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BINORMAL AND COMPLEX SYMMETRIC WEIGHTED COMPOSITION OPERATORS ON THE FOCK SPACE OVER $\mathbb{C}$

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For analytic functions $\psi, \phi : \mathbb{C} \to \mathbb{C}$, the weighted composition operator $C_{\psi,\phi}$ is the operator on the Fock space $\mathcal{F}^2$ defined as $C_{\psi,\phi} f = (\psi \cdot f) \circ \phi$ for all $f \in \mathcal{F}^2$ and the composition operator C_ϕ is the operator on the Fock space $\mathcal{F}^2$ defined as $C_\phi f = f \circ \phi$ for all $f \in \mathcal{F}^2$. A bounded operator T is on a separable Hilbert space $\mathcal{H}$ is said to be complex symmetric if there exists a conjugation operator S such that $T^* = STS$ and T is said to be binormal if T^*T and TT^* commute (i.e) $T^*TTT^* = TT^*T^*T$. Let $\mathcal{A}$ be a class of composition operators C_ϕ on $\mathcal{F}^2$ such that $C_\phi^* C_\phi$ and $C_\phi + C_\phi^*$ commute. The main results of this paper is presented in five Sections (3.1 – 3.5). In the first section, we prove that when C_ϕ is bounded and belong to $\mathcal{A}$ then C_ϕ binormal (Section 3.1). Then we describe necessary and sufficient conditions for a binormal (or) complex symmetric composition operator to have the other property (Sections 3.2, 3.3). Finally, we investigate binormality and complex symmetry of weighted composition operator $C_{\psi,\phi}$ with the weight function as a kernel function (ie) $\psi(z) = cK_p z = ce^{z\overline{p}}$ (Sections 3.4, 3.5).

1. Introduction. The Fock space $\mathcal{F}^2$ is a space of all entire functions on $\mathbb{C}$ which are square integrable with respect to Gaussian measure $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dA(z)$ where dA denotes the usual Lebesque measure on $\mathbb{C}$. It is known that $\mathcal{F}^2$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int\limits_{\mathbb{C}} f(z)\overline{g(z)} d\mu(z) = \frac{1}{\pi} \int\limits_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} dA(z)$$

for all $f, g \in \mathcal{F}^2$. It is well-known that $\mathcal{F}^2$ is a reproducing kernel Hilbert space with kernel functions of the form

$$K_w z = e^{\langle z, w \rangle} = e^{z\overline{w}}$$

for all $z, w \in \mathbb{C}$.

We denote normalized kernel function at $w \in \mathbb{C}$ as $k_w z = \frac{K_w z}{||K_w||}$.

For analytic functions ψ, ϕ on $\mathbb{C}$, the weighted composition operator $C_{\psi,\phi}$ is defined as $C_{\psi,\phi} f = (\psi \cdot f) \circ \phi$ for all $f \in \mathcal{F}^2$ and the composition operators $C_\phi f = f \circ \phi$ for all $f \in \mathcal{F}^2$.

The study of composition operators on $\mathcal{F}^2$ has been carried by many authors and characterized many of its properties. In [5], B, J. Carswell et al. characterized boundedness and compactness of composition operators on the Fock space over $\mathbb{C}^n$.

In [10], L. Zhao characterized unitary weighted composition operators and their spectrum on the Fock space over $\mathbb{C}^n$. In [11, 12], L. Zhao respectively studied isometric weighted composition operators and bounded invertible weighted composition operators on the Fock space over $\mathbb{C}^n$. In [9], T. Le investigated boundedness and compactness of weighted composition operators on $\mathcal{F}^2$ using much simpler characterization than the one in [5]. In [7], S. Jung

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et al. derived the necessary and sufficient conditions for C_ϕ to be binormal on the Hardy space with the fixed symbol ϕ is a linear fractional self map on the unit disk in the complex plane where the reproducing kernel is of the form $K_w z = \frac{1}{1-\overline{w}z}$.

2. Preliminaries. An operator T on a separable Hilbert space $\mathcal{H}$ is said to be *anti-linear* if $T(ax+by) = \overline{a}x + \overline{b}y, \forall x, y \in \mathcal{H}, \forall a, b \in \mathbb{C}$.

An anti-linear mapping S on $\mathcal{H}$ is called *conjugation* if it is
(i) Involutive: $S^2 = I$, the identity operator.
(ii) Isometry: $||Sx|| = ||x||, \forall x \in \mathcal{H}$.

An operator T on $\mathcal{H}$ is said to be *complex symmetric* if there exists a conjugation S such that $STS = T^*$.

A linear operator T is:

- *normal* if T and T^* commute $TT^* = T^*T$.
- *binormal* if T^*T and TT^* commute $T^*TTT^* = TT^*T^*T$.
- *centered* if the doubly infinite sequence $\{..., T^2(T^2)^*, TT^*, T^*T, (T^2)^*T^2, ...\}$ consists of mutually commuting operators.

Lemma 1. *Let $\psi_1, \psi_2, ...\psi_n$ be analytic functions on $\mathbb{C}$ and $\phi_1, \phi_2, ...\phi_n$ be an analytic self-map on $\mathbb{C}$. If $C_{\psi_1,\phi_1}, C_{\psi_2,\phi_2}, ...C_{\psi_n,\phi_n}$, are bounded operators on $\mathcal{F}^2$, then*

$$C_{\psi_1,\phi_1} C_{\psi_2,\phi_2} ... C_{\psi_n,\phi_n} = C_{\psi_1(\psi_2\circ\phi_1)....(\psi_n\circ\phi_{n-1}\circ...\circ\phi_1), \phi_n\circ\phi_{n-1}\circ...\circ\phi_1}.$$

Lemma 2. *Let ψ, ϕ be holomorphic functions on $\mathbb{C}$ such that $C_{\psi,\phi}$ is a bounded operator on $\mathcal{F}^2$, then $C^*_{\psi,\phi} K_w = \overline{\psi(w)} K_{\phi(w)}$ for every $w \in \mathbb{C}$,*

Theorem 1 ([5], Theorem 1)**.** *Suppose $\phi : \mathbb{C} \to \mathbb{C}$ is an analytic function and C_ϕ is bounded on $\mathcal{F}^2$ then $\phi(z) = az + b$, where $a, b \in \mathbb{C}$, $|a| \leq 1$ and if $|a| = 1$ then $b = 0$.*

Theorem 2 ([9], Theorem 2.2)**.** *Suppose ψ, ϕ be analytic functions on $\mathbb{C}$ such that ψ is not identically zero. Then $C_{\psi,\phi}$ is bounded if and only if ψ belongs to $\mathcal{F}^2$, $\phi(z) = az + \phi(0)$ with $|a| \leq 1$ and $M(\psi, \phi) := \sup\{|\psi(z)|^2 exp(|\phi(z)|^2 - |z|^2) : z \in \mathbb{C}\} < \infty$.*

Theorem 3 ([9], Theorem 3.3)**.** *Let ψ, ϕ be entire functions such that ψ is not identically zero. Then $C_{\psi,\phi}$ is a bounded normal operator on $\mathcal{F}^2$ if and only if one of the following two cases occurs:*

(i) $\phi(z) = az + b$ with $|a| = 1$ and $\psi = \psi(0)K_{-\overline{a}b}$. In this case, $C_{\psi,\phi}$ is a constant multiple of a unitary operator.

(ii) $\phi(z) = az + b$ with $|a| < 1$ and $\psi = \psi(0)K_c$, where $c = b\frac{1-\overline{a}}{1-a}$. In this case, $C_{\psi,\phi}$ is unitarily equvalent to $\psi(0)C_{az}$.

3. Main Results.

3.1. Binormal composition operators. In [1–4], S. L. Campbell studied properties of bounded linear operator T on a separable Hilbert space such that T^*T and $T+T^*$ commute. In [8], S, Jung et al. characterized the composition operators C_ϕ such that $C^*_\phi C_\phi$ and $C_\phi + C^*_\phi$ commute on Hardy space, a space of analytic functions on the unit disk in the complex plane.

Motivated by these papers, in this first section, we establish relation between binormality of composition operator C_ϕ and C_ϕ belongs to a class of composition operators such that $C^*_\phi C_\phi$ and $C_\phi + C^*_\phi$ commute.

Let $\mathcal{A}$ be a class of composition operators C_ϕ on $\mathcal{F}^2$ such that $C^*_\phi C_\phi$ and $C_\phi + C^*_\phi$ commute.

Theorem 4. *Let ϕ be an entire function on $\mathbb{C}$ such that C_ϕ is bounded on $\mathcal{F}^2$. If $C_\phi \in \mathcal{A}$ then C_ϕ is binormal.*

Proof. Since C_ϕ is bounded on $\mathcal{F}^2$, by Theorem 1, one has $\phi(z) = az + b$ with $|a| \leq 1$. Therefore, we successively have

$$\begin{aligned} C_\phi^* C_\phi C_\phi^* K_w z &= C_\phi^* C_\phi K_{\phi(w)} z = C_\phi^* C_\phi K_{(aw+b)} z = C_\phi^* K_{(aw+b)} \phi(z) = C_\phi^* e^{(az+b)\overline{(aw+b)}} = \\ &= e^{\overline{a}b\overline{w}+|b|^2} C_\phi^* K_{(|a|^2 w+\overline{a}b)} z = e^{\overline{a}b\overline{w}+|b|^2} K_{\phi(|a|^2 w+\overline{a}b)} z = \\ &= e^{\overline{a}b\overline{w}+|b|^2} K_{a(|a|^2 w+\overline{a}b)+b} z = e^{\overline{a}b\overline{w}+|b|^2+z(\overline{a}|a|^2\overline{w}+|a|^2\overline{b})}, \end{aligned} \tag{1}$$

$$\begin{aligned} C_\phi^* C_\phi C_\phi K_w z &= C_\phi^* C_\phi K_w \phi(z) = C_\phi^* C_\phi e^{(az+b)\overline{w}} = e^{b\overline{w}} C_\phi^* C_\phi K_{\overline{a}w} z = e^{b\overline{w}} C_\phi^* K_{\overline{a}w} \phi(z) = \\ &= e^{b\overline{w}} C_\phi^* e^{(az+b)a\overline{w}} = e^{b\overline{w}(1+a)} C_\phi^* K_{(\overline{a}^2 w)} z = e^{b\overline{w}(1+a)} K_{\phi(\overline{a}^2 w)} z = \\ &= e^{b\overline{w}(1+a)} K_{a(\overline{a}^2 w)+b} z = e^{b\overline{w}(1+a)+z(a|a|^2\overline{w}+\overline{b})}, \end{aligned} \tag{2}$$

$$\begin{aligned} C_\phi^* C_\phi^* C_\phi K_w z &= C_\phi^* C_\phi^* K_w \phi(z) = C_\phi^* C_\phi^* e^{(az+b)\overline{w}} = e^{b\overline{w}} C_\phi^* C_\phi^* K_{\overline{a}w} z = \\ &= e^{b\overline{w}} K_{\phi(\phi(\overline{a}w))} z = e^{b\overline{w}+z(\overline{a}|a|^2\overline{w}+\overline{b}(1+\overline{a}))}, \end{aligned} \tag{3}$$

and also

$$\begin{aligned} C_\phi C_\phi^* C_\phi K_w z &= C_\phi C_\phi^* K_w \phi(z) = C_\phi C_\phi^* e^{(az+b)\overline{w}} = e^{b\overline{w}} C_\phi C_\phi^* K_{\overline{a}w} z = e^{b\overline{w}} C_\phi K_{\phi(\overline{a}w)} z = \\ &= e^{b\overline{w}} K_{(|a|^2 w+b)} \phi(z) = e^{b\overline{w}} e^{(az+b)(|a|^2\overline{w}+\overline{b})} = e^{b\overline{w}(1+|a|^2)+|b|^2+z(a|a|^2\overline{w}+a\overline{b})} \end{aligned} \tag{4}$$

Suppose that $C_\phi \in \mathcal{A}$, then

$$C_\phi^* C_\phi (C_\phi + C_\phi^*) K_w z = (C_\phi + C_\phi^*) C_\phi^* C_\phi K_w z$$

for all $z, w \in \mathbb{C}$. Therefore, from equalities (1), (2), (3) and (4), we get

$$\begin{aligned} & e^{\overline{a}b\overline{w}+|b|^2+z(\overline{a}|a|^2\overline{w}+|a|^2\overline{b})} + e^{b\overline{w}(1+a)+z(a|a|^2\overline{w}+\overline{b})} = \\ &= e^{b\overline{w}+z(\overline{a}|a|^2\overline{w}+\overline{b}(1+\overline{a}))} + e^{b\overline{w}(1+|a|^2)+|b|^2+z(a|a|^2\overline{w}+a\overline{b})}. \end{aligned} \tag{5}$$

Taking $w = 0$ in (5), one has

$$e^{z\overline{b}(\overline{a}+1)} + e^{za\overline{b}+|b|^2} - e^{z\overline{b}} = e^{z|a|^2\overline{b}+|b|^2} \tag{6}$$

for all $z \in \mathbb{C}$.

Similarly, substituting $z = 0$ in (5) and taking conjugation on both sides of (5), we get

$$e^{w\overline{b}(\overline{a}+1)} + e^{wa\overline{b}+|b|^2} - e^{w\overline{b}} = e^{w(|a|^2\overline{b}+\overline{b})+|b|^2} \tag{7}$$

for all $w \in \mathbb{C}$.
Since (6) and (7) are true for all $z, w \in \mathbb{C}$, then for $z = w = \zeta$, we have

$$e^{\zeta|a|^2\overline{b}+|b|^2} = e^{\zeta(|a|^2\overline{b}+\overline{b})+|b|^2} \tag{8}$$

for all $\zeta \in \mathbb{C}$. Equating powers of (8), we conclude $b = 0$. This implies $\phi(z) = az$ for all $z \in \mathbb{C}$. In this case, we know C_ϕ is normal. Hence C_ϕ is binormal on $\mathcal{F}^2$. □

3.2. When are binormal composition operators complex symmetric? In this second section, we study when are binormal composition operators complex symmetric on $\mathcal{F}^2$.

Proposition 1. *Let ϕ be an analytic function on $\mathbb{C}$ such that C_ϕ is bounded on $\mathcal{F}^2$. Then C_ϕ is binormal if and only if C_ϕ is normal on $\mathcal{F}^2$.*

Proof. Since C_ϕ is bounded, using Theorem 1, we have $\phi(z) = az + b$ with $|a| \leq 1$.

$$\begin{aligned} C_\phi C_\phi^* C_\phi^* C_\phi K_w z &= C_\phi C_\phi^* C_\phi^* K_w \phi(z) = C_\phi C_\phi^* C_\phi^* e^{(az+b)\overline{w}} = e^{b\overline{w}} C_\phi C_\phi^* C_\phi^* K_{\overline{a}w} z = \\ &= e^{b\overline{w}} C_\phi C_\phi^* K_{\phi(\overline{a}w)} z = e^{b\overline{w}} C_\phi C_\phi^* K_{(|a|^2 w+b)} z = e^{b\overline{w}} C_\phi K_{\phi(|a|^2 w+b)} z = e^{b\overline{w}} C_\phi K_{(a|a|^2 w+ab+b)} z = \\ &= e^{b\overline{w}} K_{(a|a|^2 w+ab+b)} \phi(z) = e^{b\overline{w}} e^{(az+b)\overline{(a|a|^2 w+ab+b)}} = e^{|b|^2(\overline{a}+1)+b\overline{w}(\overline{a}|a|^2+1)+z(|a|^4\overline{w}+|a|^2\overline{b}+a\overline{b})}. \end{aligned} \tag{9}$$

Next consider,

$$\begin{aligned} C_\phi^* C_\phi C_\phi C_\phi^* K_w z &= C_\phi^* C_\phi C_\phi K_{\phi(w)} z = C_\phi^* C_\phi C_\phi K_{(aw+b)} z = C_\phi^* C_\phi K_{(aw+b)} \phi(z) = \\ &= C_\phi^* C_\phi e^{(az+b)\overline{(aw+b)}} = e^{|b|^2+\overline{a}b\overline{w}} C_\phi^* C_\phi K_{(|a|^2 w+\overline{a}b)} z = e^{|b|^2+\overline{a}b\overline{w}} C_\phi^* K_{(|a|^2 w+\overline{a}b)} \phi(z) = \\ &= e^{|b|^2+\overline{a}b\overline{w}} C_\phi^* e^{(az+b)\overline{(|a|^2 w+\overline{a}b)}} = e^{|b|^2(a+1)+b\overline{w}(|a|^2+\overline{a})} C_\phi^* K_{(\overline{a}|a|^2 w+\overline{a}^2 b)} z = \\ &= e^{|b|^2(a+1)+b\overline{w}(|a|^2+\overline{a})} K_{\phi(\overline{a}|a|^2 w+\overline{a}^2 b)} z = \\ &= e^{|b|^2(a+1)+b\overline{w}(|a|^2+\overline{a})} e^{z\overline{a(\overline{a}|a|^2 w+\overline{a}^2 b))+b}} = e^{|b|^2(a+1)+b\overline{w}(|a|^2+\overline{a})} e^{z(|a|^4 w+\overline{a}|a|^2\overline{b}+\overline{b})}. \end{aligned} \tag{10}$$

Suppose that C_ϕ is binormal, then by equating (9) and (10), we get

$$e^{|b|^2(\overline{a}+1)+b\overline{w}(\overline{a}|a|^2+1)+z(|a|^4\overline{w}+|a|^2\overline{b}+a\overline{b})} = e^{|b|^2(a+1)+b\overline{w}(|a|^2+\overline{a})} e^{z(|a|^4 w+\overline{a}|a|^2\overline{b}+\overline{b})} \tag{11}$$

for $z, w \in \mathbb{C}$. Taking $z = w = 0$ in (11), we get

$$|b|^2(\overline{a} - a) = 0. \tag{12}$$

Suppose that $b \neq 0$, then we have $\overline{a} = a$. Substituting this along with $w = 0$ in (11), we get

$$\overline{b}(a-1)(a^2-1) = 0.$$

This implies $|a| = 1$. Then by Theorem 1, $b = 0$ which is a contradiction. Therefore $\phi(z) = az$ for $z \in \mathbb{C}$. This implies C_ϕ is normal on $\mathcal{F}^2$. □

Theorem 5. *Let ϕ be an analytic function on $\mathbb{C}$ such that C_ϕ is bounded on $\mathcal{F}^2$. If C_ϕ is binormal then C_ϕ is complex symmetric.*

Proof. Suppose that C_ϕ is binormal on $\mathcal{F}^2$. Then by Proposition 1, C_ϕ is normal. Since every normal operator is complex symmetric, C_ϕ is complex symmetric on $\mathcal{F}^2$. □

Corollary 1. *Let ϕ be an entire function on $\mathbb{C}$ such that C_ϕ is bounded on $\mathcal{F}^2$. If C_ϕ is binormal then C_ϕ is centered.*

Proof. By Proposition 1, C_ϕ is binormal implies C_ϕ is normal. Since every normal operator is centered, C_ϕ is centered. □

3.3. When are complex symmetric composition operators binormal? In this section, we study when are complex symmetric composition operators binormal on $\mathcal{F}^2$.

Proposition 2. *Let $\phi(z)$ be an analytic function on $\mathbb{C}$ such that C_ϕ is bounded on $\mathcal{F}^2$. Then C_ϕ is normal if and only if $\phi(z) = az$ with $|a| \leq 1$.*

Proof. Since C_ϕ is bounded on $\mathcal{F}^2$, by Theorem 1, we have $\phi(z) = az + b$ with $|a| \leq 1$. Therefore,

$$C_\phi K_w z = K_w \phi(z) = e^{(az+b)\overline{w}}, \quad C_\phi^* K_w z = K_{\phi(w)} z = e^{z\overline{(aw+b)}}.$$

Suppose that C_ϕ is normal, then $e^{(az+b)\overline{w}} = e^{z\overline{(aw+b)}}$ Taking $w = 0$, we get $\overline{b}z = 0$ for all $z \in \mathbb{C}$. This implies $b = 0$.

Conversely, suppose that $\phi(z) = az$ with $|a| \leq 1$. Then

$$C_\phi^* C_\phi K_w z = C_\phi^* K_w \phi(z) = C_\phi^* K_{(\overline{a}w)} z = K_{\phi(\overline{a}w)} z = K_{(|a|^2 w)} z, \tag{13}$$

$$C_\phi C_\phi^* K_w z = C_\phi K_{\phi(w)} z = C_\phi K_{(aw)} z = K_{(aw)} \phi(z) = K_{(aw)}(az) = K_{(|a|^2 w)} z. \tag{14}$$

Comparing (13) and (14), we conclude that C_ϕ is normal on $\mathcal{F}^2$. □

Proposition 3. *Let ϕ be an analytic function on $\mathbb{C}$ such that C_ϕ is bounded on $\mathcal{F}^2$. If C_ϕ is complex symmetric with conjugation S of the form $S(f(z)) = \overline{f(\overline{z})}$ for all $f \in \mathcal{F}^2$, then $\phi(z) = az$ with $|a| \leq 1$.*

Proof. We know by ([6], Lemma 3.5), the operator S defined as $S(f(z)) = \overline{f(\overline{z})}$ is a conjugation on $\mathcal{F}^2$. Since C_ϕ is bounded, by Theorem 1, we have $\phi(z) = az + b$ with $|a| \leq 1$.

$$SC_\phi K_w z = SK_w \phi(z) = S(e^{(az+b)\overline{w}} = e^{\overline{(a\overline{z}+b)\overline{w}}} = e^{(\overline{a}z+\overline{b})w}$$

Next consider

$$C_\phi^* S K_w z = C_\phi^* S(e^{z\overline{w}}) = C_\phi^* K_{\overline{w}} z = K_{\phi(\overline{w})} z = e^{z(\overline{a}w+\overline{b})}$$

Suppose that C_ϕ is complex symmetric, then $e^{(\overline{a}z+\overline{b})w} = e^{z(\overline{a}w+\overline{b})}$ Taking $w = 0$, we get $\overline{b}z = 0$ for all $z \in \mathbb{C}$. Hence $b = 0$. □

Theorem 6. *Let ϕ be an analytic function such that C_ϕ is bounded on $\mathcal{F}^2$. If C_ϕ is complex symmetric with conjugation S of the form $S(f(z)) = \overline{f(\overline{z})}$ then C_ϕ binormal on $\mathcal{F}^2$.*

Proof. Suppose that C_ϕ is complex symmetric with conjugation S of the form $S(f(z)) = \overline{f(\overline{z})}$. Then by Proposition 3, $\phi(z) = az$ with $|a| \leq 1$. Hence C_ϕ is normal by Proposition 2. Since every normal operator is binormal. C_ϕ is binormal on $\mathcal{F}^2$. □

3.4. Binormal weighted composition operators. In this section, we study binormal weighted composition operators $C_{\psi,\phi}$ on $\mathcal{F}^2$ with $\phi(z) = az + b$ and $\psi(z) = cK_p z$ for some nonzero $p \in \mathbb{C}$ and constant c.

Theorem 7. *Let ϕ, ψ be analytic functions on $\mathbb{C}$ such that $\phi(z) = az + b$ and $\psi(z) = cK_p z$ for some nonzero $p \in \mathbb{C}$. Then $C_{\psi,\phi}$ is binormal then one of the following conditions hold: (1) $|a| = 1$, (2) a is real and $p = \phi(0)$.*

Proof. By a simple calculation, we successively have

$$C_{\psi,\phi}^* C_{\psi,\phi} C_{\psi,\phi} C_{\psi,\phi}^* K_w z = C_{\psi,\phi}^* C_{\psi,\phi} C_{\psi,\phi} \overline{\psi(w)} K_{\phi(w)} z = \overline{c} e^{\overline{w}p} C_{\psi,\phi}^* C_{\psi,\phi} C_{\psi,\phi} K_{(aw+b)} z =$$
$$= \overline{c} e^{\overline{w}p} C_{\psi,\phi}^* C_{\psi,\phi} \psi(z) K_{(aw+b)} \phi(z) = |c|^2 e^{\overline{w}(\overline{a}b+p)+|b|^2} C_{\psi,\phi}^* C_{\psi,\phi} K_{(|a|^2 w+\overline{a}b+p)} z =$$
$$= |c|^2 e^{\overline{w}(\overline{a}b+p)+|b|^2} C_{\psi,\phi}^* \psi(z) K_{(|a|^2 w+\overline{a}b+p)} \phi(z) =$$

$$= c|c|^2 e^{\overline{w}(|a|^2b+\overline{a}b+p)+|b|^2(a+1)+b\overline{p}} C^*_{\psi,\phi} K_{(\overline{a}|a|^2w+\overline{a}^2b+\overline{a}p+p)} z =$$
$$= c|c|^2 e^{\overline{w}(|a|^2b+\overline{a}b+p)+|b|^2(a+1)+b\overline{p}} \overline{\psi(\overline{a}|a|^2w+\overline{a}^2b+\overline{a}p+p)} K_{\phi(\overline{a}|a|^2w+\overline{a}^2b+\overline{a}p+p)} z =$$
$$= |c|^4 e^{\overline{w}(a|a|^2p+|a|^2b+\overline{a}b+p)+|b|^2(a+1)+b\overline{p}+a^2\overline{b}p+|p|^2(a+1)+z(|a|^4\overline{w}+|a|^2(a\overline{b}+\overline{p})+\overline{p}(\overline{a}+1))} \tag{15}$$

Next consider

$$C_{\psi,\phi}C^*_{\psi,\phi}C^*_{\psi,\phi}C_{\psi,\phi}K_w z = C_{\psi,\phi}C^*_{\psi,\phi}C^*_{\psi,\phi}\psi(z)K_w\phi(z) = ce^{\overline{w}b}C_{\psi,\phi}C^*_{\psi,\phi}C^*_{\psi,\phi}K_{(\overline{a}w+p)}z =$$
$$= ce^{\overline{w}b}C_{\psi,\phi}C^*_{\psi,\phi}\overline{\psi(\overline{a}w+p)}K_{\phi(\overline{a}w+p)}z = |c|^2 e^{\overline{w}(ap+b)+|p|^2}C_{\psi,\phi}C^*_{\psi,\phi}K_{(|a|^2w+ap+b)}z =$$
$$= |c|^2 e^{\overline{w}(ap+b)+|p|^2}C_{\psi,\phi}\overline{\psi(|a|^2w+ap+b)}K_{\phi(|a|^2w+ap+b)}z =$$
$$= \overline{c}e^{\overline{w}(|a|^2p+ap+b)+|p|^2(\overline{a}+1)+\overline{b}p}C_{\psi,\phi}K_{(a|a|^2w+a^2p+ab+b)}z =$$
$$= \overline{c}e^{\overline{w}(|a|^2p+ap+b)+|p|^2(\overline{a}+1)+\overline{b}p}\psi(z)K_{(a|a|^2w+a^2p+ab+b)}\phi(z) =$$
$$= |c|^4 e^{\overline{w}(\overline{a}|a|^2b+|a|^2p+ap+b)+\overline{a}^2\overline{p}b+\overline{b}p+(|p|^2+|b|^2)(\overline{a}+1)+z(|a|^4\overline{w}+|a|^2(\overline{ap}+\overline{b})+a\overline{b}+\overline{p})}. \tag{16}$$

Since $C_{\psi,\phi}$ is binormal, equating (15) and (16) we get

$$|c|^4 e^{\overline{w}(a|a|^2p+|a|^2b+\overline{a}b+p)+|b|^2(a+1)+b\overline{p}+a^2\overline{b}p+|p|^2(a+1)+z(|a|^4\overline{w}+|a|^2(a\overline{b}+\overline{p})+\overline{p}(\overline{a}+1))} =$$
$$= |c|^4 e^{\overline{w}(\overline{a}|a|^2b+|a|^2p+ap+b)+\overline{a}^2\overline{p}b+\overline{b}p+(|p|^2+|b|^2)(\overline{a}+1)+z(|a|^4\overline{w}+|a|^2(\overline{ap}+\overline{b})+a\overline{b}+\overline{p})} \tag{17}$$

for all $z, w \in \mathbb{C}$. Substituting $w = 0$ in (17) and equating coefficient of z, we get

$$(|a|^2 - 1)(\overline{ap} - a\overline{b}) = 0 \tag{18}$$

Next, taking $z = 0$ in (17) and equating coefficient of w, we get

$$(|a|^2 - 1)(p(a-1) - b(\overline{a}-1)) = 0. \tag{19}$$

From (18), we have either $|a| = 1$ or $ap = \overline{a}b$.

Suppose $|a| \neq 1$, then substituting $ap = \overline{a}b$ in (19), we get $p = b$. Since p is nonzero, from (18), we have $a = \overline{a}$. Hence, a is real. □

3.5. Complex symmetric weighted composition operators. In this section, we investigate when the complex symmetric weighted composition operators on $\mathcal{F}^2$ is binormal.

Theorem 8. *Let ϕ, ψ be entire functions on $\mathbb{C}$ such that $\phi(z) = az + b$ with $|a| = 1$. If $C_{\psi,\phi}$ is bounded complex symmetric with conjugation S of the form $S(f(z)) = \overline{f(\overline{z})}$ then $C_{\psi,\phi}$ is binormal on $\mathcal{F}^2$.*

Proof. We know by ([6], Lemma 3.5), the operator S defined as $S(f(z)) = \overline{f(\overline{z})}$ is a conjugation on $\mathcal{F}^2$. Since $C_{\psi,\phi}$ is bounded and complex symmetric on $\mathcal{F}^2$, by ([6], Theorem 3.15), we have $\psi(z) = ce^{bz}$ for some nonzero $c \in \mathbb{C}$ with $b + a\overline{b} = 0$. Using $|a| = 1$ and $a\overline{b} + b = 0$, we simplify

$$C^*_{\psi,\phi}C_{\psi,\phi}C_{\psi,\phi}C^*_{\psi,\phi}K_w z = C^*_{\psi,\phi}C_{\psi,\phi}C_{\psi,\phi}\overline{\psi(w)}K_{\phi(w)}z = C^*_{\psi,\phi}C_{\psi,\phi}C_{\psi,\phi}\overline{ce^{bw}}K_{(aw+b)}z =$$
$$= \overline{c}e^{\overline{bw}}C^*_{\psi,\phi}C_{\psi,\phi}C_{\psi,\phi}K_{(aw+b)}z = \overline{c}e^{\overline{bw}}C^*_{\psi,\phi}C_{\psi,\phi}\psi(z)K_{(aw+b)}\phi(z) =$$
$$= \overline{c}e^{\overline{bw}}C^*_{\psi,\phi}C_{\psi,\phi}ce^{bz}K_{(aw+b)}(az+b) = |c|^2e^{|b|^2}C^*_{\psi,\phi}C_{\psi,\phi}K_w z = |c|^2e^{|b|^2}C^*_{\psi,\phi}\psi(z)K_w\phi(z) =$$

$$= |c|^2 e^{|b|^2} C^*_{\psi,\phi} c e^{bz} K_w(az+b) = c|c|^2 e^{b\overline{w}+|b|^2} C^*_{\psi,\phi} K_{(\overline{a}w+\overline{b})} z =$$
$$= c|c|^2 e^{b\overline{w}+|b|^2} \overline{\psi(\overline{a}w+\overline{b})} K_{\phi(\overline{a}w+\overline{b})} z = c|c|^2 e^{b\overline{w}+|b|^2} \overline{c e^{b(\overline{a}w+\overline{b})}} K_{(a(\overline{a}w+\overline{b})+b)} z = |c|^4 e^{2|b|^2} K_w z. \quad (20)$$

Similarly,

$$C_{\psi,\phi} C^*_{\psi,\phi} C^*_{\psi,\phi} C_{\psi,\phi} K_w z = C_{\psi,\phi} C^*_{\psi,\phi} C^*_{\psi,\phi} \psi(z) K_w \phi(z) = C_{\psi,\phi} C^*_{\psi,\phi} C^*_{\psi,\phi} c e^{bz} K_w(az+b) =$$
$$= c e^{b\overline{w}} C_{\psi,\phi} C^*_{\psi,\phi} C^*_{\psi,\phi} K_{(\overline{a}w+\overline{b})} z = c e^{b\overline{w}} C_{\psi,\phi} C^*_{\psi,\phi} \overline{\psi(\overline{a}w+\overline{b})} K_{\phi(\overline{a}w+\overline{b})} z =$$
$$= c e^{b\overline{w}} C_{\psi,\phi} C^*_{\psi,\phi} \overline{c e^{b(\overline{a}w+\overline{b})}} K_{a(\overline{a}w+\overline{b})+b} z = |c|^2 e^{|b|^2} C_{\psi,\phi} C^*_{\psi,\phi} K_w z = |c|^2 e^{|b|^2} C_{\psi,\phi} \overline{\psi(w)} K_{\phi(w)} z =$$
$$= |c|^2 e^{|b|^2} C_{\psi,\phi} \overline{c e^{bw}} K_{(aw+b)} z = \overline{c}|c|^2 e^{\overline{b}\overline{w}+|b|^2} C_{\psi,\phi} K_{(aw+b)} z = \overline{c}|c|^2 e^{\overline{b}\overline{w}+|b|^2} \psi(z) K_{(aw+b)} \phi(z) =$$
$$= \overline{c}|c|^2 e^{\overline{b}\overline{w}+|b|^2} c e^{bz} K_{(aw+b)}(az+b) = |c|^4 e^{2|b|^2} K_w z. \quad (21)$$

From (20) and (21), we conclude $C_{\psi,\phi}$ is binormal on $\mathcal{F}^2$. □

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