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ON SOME PROPERTIES OF HASSANI TRANSFORMS

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In the present paper, based on the ideas of Algerian physicist M.E. Hassani, the generalized Hassani spatial-temporal transformations in real Hilbert space are introduced. The original transformations, introduced by M.E. Hassani, are the particular cases of the transformations, introduced in this paper. It is proven that the classes of generalized Hassani transforms do not form a group of operators in the general case. Further, using these generalized Hassani transformations as well as the theory of changeable sets and universal kinematics, the mathematically strict models of Hassani kinematics are constructed and the performance of the relativity principle in these models is discussed.

1. Introduction. Subject of constructing the theory of super-light movement had been initiated in the papers [1, 2] more than 55 years ago. Despite the fact that on today tachyons (i.e. objects moving at a velocity greater than the velocity of light) are not experimentally detected, this subject remains being actual. Initially, the theory of tachyons was considered in the framework of classical Lorentz transformations, and superlight speed for frames of reference was forbidden. But afterwards in the papers of E. Recami, V. Olkhovsky and R. Goldoni [3–5] and and later in the papers of S. Medvedev [6] as well as J. Hill and B. Cox [7] the generalized Lorentz transforms for superluminal reference frames were deduced in the case of three-dimension space of geometric variables. In the paper [8] it was proven, that the above generalized Lorentz transforms may be easy to extend to the more general case of arbitrary (in particular infinity) dimension of the space of geometric variables. M.E. Hassani in the paper [9] has proposed the another, completely different and interesting, system of coordinate transforms for superluminal reference frames in the case of three-dimension space of geometric variables. In the present paper we introduce generalized Hassani spatial-temporal transformations for real Hilbert space. The main aim of this paper is to construct universal kinematics, based on these generalized Hassani transforms and to show that these universal kinematics do not satisfy the relativity principle in the general case.

In Section 2, we introduce the generalized Hassani transforms over Hilbert space. In Section 3 we prove that the introduced classes of generalized Hassani transforms do not form a group of operators in the general case. In Section 4 we construct the generalized Hassani kinematics, based on generalized Hassani transforms and discuss the performance of the relativity principle in these kinematics.

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2. Generalized Hassani Transforms over Hilbert Space.

2.1. Generalized Hassani transforms for the special case. In the works of M. E. Hassani (see, for example, [9]) it is proposed an interesting version of the generalized Lorentz transforms for the special case, when two inertial frames are moving along the x-axis in three-dimensional space and the directions of corresponding axises "y" and "z" are parallel:

$$t' = \frac{t - \frac{Vx}{\vartheta(V)^2}}{\sqrt{1 - \frac{V^2}{\vartheta(V)^2}}}; \quad x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{\vartheta(V)^2}}}; \quad y' = y, \quad z' = z, \quad \text{where:} \tag{1}$$

- $V \in \mathbb{R}$ is the velocity of inertial reference frame $\mathfrak{l}'$, which moves relatively the fixed inertial reference frame $\mathfrak{l}$.
- (t, x, y, z) are the (space-time) coordinates of any point $\mathbf{M}$ in the fixed frame $\mathfrak{l}$,
- (t', x', y', z') are the coordinates of the same point $\mathbf{M}$ in the moving frame $\mathfrak{l}'$,
- $\vartheta(\cdot)$ is an arbitrary real function of real variable, possessing the following properties:

$$\begin{cases} \vartheta(V) = c \quad \text{for} \quad 0 \leq V < c; \\ \vartheta(V) > V \quad \text{for} \quad c \leq V < \infty; \\ \vartheta(-V) = \vartheta(V) \quad (\forall\, V \in \mathbb{R}), \end{cases} \tag{2}$$

  where c is a positive real constant, which has the physical content of the speed of light in vacuum.

First, we note that is enough to restrict ourselves to the functions $\vartheta(\cdot)$ defined on $[0, \infty)$ and to consider the expression $\vartheta(|V|)$ instead of $\vartheta(V)$ in (1). Also instead of functions, which satisfy two first conditions (2) we will consider class of functions, satisfying weaker conditions. Denote by Υ the class of functions $\vartheta\colon [0, \infty) \to \mathbb{R}$, satisfying the following conditions:

$$\begin{cases} \vartheta(\lambda) \geq \lambda \quad \text{for} \quad \lambda \in [0, \infty). \\ \exists\, \eta > 0 \quad \vartheta(\lambda) > \lambda \ (\forall \lambda \in [0, \eta)) . \end{cases} \tag{3}$$

For any function $\vartheta \in \Upsilon$ we use the following notation:

$$\mathfrak{D}_* [\vartheta] := \{\lambda \in [0, \infty) \mid \vartheta(\lambda) > \lambda\} . \tag{4}$$

According to the conditions (3), we have, $\mathfrak{D}_* [\vartheta] \neq \varnothing$, and moreover,

$$[0, \eta) \subseteq \mathfrak{D}_* [\vartheta] \quad \text{for some} \quad \eta > 0. \tag{5}$$

Then for any real number V such that $|V| \in \mathfrak{D}_* [\vartheta]$ we can introduce the following (space-temporally) coordinate transforms

$$t' = \frac{t - \frac{Vx}{\vartheta(|V|)^2}}{\sqrt{1 - \frac{V^2}{\vartheta(|V|)^2}}}; \quad x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{\vartheta(|V|)^2}}}; \quad y' = y, \quad z' = z.$$

Therefore, we have introduced the generalized Hassani transforms for the same special case as for transforms (1). In the case

$$\vartheta(\lambda) = \vartheta_c(\lambda) := \begin{cases} c, & 0 \leq \lambda < c \\ \lambda, & \lambda \geq c \end{cases} \tag{6}$$

we obtain the classical Lorentz transforms and in the case, where the function ϑ satisfies two first conditions (2) we obtain the Hassani transforms (1).

2.2. Generalized Hassani transforms for the general case of Hilbert space. Let $(\mathfrak{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a Hilbert space over the real field such, that $\dim(\mathfrak{H}) \geq 1$, where $\dim(\mathfrak{H})$ is dimension of the space $\mathfrak{H}$. Emphasize that the condition $\dim(\mathfrak{H}) \geq 1$ should be interpreted in a way that the space $\mathfrak{H}$ may be infinite-dimensional. Let $\mathcal{L}(\mathfrak{H})$ be the space of (homogeneous) linear continuous operators over the space $\mathfrak{H}$. Denote by $\mathcal{L}^{\times}(\mathfrak{H})$ the space of all operators of affine transformations over the space $\mathfrak{H}$, that is $\mathcal{L}^{\times}(\mathfrak{H}) = \left\{\mathbf{A}_{[\mathbf{a}]} \mid \mathbf{A} \in \mathcal{L}(\mathfrak{H}),\, \mathbf{a} \in \mathfrak{H}\right\}$, where $\mathbf{A}_{[\mathbf{a}]}x = \mathbf{A}x + \mathbf{a}$, $x \in \mathfrak{H}$. The *Minkowski space* over the Hilbert space $\mathfrak{H}$ is defined as the Hilbert space $\mathcal{M}(\mathfrak{H}) = \mathbb{R} \times \mathfrak{H} = \{(t,x) \mid t \in \mathbb{R},\, x \in \mathfrak{H}\}$ (with the standard definition of linear operations), equipped by the inner product and norm: $\langle \mathrm{w}_1, \mathrm{w}_2\rangle = \langle \mathrm{w}_1, \mathrm{w}_2\rangle_{\mathcal{M}(\mathfrak{H})} = t_1t_2 + \langle x_1, x_2\rangle$, $\|\mathrm{w}_1\| = \|\mathrm{w}_1\|_{\mathcal{M}(\mathfrak{H})} = \left(t_1^2 + \|x_1\|^2\right)^{1/2}$ (where $\mathrm{w}_i = (t_i, x_i) \in \mathcal{M}(\mathfrak{H})$, $i \in \{1,2\}$) ([8, 10]). In the space $\mathcal{M}(\mathfrak{H})$ we select the next subspaces: $\mathfrak{H}_0 := \{(t,\mathbf{0}) \mid t \in \mathbb{R}\}$, $\mathfrak{H}_1 := \{(0,x) \mid x \in \mathfrak{H}\}$ with $\mathbf{0}$ being zero vector. Then, $\mathcal{M}(\mathfrak{H}) = \mathfrak{H}_0 \oplus \mathfrak{H}_1$, where $\oplus$ means the orthogonal sum of subspaces. Denote $\mathbf{e}_0 := (1,\mathbf{0}) \in \mathcal{M}(\mathfrak{H})$. Let us introduce the orthogonal projectors on the subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_0$

$$\mathbf{X}\mathrm{w} = (0,x) \in \mathfrak{H}_1;\ \widehat{\mathbf{T}}\mathrm{w} = (t,\mathbf{0}) = \mathcal{T}(\mathrm{w})\,\mathbf{e}_0 \in \mathfrak{H}_0, \tag{7}$$

where $\mathcal{T}(\mathrm{w}) = t \ \ (\mathrm{w} = (t,x) \in \mathcal{M}(\mathfrak{H}))$.

Definition 1 ([8, 10])**.** The operator $S \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ is referred to as *linear coordinate transform operator* if and only if there exist the continuous inverse operator $S^{-1} \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$. The linear coordinate transform operator $S \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ is called *v-determined* if and only if $\mathcal{T}(S^{-1}\mathbf{e}_0) \neq 0$. The vector

$$\mathcal{V}(S) = \frac{\mathbf{X}S^{-1}\mathbf{e}_0}{\mathcal{T}(S^{-1}\mathbf{e}_0)} \in \mathfrak{H}_1$$

is called the *velocity* of the v-determined coordinate transform operator S.

Let $\mathbf{B}_1(\mathfrak{H}_1)$ be the unit sphere in the space $\mathfrak{H}_1$ ($\mathbf{B}_1(\mathfrak{H}_1) = \{x \in \mathfrak{H}_1 \mid \|x\| = 1\}$).

Any vector $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ generates the following orthogonal projectors, acting in $\mathcal{M}(\mathfrak{H})$:

$$\left[\begin{aligned} &\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} = \langle \mathbf{n}, \mathrm{w}\rangle\,\mathbf{n} \quad (\mathrm{w} \in \mathcal{M}(\mathfrak{H}));\\ &\ \mathbf{X}_1^{\perp}[\mathbf{n}] = \mathbf{X} - \mathbf{X}_1[\mathbf{n}]. \end{aligned}\right. \tag{8}$$

Recall, that an operator $U \in \mathcal{L}(\mathfrak{H})$ is referred to as *unitary* on $\mathfrak{H}$, if and only if $\exists\, U^{-1} \in \mathcal{L}(\mathfrak{H})$ and $\forall\, x \in \mathfrak{H}$ $\|Ux\| = \|x\|$.

Let $\mathfrak{U}(\mathfrak{H}_1)$ be the set of all *unitary* operators over the space $\mathfrak{H}_1$. Fix some real number c such, that $0 < c < \infty$. Guided by [8, 11, etc], we introduce the following operators, acting in $\mathcal{M}(\mathfrak{H})$ (for every $\lambda \in [0,c)$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$)

$$\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]\,\mathrm{w} := \frac{\left(s\mathcal{T}(\mathrm{w}) - \frac{\lambda}{c^2}\langle \mathbf{n},\mathrm{w}\rangle\right)}{\sqrt{1-\frac{\lambda^2}{c^2}}}\mathbf{e}_0 + J\left(\frac{\lambda\mathcal{T}(\mathrm{w}) - s\langle \mathbf{n},\mathrm{w}\rangle}{\sqrt{1-\frac{\lambda^2}{c^2}}}\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right); \tag{9}$$

$$\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\,\mathrm{w} := \mathbf{W}_{\lambda,c}[s,\mathbf{n},J](\mathrm{w}+\mathbf{a}) \qquad (\mathrm{w} \in \mathcal{M}(\mathfrak{H})). \tag{10}$$

Under the additional conditions $\dim(\mathfrak{H}) = 3$, $s = 1$ the right-hand part of the formula (9) is equivalent to the same part of the formula (28b) from [12, p. 43].

That is why in this case we obtain the classical Lorentz transforms for inertial reference frame in the most general form (with arbitrary orientation of axes). Now we introduce the following classes of operators:

$$\begin{aligned}
\mathfrak{O}(\mathfrak{H},c) &:= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J] \mid s\in\{-1,1\},\ \lambda\in[0,c),\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\}; && (11)\\
\mathfrak{O}_+(\mathfrak{H},c) &:= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]\in\mathfrak{O}(\mathfrak{H},c) \mid s=1\} = \\
&= \{\mathbf{W}_{\lambda,c}[1,\mathbf{n},J] \mid \lambda\in[0,c),\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\}; && (12)\\
\mathfrak{P}(\mathfrak{H},c) &:= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \mid \mathbf{W}_{\lambda,c}[s,\mathbf{n},J]\in\mathfrak{O}(\mathfrak{H},c),\ \mathbf{a}\in\mathcal{M}(\mathfrak{H})\}; && (13)\\
\mathfrak{P}_+(\mathfrak{H},c) &:= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \mid \mathbf{W}_{\lambda,c}[s,\mathbf{n},J]\in\mathfrak{O}_+(\mathfrak{H},c),\ \mathbf{a}\in\mathcal{M}(\mathfrak{H})\}.
\end{aligned}$$

It is clearly that $\mathfrak{O}(\mathfrak{H},c), \mathfrak{O}_+(\mathfrak{H},c) \subseteq \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ and $\mathfrak{P}(\mathfrak{H},c), \mathfrak{P}_+(\mathfrak{H},c) \subseteq \mathcal{L}^{\times}(\mathcal{M}(\mathfrak{H}))$.

Remark 1. It can be proven that all four classes of operators are groups of operators in the space $\mathcal{M}(\mathfrak{H})$ (see [13, Remark 4.1, Corollary 4.1]; see also [10, Assertion 2.17.1 and formula (2.94), Assertion 2.17.6, Corollary 2.19.5]). In particular $\mathfrak{O}(\mathfrak{H},c)$ coincides with the group of all linear coordinate transform operators over the space $\mathcal{M}(\mathfrak{H})$, leaving unchanged values of the functional:

$$\mathsf{M}_c(\mathrm{w}) = \|\mathbf{X}\mathrm{w}\|^2 - c^2\mathcal{T}^2(\mathrm{w}) \qquad (\mathrm{w}\in\mathcal{M}(\mathfrak{H})), \tag{14}$$

that is the set of all bijective operators $L\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ such, that:

$$\mathsf{M}_c(L\mathrm{w}) = \mathsf{M}_c(\mathrm{w}) \qquad (\forall \mathrm{w}\in\mathcal{M}(\mathfrak{H})). \tag{15}$$

In the case $\mathfrak{H}=\mathbb{R}^3$ the group of operators $\mathfrak{O}_+(\mathfrak{H},c)$ coincides with the full Lorentz group, being considered in [14]. In the case $\mathfrak{H}=\mathbb{R}^3$ the group of operators $\mathfrak{P}_+(\mathfrak{H},c)$ coincides with the famous Poincaré group [10, Remark 2.19.1].

Remark 2. It should be emphasized that for every $c\in(0,\infty)$, $\lambda\in[0,c)$, $s\in\{-1,1\}$, $J\in\mathfrak{U}(\mathfrak{H}_1)$ and $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ the operator $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]$, defined in (9) is v-determined, moreover

$$\mathcal{V}(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]) = \lambda s\mathbf{n}. \tag{16}$$

Indeed, consider the vector $\mathbf{n}_{t_0,\mu_0} = t_0\mathbf{e}_0 + \mu_0\mathbf{n} \in \mathcal{M}(\mathfrak{H})$, where $t_0 = \frac{s}{\sqrt{1-\frac{\lambda^2}{c^2}}}$, $\mu_0 = \frac{\lambda}{\sqrt{1-\frac{\lambda^2}{c^2}}}$. According to (7) we have: $\mathcal{T}(\mathbf{n}_{t_0,\mu_0}) = t_0$, $\langle\mathbf{n},\mathbf{n}_{t_0,\mu_0}\rangle = \mu_0$, $\mathbf{X}\mathbf{n}_{t_0,\mu_0} = \mu_0\mathbf{n}$, $\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathbf{n}_{t_0,\mu_0} = \mathbf{0}$. Using formula (9), we obtain:

$$\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]\,\mathbf{n}_{t_0,\mu_0} = \frac{1}{\sqrt{1-\frac{\lambda^2}{c^2}}}\left(\left(st_0 - \frac{\lambda}{c^2}\mu_0\right)\mathbf{e}_0 + (\lambda t_0 - s\mu_0)\,J\mathbf{n}\right) = \mathbf{e}_0.$$

Hence, $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]^{-1}\mathbf{e}_0 = \mathbf{n}_{t_0,\mu_0}$, $\mathcal{T}\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]^{-1}\mathbf{e}_0\right) = \mathcal{T}(\mathbf{n}_{t_0,\mu_0}) = t_0 = \frac{s}{\sqrt{1-\frac{\lambda^2}{c^2}}} \neq 0$ and $\mathcal{V}(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]) = \frac{\mathbf{X}\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]^{-1}\mathbf{e}_0}{\mathcal{T}\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]^{-1}\mathbf{e}_0\right)} = \frac{\mu_0\mathbf{n}}{t_0} = \lambda s\mathbf{n}$.

If we use the function parameter $\vartheta\in\Upsilon$ (where Υ is the class of functions, satisfying (3)) instead of the constant speed c, then we obtain the following classes of operators (for each $\vartheta \in \Upsilon$):

$$\mathfrak{O}(\mathfrak{H},[\vartheta]) := \left\{\mathbf{W}_{\lambda,\vartheta(\lambda)}[s,\mathbf{n},J] \mid s\in\{-1,1\},\ \lambda\in\mathfrak{D}_*[\vartheta],\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\right\}; \quad (17)$$

$$\mathfrak{O}_+(\mathfrak{H},[\vartheta]) := \left\{\mathbf{W}_{\lambda,\vartheta(\lambda)}[s,\mathbf{n},J] \in \mathfrak{O}(\mathfrak{H},[\vartheta]) \mid s=1\right\} =$$
$$= \left\{\mathbf{W}_{\lambda,\vartheta(\lambda)}[1,\mathbf{n},\ J] \mid \lambda\in\mathfrak{D}_*[\vartheta],\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\right\}; \quad (18)$$

$$\mathfrak{P}(\mathfrak{H},[\vartheta]) := \left\{\mathbf{W}_{\lambda,\vartheta(\lambda)}[s,\mathbf{n},J;\mathbf{a}] \mid \mathbf{W}_{\lambda,\vartheta(\lambda)}[s,\mathbf{n},J]\in\mathfrak{O}(\mathfrak{H},[\vartheta]),\ \mathbf{a}\in\mathcal{M}(\mathfrak{H})\right\}; \quad (19)$$

$$\mathfrak{P}_+(\mathfrak{H},[\vartheta]) := \left\{\mathbf{W}_{\lambda,\vartheta(\lambda)}[s,\mathbf{n},J;\mathbf{a}] \mid \mathbf{W}_{\lambda,\vartheta(\lambda)}[s,\mathbf{n},J]\in\mathfrak{O}_+(\mathfrak{H},[\vartheta]),\ \mathbf{a}\in\mathcal{M}(\mathfrak{H})\right\}, \quad (20)$$

where $\mathfrak{D}_*[\vartheta]$ is the set of ϑ-*allowed* velocities, defined by (4). It is easy to see that for each $\vartheta\in\Upsilon$ the following set-theoretic inclusions are performed:

$$\mathfrak{O}_+(\mathfrak{H},[\vartheta]) \subseteq \mathfrak{O}(\mathfrak{H},[\vartheta]) \subseteq \mathfrak{P}(\mathfrak{H},[\vartheta]); \quad (21)$$
$$\mathfrak{O}_+(\mathfrak{H},[\vartheta]) \subseteq \mathfrak{P}_+(\mathfrak{H},[\vartheta]) \subseteq \mathfrak{P}(\mathfrak{H},[\vartheta]). \quad (22)$$

- $\mathfrak{O}(\mathfrak{H},[\vartheta])$ is called a class of *generalized Hassani transforms* over Hilbert space $\mathfrak{H}$;
- $\mathfrak{O}_+(\mathfrak{H},[\vartheta])$ is called a class of *time-positive generalized Hassani transforms* over Hilbert space $\mathfrak{H}$;
- $\mathfrak{P}(\mathfrak{H},[\vartheta])$ is called a class of *Poincare-Hassani transforms* over Hilbert space $\mathfrak{H}$;
- $\mathfrak{P}_+(\mathfrak{H},[\vartheta])$ is called a class of *time-positive Poincare-Hassani transforms* over Hilbert space $\mathfrak{H}$.

3. On the group property of Hassani transforms. We have seen above that in the case of constant speed of light c all classes of operators $\mathfrak{O}(\mathfrak{H},c)$, $\mathfrak{O}_+(\mathfrak{H},c)$, $\mathfrak{P}(\mathfrak{H},c)$, $\mathfrak{P}_+(\mathfrak{H},c)$ are groups in the space $\mathcal{M}(\mathfrak{H})$.

Theorem 1. *Let $\vartheta\in\Upsilon$ and $\mathcal{Q}\in\{\mathfrak{O}(\mathfrak{H},[\vartheta]),\mathfrak{O}_+(\mathfrak{H},[\vartheta]),\mathfrak{P}(\mathfrak{H},[\vartheta]),\mathfrak{P}_+(\mathfrak{H},[\vartheta])\}$ be one of the four classes of operators introduced above. Then the class $\mathcal{Q}$ is a group of operators in the space $\mathcal{M}(\mathfrak{H})$ if and only if a number $c\in(0,\infty)$ exists such that $\vartheta(\lambda)=\vartheta_c(\lambda)$ for every $\lambda\in(0,\infty)$, where $\vartheta_c(\lambda)$ is the function defined by (6).*

Remark 3. Since (by (9), (10)) $\mathbf{W}_{0,\chi_1}[s,\mathbf{n},J]=\mathbf{W}_{0,\chi_2}[s,\mathbf{n},J]$ and $\mathbf{W}_{0,\chi_1}[s,\mathbf{n},J;\mathbf{a}]=\mathbf{W}_{0,\chi_2}[s,\mathbf{n},J;\mathbf{a}]$ $(\forall s\in\{-1,1\}\ \forall J\in\mathfrak{U}(\mathfrak{H}_1)\ \forall\chi_1,\chi_2>0\ \forall\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)\ \forall\mathbf{a}\in\mathcal{M}(\mathfrak{H}))$, in the case where there exists a number $c\in(0,\infty)$ such that $\vartheta(\lambda)=\vartheta_c(\lambda)$ $(\forall\,\lambda\in(0,\infty))$, we have

$$\begin{cases} \mathfrak{O}(\mathfrak{H},[\vartheta])=\mathfrak{O}(\mathfrak{H},[\vartheta_c])=\mathfrak{O}(\mathfrak{H},c); & \mathfrak{P}(\mathfrak{H},[\vartheta])=\mathfrak{P}(\mathfrak{H},c); \\ \mathfrak{O}_+(\mathfrak{H},[\vartheta])=\mathfrak{O}_+(\mathfrak{H},c); & \mathfrak{P}_+(\mathfrak{H},[\vartheta])=\mathfrak{P}_+(\mathfrak{H},c). \end{cases} \quad (23)$$

Hence, Theorem 1 asserts that the class

$$\mathcal{Q}\in\{\mathfrak{O}(\mathfrak{H},[\vartheta]),\ \mathfrak{O}_+(\mathfrak{H},[\vartheta]),\ \mathfrak{P}(\mathfrak{H},[\vartheta]),\ \mathfrak{P}_+(\mathfrak{H},[\vartheta])\}$$

is a group of operators in the space $\mathcal{M}(\mathfrak{H})$ if and only if $\mathcal{Q}$ coincides with the some ordinary Lorentz or Poincare group (with the constant velocity of light).

To prove Theorem 1, we need two following lemmas. To formulate these lemmas, we introduce some notations below. Let $\vartheta\in\Upsilon$. Chose any vector $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$. We will use the notation $\mathbb{I}_{\sigma,\mu}[\mathbf{n}]$ for the following operator, acting in the space $\mathfrak{H}_1$:

$$\mathbb{I}_{\sigma,\mu}[\mathbf{n}]\,x := \sigma\mathbf{X}_1[\mathbf{n}]\,x+\mu\mathbf{X}_1^{\perp}[\mathbf{n}]\,x=\sigma\langle\mathbf{n},x\rangle\,\mathbf{n}+\mu\mathbf{X}_1^{\perp}[\mathbf{n}]\,x \quad (x\in\mathfrak{H}_1;\ \sigma,\mu\in\{-1,1\}).$$

It is apparently that $\mathbb{I}_{\sigma,\mu}[\mathbf{n}]\in\mathfrak{U}(\mathfrak{H}_1)$ $(\forall\,\sigma,\mu\in\{-1,1\})$. Let us consider any numbers $t,\mu\in\mathbb{R}$. We use the notation $\mathbf{n}_{t,\mu}$ for the vector of the form

$$\mathbf{n}_{t,\mu}=t\mathbf{e}_0+\mu\mathbf{n}\in\mathcal{M}(\mathfrak{H}).$$

Lemma 1. *For any* $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$, $c_1,c_2\in(0,+\infty)$, $\lambda_1\in[0,c_1)$, $\lambda_2\in[0,c_2)$ *and* $t,\mu\in\mathbb{R}$ *the following equality holds:*

$$\mathbf{W}_{\lambda_2,c_2}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\mathbf{W}_{\lambda_1,c_1}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\mathbf{n}_{t,\mu}=\frac{1}{\widetilde{\gamma}}\big((\alpha t+\beta\mu)\,\mathbf{e}_0+(\gamma t+\delta\mu)\,\mathbf{n}\big),\qquad(24)$$

where

$$\begin{cases}\widetilde{\gamma}=\sqrt{\left(1-\frac{\lambda_1^2}{c_1^2}\right)\left(1-\frac{\lambda_2^2}{c_2^2}\right)},\\ \alpha=1+\frac{\lambda_1\lambda_2}{c_2^2},\quad \beta=-\left(\frac{\lambda_2}{c_2^2}+\frac{\lambda_1}{c_1^2}\right),\\ \gamma=-\left(\lambda_2+\lambda_1\right),\quad \delta=1+\frac{\lambda_1\lambda_2}{c_1^2},\end{cases}\qquad(25)$$

$$\alpha,\ \delta,\ \widetilde{\gamma}>0.\qquad(26)$$

Proof. Denote, $\mathbf{L}^{(c_1,c_2)}_{(\lambda_1,\lambda_2)}:=\mathbf{W}_{\lambda_2,c_2}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\mathbf{W}_{\lambda_1,c_1}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]$.

It is easy to see that $\mathbf{L}^{(c_1,c_2)}_{(\lambda_1,\lambda_2)}\mathbf{n}_{t,\mu}=\mathbf{W}_{\lambda_2,c_2}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\mathrm{w}_{t,\mu}$, where $\mathrm{w}_{t,\mu}=\mathbf{W}_{\lambda_1,c_1}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\mathbf{n}_{t,\mu}$. Hence, taking into account the formulas (9) and (8), we successively deduce

$$\mathrm{w}_{t,\mu}=\mathbf{W}_{\lambda_1,c_1}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\mathbf{n}_{t,\mu}=\frac{1}{\sqrt{1-\frac{\lambda_1^2}{c_1^2}}}\left(\left(t-\frac{\lambda_1\mu}{c_1^2}\right)\mathbf{e}_0+(\mu-\lambda_1 t)\,\mathbf{n}\right);$$

$$\mathcal{T}(\mathrm{w}_{t,\mu})=\frac{t-\frac{\lambda_1\mu}{c_1^2}}{\sqrt{1-\frac{\lambda_1^2}{c_1^2}}},\quad \langle\mathbf{n},\mathrm{w}_{t,\mu}\rangle=\frac{\mu-\lambda_1 t}{\sqrt{1-\frac{\lambda_1^2}{c_1^2}}},\quad \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}_{t,\mu}=\mathbf{0};$$

$$\mathbf{L}^{(c_1,c_2)}_{(\lambda_1,\lambda_2)}\mathbf{n}_{t,\mu}=\mathbf{W}_{\lambda_2,c_2}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\mathrm{w}_{t,\mu}=$$

$$=\frac{1}{\sqrt{\left(1-\frac{\lambda_1^2}{c_1^2}\right)\left(1-\frac{\lambda_2^2}{c_2^2}\right)}}\left(\left(t-\frac{\lambda_1\mu}{c_1^2}-\frac{(\mu-\lambda_1 t)\,\lambda_2}{c_2^2}\right)\mathbf{e}_0+\right.$$

$$\left.+\left((\mu-\lambda_1 t)-\lambda_2\left(t-\frac{\lambda_1\mu}{c_1^2}\right)\right)\mathbf{n}\right)=$$

$$=\frac{1}{\sqrt{\left(1-\frac{\lambda_1^2}{c_1^2}\right)\left(1-\frac{\lambda_2^2}{c_2^2}\right)}}\left(\left(t\left(1+\frac{\lambda_1\lambda_2}{c_2^2}\right)-\mu\left(\frac{\lambda_2}{c_2^2}+\frac{\lambda_1}{c_1^2}\right)\right)\mathbf{e}_0+\right.$$

$$\left.+\left(\mu\left(1+\frac{\lambda_1\lambda_2}{c_1^2}\right)-t\,(\lambda_2+\lambda_1)\right)\mathbf{n}\right)=\frac{1}{\widetilde{\gamma}}\big((\alpha t+\beta\mu)\,\mathbf{e}_0+(\gamma t+\delta\mu)\,\mathbf{n}\big).$$

The lemma is proved. □

Lemma 2. *Assume that* $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$, $c_1,c_2\in(0,+\infty)$, $\lambda_1\in(0,c_1)$, $\lambda_2\in(0,c_2)$, *and, moreover,* $c_1\neq c_2$. *Then there does not exist any number* $c\in(0,\infty)$ *such that* $\mathbf{W}_{\lambda_2,c_2}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\mathbf{W}_{\lambda_1,c_1}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\in\mathfrak{P}(\mathfrak{H},c)$.

Proof. Denote, $\mathbf{L}^{(c_1,c_2)}_{(\lambda_1,\lambda_2)}:=\mathbf{W}_{\lambda_2,c_2}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\mathbf{W}_{\lambda_1,c_1}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]$.

Assume on the contrary $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} \in \mathfrak{P}(\mathfrak{H},c)$ for some $c \in (0,\infty)$. Then, according to (13), (11) the elements $\lambda \in [0,c)$, $s \in \{-1,1\}$, $\mathbf{n}_0 \in \mathbf{B}_1(\mathfrak{H}_1)$, $J \in \mathfrak{U}(\mathfrak{H}_1)$ and $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$ must exist such that $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} = \mathbf{W}_{\lambda,c}[s,\mathbf{n}_0,J;\mathbf{a}]$. But by (9), $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}\mathbf{0} = \mathbf{W}_{\lambda_2,c_2}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathbf{W}_{\lambda_1,c_1}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathbf{0} = \mathbf{0}$. Hence, we see that $\mathbf{W}_{\lambda,c}[s,\mathbf{n}_0,J]\,\mathbf{a} = \mathbf{W}_{\lambda,c}[s,\mathbf{n}_0,J;\mathbf{a}]\,\mathbf{0} = \mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}\mathbf{0} = \mathbf{0}$, that is (since, by Remark 1, the linear operator $\mathbf{W}_{\lambda,c}[s,\mathbf{n}_0,J]$ is bijective), we get, $\mathbf{a} = \mathbf{0}$. Hence, (according to (11)), we have $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} = \mathbf{W}_{\lambda,c}[s,\mathbf{n}_0,J;\mathbf{0}] = \mathbf{W}_{\lambda,c}[s,\mathbf{n}_0,J] \in \mathfrak{O}(\mathfrak{H},c)$. So, according to Remark 1, for the operator $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}$ the condition (15) is fulfilled. Hence, taking into account (14), we obtain

$$\mathsf{M}_c\left(\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}\mathbf{n}_{t,\mu}\right) = \mathsf{M}_c(\mathbf{n}_{t,\mu}) = \mu^2 - c^2t^2 \qquad (\forall t,\mu \in \mathbb{R}). \tag{27}$$

On the other hand, using Lemma 1 (formula (24)) and (14), we deduce

$$\mathsf{M}_c\left(\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}\mathbf{n}_{t,\mu}\right) = \frac{1}{\widetilde{\gamma}^2}\left((\gamma t+\delta\mu)^2 - c^2(\alpha t+\beta\mu)^2\right) \qquad (\forall t,\mu \in \mathbb{R}). \tag{28}$$

From equalities (27), (28) we deduce the equality:

$$\frac{1}{\widetilde{\gamma}^2}\left((\gamma t+\delta\mu)^2 - c^2(\alpha t+\beta\mu)^2\right) = \mu^2 - c^2t^2 \quad (\forall t,\mu \in \mathbb{R}). \tag{29}$$

Substituting the values $t := 1$, $\mu := c$ and $t := 1$, $\mu := -c$ as well as $t := 1$, $\mu := 0$ into (29) we conclude that the following equalities are true

$$\begin{cases} (\gamma + c\delta)^2 & = & c^2(\alpha + c\beta)^2; \\ (\gamma - c\delta)^2 & = & c^2(\alpha - c\beta)^2; \\ \gamma^2 - c^2\alpha^2 & = & -c^2\widetilde{\gamma}^2. \end{cases} \tag{30}$$

System (30) is satisfied in the following four cases considered below.

$$\textbf{Case 1}: \quad \begin{cases} \gamma + c\delta & = & -c(\alpha + c\beta); \\ \gamma - c\delta & = & c(\alpha - c\beta); \\ \gamma^2 - c^2\alpha^2 & = & -c^2\widetilde{\gamma}^2. \end{cases} \tag{31}$$

In this case, from the first two equalities of (31) we obtain the equality $2c\delta = -2c\alpha$, that is $\delta = -\alpha$, which contradicts to (26), because, according to (26), the both numbers δ and α must be positive.

$$\textbf{Case 2}: \quad \begin{cases} \gamma + c\delta & = & -c(\alpha + c\beta); \\ \gamma - c\delta & = & -c(\alpha - c\beta); \\ \gamma^2 - c^2\alpha^2 & = & -c^2\widetilde{\gamma}^2. \end{cases} \tag{32}$$

In this case, from the first two equalities of (32) we obtain the equality $\gamma = -c\alpha$. And substituting this value of γ into the third equality of (32) we obtain the equality $\widetilde{\gamma} = 0$, which contradicts to (26), because, according to (26), the number $\widetilde{\gamma}$ is positive.

$$\textbf{Case 3}: \quad \begin{cases} \gamma + c\delta & = & c(\alpha + c\beta); \\ \gamma - c\delta & = & c(\alpha - c\beta); \\ \gamma^2 - c^2\alpha^2 & = & -c^2\widetilde{\gamma}^2. \end{cases} \tag{33}$$

In this case from the first two equalities of (33) we obtain the equality $\gamma = c\alpha$. And substituting this value of γ into the third equality of (33) we obtain the equality $\widetilde{\gamma} = 0$, which contradicts to (26), because, according to (26), this number must be positive.

$$\textbf{Case 4}: \qquad \begin{cases} \gamma + c\delta & = \ c\,(\alpha + c\beta)\,; \\ \gamma - c\delta & = \ -c\,(\alpha - c\beta)\,; \\ \gamma^2 - c^2\alpha^2 & = \ -c^2\,\widetilde{\gamma}^{\,2}. \end{cases} \tag{34}$$

In this case, from the first two equalities of (34) we obtain the equality $\delta = \alpha$, where, according to (25), $\alpha = 1 + \frac{\lambda_1\lambda_2}{c_2^2}$, $\delta = 1 + \frac{\lambda_1\lambda_2}{c_1^2}$. So, taking into account that $\lambda_i > 0$ $\left(i \in \overline{1,2}\right)$ (according to the conditions of this lemma), we obtain the equality, $c_1 = c_2$, which contradicts to the hypotheses of this lemma. Therefore, the assumption that $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} \in \mathfrak{P}\,(\mathfrak{H}, c)$ leads to contradiction in the all possible cases. Thus, $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} \notin \mathfrak{P}\,(\mathfrak{H}, c)$, that it was necessary to prove. □

Proof of Theorem 1. Let $\vartheta \in \Upsilon$. Assume that the operator class $\mathcal{Q} \in \{\mathfrak{O}\,(\mathfrak{H}, [\vartheta]),$ $\mathfrak{O}_+\,(\mathfrak{H}, [\vartheta]),\ \mathfrak{P}\,(\mathfrak{H}, [\vartheta]),\ \mathfrak{P}_+\,(\mathfrak{H}, [\vartheta])\}$ is a group. Since $\vartheta \in \Upsilon$, according to (5), we have $\mathfrak{D}_*\,[\vartheta] \setminus \{0\} \neq \varnothing$. First, we prove that

$$\vartheta(\lambda) \equiv \mathrm{const} \qquad (\forall \lambda \in \mathfrak{D}_*\,[\vartheta] \setminus \{0\})\,. \tag{35}$$

Let us consider any numbers $\lambda_1, \lambda_2 \in \mathfrak{D}_*\,[\vartheta] \setminus \{0\}$ $(\lambda_1, \lambda_2 > 0)$. Chose any vector $\mathbf{n} \in \mathbf{B}_1\,(\mathfrak{H}_1)$. Denote

$$c_i := \vartheta\,(\lambda_i) \qquad \left(i \in \overline{1,2}\right), \tag{36}$$

where $\overline{1,n} = \{1, \ldots, n\}$ $(n \in \mathbb{N})$. Then we obtain $0 < \lambda_i < c_i$ $\left(i \in \overline{1,2}\right)$ (according to (4)) and $\mathbf{W}_{\lambda_i,c_i}\,[1, \mathbf{n}, \mathbb{I}_{-1,1}\,[\mathbf{n}]] \in \mathfrak{O}_+\,(\mathfrak{H}, [\vartheta])$ $\left(i \in \overline{1,2}\right)$. So, according to the inclusions (21), (22), we have:

$$\mathbf{W}_{\lambda_i,c_i}\,[1, \mathbf{n}, \mathbb{I}_{-1,1}\,[\mathbf{n}]] \in \mathcal{Q} \qquad \left(i \in \overline{1,2}\right).$$

Since, according to the above assumption, the class $\mathcal{Q}$ is a group, the operator $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} := \mathbf{W}_{\lambda_2,c_2}\,[1, \mathbf{n}, \mathbb{I}_{-1,1}\,[\mathbf{n}]]\,\mathbf{W}_{\lambda_1,c_1}\,[1, \mathbf{n}, \mathbb{I}_{-1,1}\,[\mathbf{n}]]$ belongs to $\mathcal{Q}$. Since $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} \in \mathcal{Q}$ then, according to inclusions (21), (22), we get $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} \in \mathfrak{P}\,(\mathfrak{H}, [\vartheta])$. So, according to (19), (17) the number $c_0 \in (0, \infty)$ must exist such that $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} \in \mathfrak{P}\,(\mathfrak{H}, c_0)$, i.e. $\mathbf{W}_{\lambda_2,c_2}\,[1, \mathbf{n}, \mathbb{I}_{-1,1}\,[\mathbf{n}]]\,\mathbf{W}_{\lambda_1,c_1}\,[1, \mathbf{n}, \mathbb{I}_{-1,1}\,[\mathbf{n}]] \in \mathfrak{P}\,(\mathfrak{H}, c_0)$.

The last correlation, according to Lemma 2 is possible only in the case $c_1 = c_2$. And, taking into account (36) we deduce $\vartheta\,(\lambda_1) = \vartheta\,(\lambda_2)$ $(\forall \lambda_1\lambda_2 \in \mathfrak{D}_*\,[\vartheta] \setminus \{0\})$. So the correlation (35) is valid. And, in accordance with (35) the number $c \in (0, \infty)$ exists such that

$$\vartheta(\lambda) = c \qquad (\forall \lambda \in \mathfrak{D}_*\,[\vartheta] \setminus \{0\})\,. \tag{37}$$

The next aim is to prove that: $\mathfrak{D}_*\,[\vartheta] = [0, c)$. Using correlations (4) and (37) for each $\lambda \in \mathfrak{D}_*\,[\vartheta] \setminus \{0\}$ we obtain, $0 \leq \lambda < \vartheta\,(\lambda) = c$. So, we have the inclusion

$$\mathfrak{D}_*\,[\vartheta] \subseteq [0, c). \tag{38}$$

So it remains to prove the inverse inclusion. Chose any fixed $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$. Applying the formula (24) for the case $c_1=c_2=c$ and $\lambda_1=\lambda_2=\lambda\in\mathfrak{D}_*[\vartheta]\subseteq[0,c)$ we obtain

$$\begin{aligned}\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2\mathbf{n}_{t,\mu}&=\mathbf{L}^{(c,c)}_{(\lambda,\lambda)}\mathbf{n}_{t,\mu}=\\&=\frac{1}{\sqrt{\left(1-\frac{\lambda^2}{c^2}\right)^2}}\left(\left(t\left(1+\frac{\lambda^2}{c^2}\right)-\mu\left(\frac{2\lambda}{c^2}\right)\right)\mathbf{e}_0+\left(\mu\left(1+\frac{\lambda^2}{c^2}\right)-t\cdot2\lambda\right)\mathbf{n}\right)=\\&=\frac{1}{\sqrt{1-\frac{\xi(\lambda)^2}{c^2}}}\left(\left(t-\mu\frac{\xi(\lambda)}{c^2}\right)\mathbf{e}_0+\left(\mu-t\,\xi(\lambda)\right)\mathbf{n}\right)\quad(\forall t,\mu\in\mathbb{R}),\end{aligned}\tag{39}$$

where $\xi(x)=\frac{2x}{1+\frac{x^2}{c^2}}$ $(x\in\mathbb{R})$. Taking into account (37) and (18), we see that $\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]=\mathbf{W}_{\lambda,\vartheta(\lambda)}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\in\mathfrak{D}_+(\mathfrak{H},[\vartheta])$. So, according to inclusions (21), (22), we have: $\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]\in\mathcal{Q}$. Since, by the above assumption, the class $\mathcal{Q}$ is a group, the operator $\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2$ also must belong to $\mathcal{Q}$, and, according to inclusions (21), (22), we get $\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2\in\mathfrak{P}(\mathfrak{H},[\vartheta])$. So, by (19), (17) the elements $\lambda_*\in\mathfrak{D}_*[\vartheta]$, $s\in\{-1,1\}$, $\mathbf{n}_*\in\mathbf{B}_1(\mathfrak{H}_1)$, $J\in\mathfrak{U}(\mathfrak{H}_1)$ and $\mathbf{a}\in\mathcal{M}(\mathfrak{H})$ must exist such that $\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2=\mathbf{W}_{\lambda_*,\vartheta(\lambda_*)}[s,\mathbf{n}_*,J;\mathbf{a}]$. But by (9), $\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2\mathbf{0}=\mathbf{0}$. Hence, $\mathbf{W}_{\lambda_*,\vartheta(\lambda_*)}[s,\mathbf{n}_*,J]\,\mathbf{a}=\mathbf{W}_{\lambda_*,\vartheta(\lambda_*)}[s,\mathbf{n}_*,J;\mathbf{a}]\,\mathbf{0}=\mathbf{0}$. Therefore, we have, $\mathbf{a}=\mathbf{0}$. Thus, $\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2=\mathbf{W}_{\lambda_*,\vartheta(\lambda_*)}[s,\mathbf{n}_*,J]$. And, taking into account (37), we get $\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2=\mathbf{W}_{\lambda_*,c}[s,\mathbf{n}_*,J]$.

Thence, using the formula (16), we deduce

$$\mathcal{V}\left(\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2\right)=\lambda_*s\mathbf{n}_*.\tag{40}$$

From the other hand, applying equality (39) for the vector $\mathbf{n}_{t_0^*,\mu_0^*}$, where $t_0^*=\frac{1}{\sqrt{1-\frac{\xi(\lambda)^2}{c^2}}}$, $\mu_0^*=\frac{\xi(\lambda)}{\sqrt{1-\frac{\xi(\lambda)^2}{c^2}}}$, we obtain

$$\begin{aligned}&\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2\mathbf{n}_{t_0^*,\mu_0^*}=\\&\qquad=\frac{1}{\sqrt{1-\frac{\xi(\lambda)^2}{c^2}}}\left(\left(t_0^*-\mu_0^*\frac{\xi(\lambda)}{c^2}\right)\mathbf{e}_0+\left(\mu_0^*-t_0^*\,\xi(\lambda)\right)\mathbf{n}\right)=\mathbf{e}_0.\end{aligned}$$

So, by Definition 1, we deduce

$$\begin{aligned}\mathcal{V}\left(\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2\right)&=\frac{\mathbf{X}\left(\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2\right)^{-1}\mathbf{e}_0}{\mathcal{T}\left(\left(\mathbf{W}_{\lambda,c}\left[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right]^2\right)^{-1}\mathbf{e}_0\right)}=\\&=\frac{\mathbf{X}\mathbf{n}_{t_0^*,\mu_0^*}}{\mathcal{T}\left(\mathbf{n}_{t_0^*,\mu_0^*}\right)}=\frac{\mu_0^*\mathbf{n}}{t_0^*}=\xi(\lambda)\,\mathbf{n}.\end{aligned}\tag{41}$$

From equalities (40) and (41) it follows that $\lambda_*s\mathbf{n}_*=\xi(\lambda)\,\mathbf{n}$. And taking into account that $\lambda,\lambda_*\geq0$, $\xi(\lambda)=\frac{2\lambda}{1+\frac{\lambda^2}{c^2}}\geq0$, $|s|=1$, $\|\mathbf{n}\|=\|\mathbf{n}_*\|=1$, we obtain the equality $\xi(\lambda)=\lambda_*$. Since $\lambda_*\in\mathfrak{D}_*[\vartheta]$, we have $\xi(\lambda)\in\mathfrak{D}_*[\vartheta]$. Thus, we have proven the following statement

$$\forall\lambda\in\mathfrak{D}_*[\vartheta]\quad\left(\xi(\lambda)\in\mathfrak{D}_*[\vartheta]\right).\tag{42}$$

The function $\xi(x)$ obeys the following three properties:
1^0. $\xi(x)$ *is continuous and increasing on* $[0,c]$.

Indeed, it is apparently that $\xi(x)$ is differentiable and thus continuous on $[0,c]$. Moreover, simple calculation shows that $\frac{d}{dx}\xi(x) = 2\frac{1-\frac{x^2}{c^2}}{\left(1+\frac{x^2}{c^2}\right)^2} > 0$ for $x \in [0,c)$.

2^0. $\xi(\lambda) > \lambda$ *for* $\lambda \in (0,c)$.

Indeed, for $\lambda \in (0,c)$ we have, $\xi(\lambda) = \frac{2\lambda}{1+\frac{\lambda^2}{c^2}} > \frac{2\lambda}{1+\frac{c^2}{c^2}} = \lambda$.

3^0. $\xi(\lambda) \in [0,c)$ *for* $\lambda \in [0,c)$.

Indeed, using Property 1^0, for $\lambda \in [0,c)$ we obtain, $0 \le \xi(\lambda) < \xi(c) = \frac{2c}{1+\frac{c^2}{c^2}} = c$.

According to (5), the number $\eta_0 \in (0,c)$ exists such that:

$$[0,\eta_0) \subseteq \mathfrak{D}_*[\vartheta] \tag{43}$$

Taking into account that $\xi(0) = 0$ and using Property 1^0 as well as statement (42), from inclusion (43) we deduce that $[0,\eta_1) \subseteq \mathfrak{D}_*[\vartheta]$, where $\eta_1 = \xi(\eta_0)$. Thus, recursively we obtain the inclusions:

$$[0,\eta_n) \subseteq \mathfrak{D}_*[\vartheta] \qquad (\forall n \in \mathbb{N}), \tag{44}$$

where $\eta_{k+1} = \xi(\eta_k)\ \ (k \in \mathbb{N})$. From Property 2^0 it follows that the sequence $(\eta_n)_{n=0}^{\infty}$ is increasing. From Property 3^0, taking into account that $\eta_0 \in (0,c)$, we obtain

$$\eta_n \in (0,c) \quad (n \in \mathbb{N}). \tag{45}$$

Thus, the sequence $(\eta_n)_{n=0}^{\infty}$ is monotonous and bounded. And, by the Weierstrass theorem, there exists the number $\eta_\infty \in \mathbb{R}$ such that $\eta_\infty = \lim\limits_{n\to\infty} \eta_n$.

Since the sequence $(\eta_n)_{n=0}^{\infty}$ is increasing, from (45) we obtain the inequality:

$$0 < \eta_\infty \le c \tag{46}$$

and from (44) we obtain the inclusion

$$[0,\eta_\infty) = \bigcup_{n=1}^{\infty} [0,\eta_n) \subseteq \mathfrak{D}_*[\vartheta]. \tag{47}$$

Since $\eta_{k+1} = \xi(\eta_k)$ $(\forall k \in \mathbb{N})$, we obtain the following equality $\eta_\infty = \xi(\eta_\infty)$. In view of inequality (46), the number η_∞ is the *positive* solution of the equation $x = \frac{2x}{1+\frac{x^2}{c^2}}$. Simple calculation shows that the last equation has only one positive solution $x = c$. Therefore, $\eta_\infty = c$. And taking into account the inclusion (47), we obtain $[0,c) \subseteq \mathfrak{D}_*[\vartheta]$. The last inclusion together with the inclusion (38) proves the equality

$$\mathfrak{D}_*[\vartheta] = [0,c),. \tag{48}$$

Using (48), for $\lambda \in (0,c) = \mathfrak{D}_*[\vartheta] \setminus \{0\}$ we obtain $\vartheta(\lambda) = c$ (by (37)). For $\lambda \in [c,\infty)$, according to (48), we have $\lambda \notin \mathfrak{D}_*[\vartheta]$. Therefore, in this case by (4) and (3), we obtain $\vartheta(\lambda) = \lambda$. Thus, by (6), it follows $\vartheta(\lambda) = \vartheta_c(\lambda)$ $(\forall\,\lambda \in (0,\infty))$.

Conversely, if $\vartheta(\lambda) = \vartheta_c(\lambda)$ $(\forall\,\lambda \in (0,\infty))$ then by equalities (23) and Remark 1 the class $\mathcal{Q} \in \{\mathfrak{O}(\mathfrak{H},[\vartheta]),\ \mathfrak{O}_+(\mathfrak{H},[\vartheta]),\ \mathfrak{P}(\mathfrak{H},[\vartheta]),\ \mathfrak{P}_+(\mathfrak{H},[\vartheta])\}$ is a group of operators in the space $\mathcal{M}(\mathfrak{H})$. $\square$

Remark 4. Thus, Theorem 1 asserts that in the case of non-constant velocity of light (where there is no number $c \in (0,\infty)$ such that $\vartheta(\lambda) = \vartheta_c(\lambda)$ $(\forall\, \lambda \in (0,\infty))$) each of the classes of operators $\mathfrak{O}\left(\mathfrak{H},[\vartheta]\right)$, $\mathfrak{O}_+\left(\mathfrak{H},[\vartheta]\right)$, $\mathfrak{P}\left(\mathfrak{H},[\vartheta]\right)$, $\mathfrak{P}_+\left(\mathfrak{H},[\vartheta]\right)$ is not a group of operators over the space $\mathcal{M}\left(\mathfrak{H}\right)$, because the product (composition) of two operators from the class $\mathcal{Q} \in \{\mathfrak{O}\left(\mathfrak{H},[\vartheta]\right),\ \mathfrak{O}_+\left(\mathfrak{H},[\vartheta]\right),\ \mathfrak{P}\left(\mathfrak{H},[\vartheta]\right),\ \mathfrak{P}_+\left(\mathfrak{H},[\vartheta]\right)\}$ does not belong to $\mathcal{Q}$ in the general case.

From the other hand, these classes of operators have the following properties.

Properties. *Let $\vartheta \in \Upsilon$ and $\mathcal{Q} \in \{\mathfrak{O}\left(\mathfrak{H},[\vartheta]\right), \mathfrak{O}_+\left(\mathfrak{H},[\vartheta]\right), \mathfrak{P}\left(\mathfrak{H},[\vartheta]\right), \mathfrak{P}_+\left(\mathfrak{H},[\vartheta]\right)\}$. Then:*

1. *$\mathbb{I} \in \mathcal{Q}$, where $\mathbb{I} = \mathbb{I}_{\mathcal{M}(\mathfrak{H})}$ is the identity operator over the space $\mathcal{M}\left(\mathfrak{H}\right)$.*
2. *If $U \in \mathcal{Q}$ then $U^{-1} \in \mathcal{Q}$.*
3. *If $\mathcal{Q} \in \{\mathfrak{O}\left(\mathfrak{H},[\vartheta]\right), \mathfrak{P}\left(\mathfrak{H},[\vartheta]\right)\}$ and $U \in \mathcal{Q}$ then $-U \in \mathcal{Q}$.*

Proof. **1**. Chose any vector $\mathbf{n} \in \mathbf{B}_1\left(\mathfrak{H}_1\right)$. Using formula (9) we readily obtain the equality, $\mathbf{W}_{0,\vartheta(0)}\left[1, \mathbf{n}, \mathbb{I}_{-1,1}\left[\mathbf{n}\right]\right] = \mathbb{I}$. In view of inclusions (21), (22), we have $\mathbb{I} \in \mathfrak{O}_+\left(\mathfrak{H},[\vartheta]\right) \subseteq \mathcal{Q} \in \{\mathfrak{O}\left(\mathfrak{H},[\vartheta]\right), \mathfrak{O}_+\left(\mathfrak{H},[\vartheta]\right), \mathfrak{P}\left(\mathfrak{H},[\vartheta]\right), \mathfrak{P}_+\left(\mathfrak{H},[\vartheta]\right)\}$.

2. Using [15, Corollary 4] for any $s \in \{-1,1\}$, $\lambda \in \mathfrak{D}_*\left[\vartheta\right]$, $\mathbf{n} \in \mathbf{B}_1\left(\mathfrak{H}_1\right)$, $J \in \mathfrak{U}\left(\mathfrak{H}_1\right)$ and $\mathbf{a} \in \mathcal{M}\left(\mathfrak{H}\right)$ we obtain

$$\left(\mathbf{W}_{\lambda,\vartheta(\lambda)}\left[s, \mathbf{n}, J; \mathbf{a}\right]\right)^{-1} = \mathbf{W}_{\lambda,\vartheta(\lambda)}\left[s,\ J\mathbf{n},\ J^{-1}; \widetilde{\mathbf{a}}\right],$$

where $\widetilde{\mathbf{a}} = -\left(\mathbf{W}_{\lambda,\vartheta(\lambda)}\left[s,\ J\mathbf{n}, J^{-1}\right]\right)^{-1}\mathbf{a} \in \mathcal{M}\left(\mathfrak{H}\right)$ (so if $\mathbf{a} = \mathbf{0}$ then $\widetilde{\mathbf{a}} = \mathbf{0}$). The last formula shows that the operation of taking the inverse operator is closed in the class $\mathcal{Q} \in \{\mathfrak{O}\left(\mathfrak{H},[\vartheta]\right), \mathfrak{O}_+\left(\mathfrak{H},[\vartheta]\right), \mathfrak{P}\left(\mathfrak{H},[\vartheta]\right), \mathfrak{P}_+\left(\mathfrak{H},[\vartheta]\right)\}$.

3. In the case $\mathcal{Q} = \mathfrak{O}\left(\mathfrak{H},[\vartheta]\right)$ the operator $U \in \mathcal{Q}$ can be represented in the form $U = \mathbf{W}_{\lambda,\vartheta(\lambda)}\left[s, \mathbf{n}, J\right]$, where $s \in \{-1,1\}$, $\lambda \in \mathfrak{D}_*\left[\vartheta\right]$, $\mathbf{n} \in \mathbf{B}_1\left(\mathfrak{H}_1\right)$ and $J \in \mathfrak{U}\left(\mathfrak{H}_1\right)$. Hence, using elementary calculations and formula (9), we obtain $-U = \mathbf{W}_{\lambda,\vartheta(\lambda)}\left[-s, -\mathbf{n}, J\mathbb{I}_{1,-1}\left[\mathbf{n}\right]\right] \in \mathfrak{O}\left(\mathfrak{H},[\vartheta]\right) = \mathcal{Q}$.

In the case $\mathcal{Q} = \mathfrak{P}\left(\mathfrak{H},[\vartheta]\right)$ the operator $U \in \mathcal{Q}$ can be represented in the form $U = \mathbf{W}_{\lambda,\vartheta(\lambda)}\left[s, \mathbf{n}, J; \mathbf{a}\right]$, where $s \in \{-1,1\}$, $\lambda \in \mathfrak{D}_*\left[\vartheta\right]$, $\mathbf{n} \in \mathbf{B}_1\left(\mathfrak{H}_1\right)$, $J \in \mathfrak{U}\left(\mathfrak{H}_1\right)$ and $\mathbf{a} \in \mathcal{M}\left(\mathfrak{H}\right)$. Therefore, (according to (10)) for $\mathrm{w} \in \mathcal{M}\left(\mathfrak{H}\right)$ we deduce

$$-U\mathrm{w} = -\mathbf{W}_{\lambda,\vartheta(\lambda)}\left[s, \mathbf{n}, J\right]\left(\mathrm{w} + \mathbf{a}\right) = \mathbf{W}_{\lambda,\vartheta(\lambda)}\left[s_1, \mathbf{n}_1, J_1\right]\left(\mathrm{w} + \mathbf{a}\right) = \mathbf{W}_{\lambda,\vartheta(\lambda)}\left[s_1, \mathbf{n}_1, J_1; \mathbf{a}\right]\mathrm{w},$$

where $s_1 = -s, \mathbf{n}_1 = -\mathbf{n}, J_1 = J\mathbb{I}_{1,-1}\left[\mathbf{n}\right]$. Hence,

$$-U = \mathbf{W}_{\lambda,\vartheta(\lambda)}\left[s_1, \mathbf{n}_1, J_1; \mathbf{a}\right] \in \mathfrak{P}\left(\mathfrak{H},[\vartheta]\right) \ = \ \mathcal{Q}.$$

□

4. Notes on generalized Hassani kinematics and Relativity Principle. In this section, we want to construct universal kinematics, based on generalized Hassani transforms and demonstrate that these kinematics does not satisfy of the relativity principle in the general case. Further, we use the system of notations and definitions from the theory of changeable sets and universal kinematics [11, 13, 16–18] (the most complete and detailed explanation of these theories can be found in [10]). Let $\left(\mathfrak{H}, \|\cdot\|, \langle\cdot,\cdot\rangle\right)$ be a real Hilbert space with $\dim\left(\mathfrak{H}\right) \geq 1$, $\mathcal{B}$ be any base changeable set such that $\mathfrak{Bs}(\mathcal{B}) \subseteq \mathfrak{H}$ and $\mathbb{T}\mathbf{m}(\mathcal{B}) = (\mathbb{R}, \leq)$, where $\leq$ is the standard order in the field of real numbers $\mathbb{R}$ and ϑ be a function from the class Υ, (see (3)).

Applying the results of [10, 11] to the classes of operators $\mathfrak{P}(\mathfrak{H},[\vartheta])$ and $\mathfrak{P}_+(\mathfrak{H},[\vartheta])$ we can introduce the following universal kinematics:

$$\mathfrak{UH}_0(\mathfrak{H},\mathcal{B},\vartheta) := \mathfrak{Ku}\left(\mathfrak{P}(\mathfrak{H},[\vartheta]),\mathcal{B};\,\mathfrak{H}\right); \quad \mathfrak{UH}(\mathfrak{H},\mathcal{B},\vartheta) := \mathfrak{Ku}\left(\mathfrak{P}_+(\mathfrak{H},[\vartheta]),\mathcal{B};\,\mathfrak{H}\right),$$

where the notation $\mathfrak{Ku}(\cdot,\cdot;\,\cdot)$ was introduced in [11, pg. 112], [10, pg. 166]. In Theorem 1, it was proven that in the case where there does not exist the number $c\in(0,\infty)$ such that $\vartheta(\lambda)=\vartheta_c(\lambda)$ $(\forall\,\lambda\in(0,\infty))$, the classes of operators $\mathfrak{P}(\mathfrak{H},[\vartheta])$ and $\mathfrak{P}_+(\mathfrak{H},[\vartheta])$ do not form groups with the operation of multiplication of operators over $\mathcal{M}(\mathfrak{H})$. This means that the kinematics $\mathfrak{UH}_0(\mathfrak{H},\mathcal{B},\vartheta)$ and $\mathfrak{UH}(\mathfrak{H},\mathcal{B},\vartheta)$, constructed on the basis of these classes do not satisfy the relativity principle. Indeed, according to [10, Property 3.23.1(1)] for universal kinematics $\mathcal{F}\in\{\mathfrak{UH}_0(\mathfrak{H},\mathcal{B},\vartheta),\,\mathfrak{UH}(\mathfrak{H},\mathcal{B},\vartheta)\}$ any reference frame $\mathfrak{l}\in\mathcal{L}k(\mathcal{F})$ can be represented in the form $\mathfrak{l}=(U,U[\mathcal{B}])$, where

$$U\in\mathcal{Q}_{\mathcal{F}}=\begin{cases}\mathfrak{P}(\mathfrak{H},[\vartheta]), & \mathcal{F}=\mathfrak{UH}_0(\mathfrak{H},\mathcal{B},\vartheta)\\ \mathfrak{P}_+(\mathfrak{H},[\vartheta]), & \mathcal{F}=\mathfrak{UH}(\mathfrak{H},\mathcal{B},\vartheta).\end{cases}$$

So, according to [10, Property 3.23.1(7)], the set of universal coordinate transforms

$$\mathbb{UP}(\mathfrak{l})=\{[\mathfrak{m}\leftarrow\mathfrak{l},\mathcal{F}]\mid\mathfrak{m}\in\mathcal{L}k(\mathcal{F})\}=\left\{VU^{-1}\mid V\in\mathcal{Q}_{\mathcal{F}}\right\},$$

providing transition from some reference frame $\mathfrak{l}=(U,U[\mathcal{B}])\in\mathcal{L}k(\mathcal{F})$ to all other frames $\mathfrak{m}\in\mathcal{L}k(\mathcal{F})$, is different for different frames $\mathfrak{l}$. Moreover, assume that there does not exist the number $c\in(0,\infty)$ such that $\vartheta(\lambda)=\vartheta_c(\lambda)$ $(\forall\,\lambda\in(0,\infty))$. Then, taking into account Remark 4 and the Properties, it can be proven that the set $\mathbb{UP}(\mathfrak{l})$ coincides with the starting class of transforms $\mathcal{Q}_{\mathcal{F}}$ only for some (but not all) reference frames, for example for the frame $\mathfrak{l}_{0,\mathcal{B}}=(\mathbb{I},\mathbb{I}[\mathcal{B}])=(\mathbb{I},\mathcal{B})\in\mathcal{L}k(\mathcal{F})$ (that is for this reference frame it is valid the equality $\mathbb{UP}(\mathfrak{l}_{0,\mathcal{B}},\mathcal{F})=\mathcal{Q}_{\mathcal{F}}$ but for other frames the similar equality may be not true). But, the principle of relativity is only one of the experimentally established facts, which must not be satisfied when we exit out of the light barrier or may be satisfied only approximatively with the great accuracy even in subluminal case. It is useful to consider the kinematics $\mathfrak{UH}(\mathfrak{H},\mathcal{B},\vartheta)$ in the case, where the continuous function ϑ satisfies the following conditions 1) $\vartheta(\lambda)>\lambda$ $(\forall\lambda\in[0,\infty))$; 2) $c-\varepsilon_1<\vartheta(\lambda)\le c$ for $0\le\lambda<c-\varepsilon$, where c is the speed of light in vacuum and $\varepsilon,\varepsilon_1\in(0,c)$ are some small positive numbers. In this case we obtain the kinematics, which may be arbitrarily close to classical special relativity one and in which the hard-surmountable light barrier may be overcome by means of the continuous change of the velocity of particle. And in this kinematics the principle of relativity is satisfied only approximatively for the speeds of reference frames that are in the range $[0,\,c-\varepsilon)$ relatively to the reference frame $\mathfrak{l}_{0,\mathcal{B}}\in\mathcal{L}k(\mathfrak{UH}(\mathfrak{H},\mathcal{B},\vartheta))$. The possibility of violation the relativity principle is discussed in the physical literature (see for example [19–27]).

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