УДК 517.98, 517.5

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# FACTORISATION OF ORTHOGONAL PROJECTORS

N. S. Sushchyk, V. M. Degnerys. *Factorisation of orthogonal projectors*, Mat. Stud. **55** (2021), 181–187.

We study the problem of a special factorisation of an orthogonal projector $P$ acting in the Hilbert space $L_2(\mathbb{R})$ with $\dim\ker P < \infty$. In particular, we prove that the orthogonal projector $P$ admits a special factorisation in the form $P = VV^*$, where $V$ is an isometric upper-triangular operator in the Banach algebra of all linear continuous operators in $L_2(\mathbb{R})$. Moreover, we give an explicit formula for the operator $V$.

**1. Introduction.** Let $H := L_2(\mathbb{R})$ be a Hilbert space with the standard norm $\|\cdot\|$ and the inner product $(\cdot \mid \cdot)$, and let $\mathcal{B} := \mathcal{B}(H)$ be the Banach algebra of all linear continuous operators in $H$. Let us fix in the algebra $\mathcal{B}$ the continuous chain of orthoprojectors $\mathfrak{E} := \{E(\xi)\}_{\xi\in\mathbb{R}}$, where $E(\xi)$ is the multiplication operator on the characteristic function of the interval $(-\infty, \xi)$.

An operator $A \in \mathcal{B}$ is called an upper-triangular operator with respect to the chain $\mathfrak{E}$ if for every $E \in \mathfrak{E}$ the subspace $EH$ is an invariant subspace of $A$, i,e.,

$$E^{\perp}AE = 0, \quad E \in \mathfrak{E} \quad (E^{\perp} := I - E).$$

Similarly, an operator $A \in \mathcal{B}$ is called a lower-triangular operator with respect to the chain $\mathfrak{E}$ if for every $E \in \mathfrak{E}$ the subspace $E^{\perp}H$ is an invariant subspace of $A$, i.e.,

$$EAE^{\perp} = 0, \quad E \in \mathfrak{E}.$$

We set

$$\mathcal{B}^+ := \{B \in \mathcal{B} : \forall E \in \mathfrak{E} \quad E^{\perp}BE = 0\},$$
$$\mathcal{B}^- := \{B \in \mathcal{B} : \forall E \in \mathfrak{E} \quad EBE^{\perp} = 0\}.$$

$\mathcal{B}^+$ and $\mathcal{B}^-$ are closed subalgebras in the algebra $\mathcal{B}$. It is easy to see that if $A \in \mathcal{B}^+$, then the adjoint operator $A^*$ belongs to the algebra $\mathcal{B}^-$.

**Definition 1.** We say that an operator $A \in \mathcal{B}$ *admits $UL$-factorisation* if there exist $A_+ \in \mathcal{B}^+$, $A_- \in \mathcal{B}^-$ such that $A = A_+A_-$.

2010 *Mathematics Subject Classification:* 47A68, 47A46.
*Keywords:* special factorisation; orthogonal projector.
doi:10.30970/ms.55.2.181-187

Definition 1 is wider than usually accepted in the mathematical literature since it does not assume invertibility of $A$ (see, e.g., [1]). We only know the paper [2], where the factorisation problem was studied for nonnegative non-invertible operators.

In the case when $A \in \mathcal{B}$ is a nonnegative self-adjoint operator, we consider a special factorisation.

**Definition 2.** Let $A \in \mathcal{B}$ and $A \geq 0$. We say that $A$ *admits a special factorisation* if there exists $A_+ \in \mathcal{B}^+$ such that $A = A_+ A_+^*$.

In the present paper, we study the following problem.

**Problem 1.** Does every orthogonal projector $P \in \mathcal{B}$ with $\dim\ker P < \infty$ admit a special factorisation in the form $P = VV^*$, where $V$ is an isometric operator in $\mathcal{B}^+$?

It follows from the results of Larson [3] that not every uniformly positive operator $A \in \mathcal{B}$ admits a factorisation $A = BB^*$, where $B \in \mathcal{B}^+$. In the case when an operator is non-invertible the problem of its special factorisation is much more difficult. In the mentioned work [2], a special factorisation of an orthogonal projector $P$ with $\dim\ker P < \infty$ was considered in the Hilbert space $L_2(0,1)$ with chain of orthoprojectors $\{\widetilde{E}(\xi) \mid \xi \in [0,1]\}$, where $\widetilde{E}(\xi)$ is the multiplication operator on the characteristic function of the interval $[0,\xi)$. In [2], it was proved that an orthogonal projector $P$ with $\dim\ker P < \infty$ admits a special factorisation if the additional condition

$$\forall \xi \in [0,1] \quad \dim \widetilde{E}(\xi)\ker P = \dim\ker P$$

holds.

The main result of this paper gives an explicit formula for an isometric operator $V \in \mathcal{B}^+$ such that $VV^* = P$ for an orthogonal projector $P \in \mathcal{B}$ with $\dim\ker P < \infty$.

**2. Factorisation of an orthogonal projector.** Denote by $G$ the Hilbert space $\mathbb{C}^n$ with the standard inner product

$$(x \mid y)_G := \sum_{j=1}^{n} x_j \bar{y}_j, \quad x = (x_j)_{j=1}^n, \ y = (y_j)_{j=1}^n.$$

Let $P$ be an orthogonal projector in $\mathcal{B}$ with $\dim\ker P = n \in \mathbb{N}$, and let $(\varphi_j)_{j=1}^n$ be an orthonormal basis in the space $\ker P$.

Let us consider the function $\Phi\colon \mathbb{R} \to G'$ ($G'$ is the dual space to $G$) that is defined by the formula

$$\Phi(t)c = \sum_{j=1}^{n} c_j \varphi_j(t), \qquad t \in \mathbb{R}, \quad c = (c_j)_{j=1}^n \in G.$$

It is easy to see that $\Phi \in L_2(\mathbb{R}, G')$ . Denote by $\Phi^*(t)$ the operator that is adjoint to $\Phi(t)$. It acts from $\mathbb{C}$ into $G$ by the formula

$$\Phi^*(t)c = c(\overline{\varphi_1(t)}, \ldots, \overline{\varphi_n(t)}), \qquad t \in \mathbb{R}, \quad c \in \mathbb{C}.$$

We also define the function

$$A(x) := \int_{-\infty}^{x} \Phi^*(t)\Phi(t)dt, \quad x \in \mathbb{R}, \tag{1}$$

which plays an important role in the study. Clearly, this function is absolutely continuous, and $A'(x) = \Phi^*(x)\Phi(x)$ for almost every $x \in \mathbb{R}$. Moreover, $A^*(x) = A(x)$ for all $x \in \mathbb{R}$, and

$$A(x_1) \leq A(x_2) \quad \text{for} \quad x_1 \leq x_2. \tag{2}$$

Let us consider the subspaces

$$F(x) := \ker A(x), \qquad G(x) := \operatorname{ran} A(x), \quad x \in \mathbb{R}.$$

Since $G(x) \oplus F(x) = G$ for all $x \in \mathbb{R}$, it follows by (2) that

$$F(x_1) \supset F(x_2), \quad G(x_1) \subset G(x_2) \qquad \text{for} \quad x_1 \leq x_2.$$

It is easy to check that the function

$$\rho(x) := \dim F(x), \quad x \in \mathbb{R},$$

is nonincreasing, left-continuous, and piecewise-constant. We denote by $(\xi_k)_{k=1}^m$ a strictly increasing sequence of all points of discontinuity of the function $\rho$, and let

$$\Delta_s := \begin{cases} (-\infty, \xi_1), & \text{if } s = 1; \\ (\xi_{s-1}, \xi_s), & \text{if } 1 < s \leq m; \\ (\xi_m, +\infty), & \text{if } s = m+1. \end{cases}$$

Put $F_k := F(\xi_k),\ G_k := G(\xi_k),\ k = 1, \dots, m$. It follows from the above that

$$F(x) = F_k, \qquad G(x) = G_k, \qquad x \in \Delta_k, \quad k = 1, \dots, m,$$

and, moreover, $F(x) = \{0\} =: F_{m+1},\ G(x) = G =: G_{m+1},\ \ x \in \Delta_{m+1}$. Thus the equalities

$$\operatorname{ran} A(x) = G_k, \quad x \in \Delta_k, \quad k = 1, \dots, m+1, \tag{3}$$

hold.

**Lemma 1.** *For an arbitrary $x \in \Delta_k$ $(k = 1, \dots, m+1)$ the operator*

$$A_k(x) := A(x)|_{G_k}$$

*is invertible in $G_k$. Moreover, the function $x \mapsto A_k^{-1}(x)$ is continuous on $\Delta_k$.*

*Proof.* Let $x \in \Delta_k$. Since the operator $A(x)$ is self-adjoint, it follows from (3) that $A$ maps $G_k$ onto itself. Hence the operator $A_k(x)$ is invertible. As already mentioned, the function $x \to A(x)$ is continuous. Thus the function $\Delta_k \ni x \to A_k(x)$ is continuous, too. Taking into account that the operators $A_k(x)$, $x \in \Delta_k$, are invertible, we obtain that the functions $x \mapsto A_k^{-1}(x)$ are continuous on $\Delta_k$. □

**Definition 3.** Denote by $A^\flat \colon \mathbb{R} \to \mathcal{B}(G)$ the function acting by the formula

$$A^\flat(x) := A_k^{-1}(x) \oplus O_k, \quad x \in \Delta_k, \quad k = 1, \dots, m+1, \tag{4}$$

where $O_k$ is a null-operator in $F_k$.

**Remark 1.** It follows from Lemma 1 that the function $x \mapsto A^\flat(x)$ is continuous on every interval $\Delta_j$, and its points of discontinuity can only be points $\xi_j$, $j = 1, \dots, m$. Moreover, in view of Lemma 1 and (1), we have for almost every $x \in \Delta_k$ $(k = 1, \dots, m+1)$ the equality

$$A(x)A^\flat(x)\Phi^*(x) = \Phi^*(x).$$

The main result of this paper is:

**Theorem 1.** *The formula*

$$(Vf)(x) := f(x) - \int_x^\infty \Phi(x)A^\flat(t)\Phi^*(t)f(t)dt, \quad x \in \mathbb{R}, \quad f \in H, \tag{5}$$

*defines an isometric upper-triangular operator such that* $VV^* = P$*.*

The proof of Theorem 1 will be divided into parts presented below as lemmas. Denote by $C_0$ the set of all continuous functions $f\colon \mathbb{R} \to \mathbb{C}$ with compact support not intersecting the set $\{\xi_j\}_{j=1}^m$. Note that the set $C_0$ is everywhere dense in $H$.

**Lemma 2.** *Assume that the operator* $V$ *is introduced by the formula* (5) *and* $U = I - V$*. For an arbitrary* $f \in C_0$ *the equality*

$$\|Uf\|^2 = (Uf \mid f) + (f \mid Uf)$$

*holds.*

*Proof.* Let $f \in C_0$ . Taking into account Remark 1, we conclude that the vector-valued function

$$h(t) := A^\flat(t)\Phi^*(t)f(t), \quad t \in \mathbb{R},$$

is square integrable and has a compact support; as a result, it is integrable on $\mathbb{R}$. Hence the function

$$x \to \int_x^\infty h(t)dt$$

is continuous and bounded on the whole line. Thus, since the function $x \mapsto \|\Phi(x)\|$ belongs to $H$, we obtain that $Uf \in H$ and

$$(Uf)(x) = \int_x^\infty \Phi(x)h(t)dt, \quad x \in \mathbb{R}.$$

It follows from the last formula that

$$|(Uf)(x)|^2 = \int_x^\infty \int_x^\infty (\Phi^*(x)\Phi(x)h(t) \mid h(\xi))_G dt d\xi.$$

Let us calculate the integral

$$J := \int_{\mathbb{R}} |(Uf)(x)|^2 dx = \int_{-\infty}^\infty \int_x^\infty \int_x^\infty (\Phi^*(x)\Phi(x)h(t) \mid h(\xi))_G dt d\xi dx.$$

We see that $J = J_1 + J_2$, where

$$J_1 := \iiint\limits_{x \le t \le \xi} (\Phi^*(x)\Phi(x)h(t) \mid h(\xi))_G dx dt d\xi,$$

$$J_2 := \iiint\limits_{x \le \xi \le t} (\Phi^*(x)\Phi(x)h(t) \mid h(\xi))_G dx dt d\xi.$$

Integrating over the variable $x$ the integrals $J_1$ and $J_2$, and taking into account the definition of the operator $A(x)$, we get that

$$J_1 = \iint\limits_{t\le\xi} (A(t)h(t) \mid h(\xi))_G dt d\xi,$$

$$J_2 = \iint\limits_{\xi\le t} (A(\xi)h(t) \mid h(\xi))_G dt d\xi = \iint\limits_{\xi\le t} (h(t) \mid A(\xi)h(\xi))_G dt d\xi.$$

Note that, for almost every $t \in \mathbb{R}$,

$$A(t)h(t) = A(t)A^\flat(t)\Phi^*(t)f(t) = \Phi^*(t)f(t).$$

Thus

$$J_1 = \iint\limits_{t\le\xi} (\Phi^*(t)f(t) \mid h(\xi))_G dt d\xi = \iint\limits_{t\le\xi} (f(t) \mid \Phi(t)h(\xi))_G dt d\xi = (f \mid Uf).$$

Similarly, we obtain that

$$J_2 = \iint\limits_{\xi\le t} (h(t) \mid \Phi^*(\xi)f(\xi))_G dt d\xi = \iint\limits_{\xi\le t} (\Phi(\xi)h(t) \mid f(\xi))_G dt d\xi = (Uf \mid f).$$

Therefore, $J = (Uf \mid f) + (f \mid Uf)$ as claimed. □

**Corollary 1.** *The operator $V$ that is defined by the formula* (5) *is an isometric upper-triangular operator.*

*Proof.* According to Lemma 2, we get that for an arbitrary $f \in C_0$

$$\|Vf\|^2 = (f - Uf \mid f - Uf) = \|f\|^2 + \|Uf\|^2 - (Uf \mid f) - (f \mid Uf) = \|f\|^2.$$

Since the set $C_0$ is everywhere dense in $H$, we have that the operator $V$ is continuously extended to an isometric operator on the whole space $H$. Obviously, the extended operator acts by the formula (5) and it is upper-triangular. □

**Lemma 3.** *For every $g \in C_0$ the equality*

$$\|U^*g\|^2 = (U^*g \mid g) + (g \mid U^*g) - \|P^\perp g\|^2 \qquad (P^\perp := I - P)$$

*holds.*

*Proof.* Using elementary calculations, we get that the adjoint to $U$ operator acts on functions $g \in C_0$ by the formula

$$(U^*g)(x) = \int_{-\infty}^{x} \Phi(x)A^\flat(x)\Phi^*(t)g(t)dt, \quad x \in \mathbb{R}.$$

Thus for an arbitrary $g \in C_0$

$$|(U^*g)(x)|^2 = \left| \int_{-\infty}^{x} \Phi(x)A^\flat(x)\Phi^*(t)g(t)dt \right|^2 = (A^\flat(x)\Phi^*(x)\Phi(x)A^\flat(x)r(x) \mid r(x))_G,$$

where

$$r(x) := \int_{-\infty}^{x} \Phi^*(t)g(t)dt, \quad x \in \mathbb{R}.$$

Taking into account (4), it is easy to check that for almost every $x \in \mathbb{R}$

$$(A^\flat(x))' = -A^\flat(x)A'(x)A^\flat(x) = -A^\flat(x)\Phi^*(x)\Phi(x)A^\flat(x).$$

Thus

$$|(U^*g)(x)|^2 = -((A^\flat(x))'r(x) \mid r(x))_G = -\int_{-\infty}^{x}\int_{-\infty}^{x} ((A^\flat(x))'\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi.$$

Using this fact, we get

$$\|U^*g\|^2 = J := -\int_{-\infty}^{\infty}\int_{-\infty}^{x}\int_{-\infty}^{x} ((A^\flat(x))'\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi dx.$$

Similarly to the proof of Lemma 2, we rewrite the integral $J$ as a sum of the integrals

$$J_1 := -\iiint\limits_{\xi \le t \le x} (A^\flat(x))'\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi dx,$$

$$J_2 := -\iiint\limits_{t \le \xi \le x} (A^\flat(x))'\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi dx.$$

Integrating over the variable $x$ in both integrals and using that

$$\int_{t}^{+\infty} (A^\flat(x))'dx = A^\flat(+\infty) - A^\flat(t) = I_G - A^\flat(t),$$

($I_G$ is the identity operator in $G$), we obtain that

$$J_1 = \iint\limits_{\xi \le t} (A^\flat(t)\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi - \iint\limits_{\xi \le t} (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi =$$

$$= (Ug \mid g) - \iint\limits_{\xi \le t} (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi;$$

$$J_2 = \iint\limits_{t \le \xi} (A^\flat(\xi)\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi - \iint\limits_{t \le \xi} (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi =$$

$$= (U^*g \mid g) - \iint\limits_{t \le \xi} (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi.$$

It thus follows that

$$J = J_1 + J_2 = (g \mid U^*g) + (U^*g \mid g) - \int_{\mathbb{R}}\int_{\mathbb{R}} (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi.$$

Since

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (\Phi^*(t)g(t) \mid \Phi^*(\xi)g(\xi))_G dtd\xi = \sum_{j=1}^{n} |(g \mid \varphi_j)|^2 = \|P^\perp g\|^2,$$

we have $J = (g \mid U^*g) + (U^*g \mid g) - \|P^\perp g\|^2$. □

*Proof of Theorem 1.* It follows from Lemma 3 that for an arbitrary $g \in C_0$

$$\|V^*g\|^2 = ((I-U)^*g \mid (I-U)^*g) = \|g\|^2 + \|U^*g\|^2 - (U^*g \mid g) - (g \mid U^*g) = \\ = \|g\|^2 - \|P^{\perp}g\|^2 = \|Pg\|^2,$$

i.e., $(VV^*g \mid g) = (Pg \mid g)$. Therefore, we get the equality $VV^* = P$. In view of Corollary 1, the operator $V$ is isometric and upper-triangular. The theorem is proved. □

**Acknowledgements.** The authors express their gratitude to Dr. Ya. V. Mykytyuk for the formulation of the problem and thank him for his ideas to solve it.

## REFERENCES

1. I. Gohberg, M. Krein, Theory of Volterra operators in Hilbert space and its applications, Nauka Publ., Moscow, 1967 (in Russian); Engl. transl.: Amer. Math. Soc. Transl. Math. Monographs, V.24, Amer. Math. Soc., Providence, RI, 1970.
2. S. Albeverio, R. Hryniv, Ya. Mykytyuk, *Factorisation of non-negative Fredholm operators and inverse spectral problems for Bessel operators*, Integr. equ. oper. theory, **64** (2009), 301–323.
3. D.R. Larson, *Nest algebras and similarity transformations*, Ann of Math. (2), **121** (1985), №2, 409–427.

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*Received 14.02.2021*