]
where $O_k$ is a null-operator in $F_k$. 
Remark 1. It follows from Lemma 1 that the function \( x \mapsto A^\flat(x) \) is continuous on every interval \( \Delta_j \), and its points of discontinuity can only be points \( \xi_j \), \( j = 1, \ldots, m \). Moreover, in view of Lemma 1 and (1), we have for almost every \( x \in \Delta_k \) (\( k = 1, \ldots, m + 1 \)) the equality

\[
A(x)A^\flat(x)\Phi(x) = \Phi^*(x).
\]

The main result of this paper is:

**Theorem 1.** The formula

\[
(V f)(x) := f(x) - \int_x^\infty \Phi(x)A^\flat(t)\Phi^*(t)f(t)dt, \quad x \in \mathbb{R}, \quad f \in H,
\]

defines an isometric upper-triangular operator such that \( VV^* = P \).

The proof of Theorem 1 will be divided into parts presented below as lemmas. Denote by \( C_0 \) the set of all continuous functions \( f: \mathbb{R} \to \mathbb{C} \) with compact support not intersecting the set \( \{ \xi_j \}_{j=1}^m \). Note that the set \( C_0 \) is everywhere dense in \( H \).

**Lemma 2.** Assume that the operator \( V \) is introduced by the formula (5) and \( U = I - V \). For an arbitrary \( f \in C_0 \) the equality

\[
\|Uf\|^2 = (Uf | f) + (f | Uf)
\]

holds.

**Proof.** Let \( f \in C_0 \). Taking into account Remark 1, we conclude that the vector-valued function

\[
h(t) := A^\flat(t)\Phi^*(t)f(t), \quad t \in \mathbb{R},
\]

is square integrable and has a compact support; as a result, it is integrable on \( \mathbb{R} \). Hence the function

\[
x \mapsto \int_x^\infty h(t)dt
\]

is continuous and bounded on the whole line. Thus, since the function \( x \mapsto \|\Phi(x)\| \) belongs to \( H \), we obtain that \( Uf \in H \) and

\[
(Uf)(x) = \int_x^\infty \Phi(x)h(t)dt, \quad x \in \mathbb{R}.
\]

It follows from the last formula that

\[
|(Uf)(x)|^2 = \int_x^\infty \int_x^\infty (\Phi^*(x)\Phi(x)h(t) | h(\xi))Gdt\,d\xi.
\]

Let us calculate the integral

\[
J := \int_\mathbb{R} |(Uf)(x)|^2dx = \int_{-\infty}^\infty \int_x^\infty \int_x^\infty (\Phi^*(x)\Phi(x)h(t) | h(\xi))Gdt\,d\xi\,dx.
\]

We see that \( J = J_1 + J_2 \), where

\[
J_1 := \int_{x \leq t \leq \xi} (\Phi^*(x)\Phi(x)h(t) | h(\xi))Gdxdt\,d\xi,
\]

\[
J_2 := \int_{x \leq \xi \leq t} (\Phi^*(x)\Phi(x)h(t) | h(\xi))Gdxdt\,d\xi.
\]
Integrating over the variable $x$ the integrals $J_1$ and $J_2$, and taking into account the definition of the operator $A(x)$, we get that

$$J_1 = \iint_{t \leq \xi} (A(t)h(t) | h(\xi))_{G} dtd\xi,$$

$$J_2 = \iint_{\xi \leq t} (A(\xi)h(t) | h(\xi))_{G} dtd\xi = \iint_{\xi \leq t} (h(t) | A(\xi)h(\xi))_{G} dtd\xi.$$

Note that, for almost every $t \in \mathbb{R}$,

$$A(t)h(t) = A(t)A^\flat(t)\Phi^*(t)f(t) = \Phi^*(t)f(t).$$

Thus

$$J_1 = \iint_{t \leq \xi} (\Phi^*(t)f(t) | h(\xi))_{G} dtd\xi = \iint_{t \leq \xi} (f(t) | \Phi(t)h(\xi))_{G} dtd\xi = (f | Uf).$$

Similarly, we obtain that

$$J_2 = \iint_{\xi \leq t} (h(t) | \Phi^*(\xi)f(\xi))_{G} dtd\xi = \iint_{\xi \leq t} (\Phi(\xi)h(t) | f(\xi))_{G} dtd\xi = (Uf | f).$$

Therefore, $J = (Uf | f) + (f | Uf)$ as claimed. \hfill $\Box$

**Corollary 1.** The operator $V$ that is defined by the formula (5) is an isometric upper-triangular operator.

**Proof.** According to Lemma 2, we get that for an arbitrary $f \in C_0$

$$\|Vf\|^2 = (f - Uf | f - Uf) = \|f\|^2 + \|Uf\|^2 - (Uf | f) - (f | Uf) = \|f\|^2.$$

Since the set $C_0$ is everywhere dense in $H$, we have that the operator $V$ is continuously extended to an isometric operator on the whole space $H$. Obviously, the extended operator acts by the formula (5) and it is upper-triangular. \hfill $\Box$

**Lemma 3.** For every $g \in C_0$ the equality

$$\|U^*g\|^2 = (U^*g | g) + (g | U^*g) - \|P^g\|^2 \quad (P^g := I - P)$$

holds.

**Proof.** Using elementary calculations, we get that the adjoint to $U$ operator acts on functions $g \in C_0$ by the formula

$$(U^*g)(x) = \int_{-\infty}^{x} \Phi(x)A^\flat(x)\Phi^*(t)g(t)dt, \quad x \in \mathbb{R}.$$ 

Thus for an arbitrary $g \in C_0$

$$|(U^*g)(x)|^2 = \left| \int_{-\infty}^{x} \Phi(x)A^\flat(x)\Phi^*(t)g(t)dt \right|^2 = (A^\flat(x)\Phi^*(x)\Phi(x)A^\flat(x)r(x) | r(x))_{G},$$
where
\[ r(x) := \int_{-\infty}^{x} \Phi^*(t)g(t)dt, \quad x \in \mathbb{R}. \]
Taking into account (4), it is easy to check that for almost every \( x \in \mathbb{R} \)
\[ (A^x(x))' = -A^x(x)A'(x)A^x(x) = -A^x(x)\Phi^*(x)\Phi(x)A^x(x). \]
Thus
\[ |(U^*g)(x)|^2 = -((A^x(x))'r(x) | r(x))_G = -\int_{-\infty}^{x} \int_{-\infty}^{x} ((A^x(x))'\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_Gdtd\xi. \]
Using this fact, we get
\[ \|U^*g\|^2 = J := -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ((A^x(x))'\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_Gdtd\xi dx. \]
Similarly to the proof of Lemma 2, we rewrite the integral \( J \) as a sum of the integrals
\[ J_1 := -\int\int\int (A^x(x))'\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi)_Gdtd\xi dx, \]
\[ J_2 := -\int\int\int (A^x(x))'\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi)_Gdtd\xi dx. \]
Integrating over the variable \( x \) in both integrals and using that
\[ \int_{t}^{+\infty} (A^x(x))'dx = A^x(+\infty) - A^x(t) = I_G - A^x(t), \]
\( (I_G \) is the identity operator in \( G \)), we obtain that
\[ J_1 = \int\int (A^t(x)\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_Gdtd\xi - \int\int (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_Gdtd\xi = \]
\[ = (Ug \mid g) - \int\int (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_Gdtd\xi; \]
\[ J_2 = \int\int (A^t(\xi)\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_Gdtd\xi - \int\int (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_Gdtd\xi = \]
\[ = (U^*g \mid g) - \int\int (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_Gdtd\xi. \]
It thus follows that
\[ J = J_1 + J_2 = (g \mid U^*g) + (U^*g \mid g) - \int_{\mathbb{R}} \int_{\mathbb{R}} (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_Gdtd\xi. \]
Since
\[ \int_{\mathbb{R}} \int_{\mathbb{R}} (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_Gdtd\xi = \sum_{j=1}^{n} |(g \mid \varphi_j)|^2 = \|P^\perp g\|^2, \]
we have \( J = (g \mid U^*g) + (U^*g \mid g) - \|P^\perp g\|^2. \) \qed
Proof of Theorem 1. It follows from Lemma 3 that for an arbitrary \( g \in C_0 \)
\[
\|V^*g\|^2 = ((I - U)^*g \mid (I - U)^*g) = \|g\|^2 + \|U^*g\|^2 - (U^*g \mid g) - (g \mid U^*g) = \\
= \|g\|^2 - \|P^\perp g\|^2 = \|Pg\|^2,
\]
i.e., \((VV^*g \mid g) = (Pg \mid g)\). Therefore, we get the equality \(VV^* = P\). In view of Corollary 1, the operator \( V \) is isometric and upper-triangular. The theorem is proved.

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REFERENCES