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INTERPOLATION RATIONAL INTEGRAL FRACTION OF THE HERMITE TYPE ON A CONTINUAL SET OF NODES

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The paper is devoted to approximation of functionals on a continual set of nodes. A frame of this set is arbitrary and we fix elements from the space of piecewise continuous functions on the segment $[0,1]$ with a finite number of jump discontinuity points. At first, a number of approaches to the construction of interpolation rational approximations with arbitrary multiplicity of interpolation nodes are analyzed. Such rational Hermitian interpolants are obtained by means of a limit transition from a suitable interpolating fraction. An integral rational Hermitian interpolant of the third order on a continual set of nodes is constructed and investigated. This interpolant is the ratio of a functional polynomial of the first degree to a functional polynomial of the second degree. An integral equation is obtained from interpolation conditions. This equation is reduced by elementary transformations to the standard form of integral Volterra equation of the second kind.

The lemma on the existence of a unique continuous solution of this equation is proved. We also prove the theorem that the constructed rational fraction is interpolating. To obtain a functional interpolation rational interpolant with two double interpolation nodes, it is not possible to use the above technique via limit transition. Therefore, we use continual interpolation conditions of the Hermite type. The resulting interpolant is one that retains any rational functional of the resulting form. Therefore, this interpolant is the ratio of a functional polynomial of the first degree to a functional polynomial of the second degree.

The number of publications (see for example [1]–[7]) are devoted to approximation of functionals $F\colon L_1(0,1)\to\mathbb{R}^1$ on a continual set of nodes

$$\begin{gathered} x^n(z,\boldsymbol{\xi}^n)=x_0(z)+\sum_{i=1}^{n}H(z-\xi_i)\big[x_i(z)-x_{i-1}(z)\big],\\ \boldsymbol{\xi}^n=(\xi_1,\xi_2,\dots,\xi_n)\in\boldsymbol{\Omega}_{\boldsymbol{z}^n}=\{\boldsymbol{z}^n\colon 0\le z_1\le\ldots\le z_n\le 1\}. \end{gathered} \tag{1}$$

Let $x_i(z)\in Q[0,1]$, $i=0,1,\ldots$, be arbitrary fixed elements of the space $Q[0,1]$ of piecewise continuous functions on a segment $[0,1]$ with a finite number of jump discontinuity points. The set of such functions is called the interpolant frame and $H(t)$ is the Heaviside function.

In the papers [1]–[2] the constructions of functional Newton type polynomials of the third and fourth degrees are substantiated on the continual set of nodes (1), which do not require the substitution rule. In [3] we consider a functional polynomial of Newton type, which is

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built on a continual set of nodes (1). A sufficient condition for the interpolation of this polynomial is the fulfillment of the substitution rule. On the basis of Newton's interpolation formulas an interpolating functional Taylor type polynomial is constructed with using the multiplicity of nodes by means of a limit transition.

In the paper [4], an interpolating integral continued fraction has been constructed and investigated on a continual set of nodes (1), which is a natural generalization of an interpolation continued fraction. The optimal choice of the sequence of interpolation nodes is indicated.

In [5]–[6], an abstract continued fraction of Thiele type that is interpolating for a nonlinear operator acting from the linear topological space X to an algebra Y with unity is constructed. In some particular cases, it transforms into a classical Thiele fraction or a matrix-valued fraction of Thiele type depending on many variables. In the paper [7], for a functional given on a continual set of nodes on the basis of the previously constructed interpolation integral continued fraction of the Newton type, an interpolant with a k-th twofold node has been constructed and investigated. It is proved that the constructed integral continued fraction is an interpolant of Hermite type.

We will use the following notation for a finite continued fraction

$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cdots+\cfrac{a_n}{b_n}}}} = \mathop{\mathrm{D}}_{i=1}^{n} \frac{a_i}{b_i} = \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ldots + \frac{a_n|}{|b_n}. \tag{2}$$

Lemma 1. *n-th successive convergent fraction $Q_n = q_0 + \mathop{\mathrm{D}}\limits_{i=1}^{n} \frac{q_i}{1}$ of a continued fraction Q_∞ coincides with the fraction $Q_n = \frac{A_n}{B_n}$, where n-th numerator A_n and n-th denominator B_n are determined by the recurrent formulas*

$$A_k = A_{k-1} + q_k A_{k-2},\ k = 1, 2, \ldots, \quad A_{-1} = 1, \quad A_0 = K_0^I,$$
$$B_k = B_{k-1} + q_k B_{k-2},\ k = 1, 2, \ldots, \quad B_{-1} = 0, \quad B_0 = 1.$$

We note, if $q_i = q_i\big(x(\cdot)\big)$, $i = 0, 1, \ldots, n$, are defined by formulas (2), then A_n, B_n are the functional polynomials in variable $x(z)$ of degrees $\left[\frac{n+1}{2}\right]\left[\frac{n+2}{2}\right]$ and $\left[\frac{n}{2}\right]\left[\frac{n+3}{2}\right]$, respectively, and $Q_n\big(x(\cdot)\big)$ is a rational functional interpolant (in the previous considerations square brackets denote the integer part of the number). However, the total functional degree of the numerator and denominator is $\left[\frac{n+1}{2}\right]\left[\frac{n+2}{2}\right] + \left[\frac{n}{2}\right]\left[\frac{n+3}{2}\right]$, and the number of nodes in the frame of continuous interpolation nodes is equal to $n+1$. Therefore, it is natural to require that the sum of the numerator and denominator is a functional polynomial of degree n.

Le u's illustrate these considerations with the following example.

The rational interpolating functional fraction will have the form

$$Q_n\big(x(\cdot)\big) = \frac{F\big(x_0(\cdot)\big) + \sum\limits_{j=1}^{n-1} \int\limits_0^1 \int\limits_{z_1}^1 \dots \int\limits_{z_{j-1}}^1 K_j^I(\boldsymbol{z}^j) \prod\limits_{p=1}^{j} \big(x(z_p) - x_{p-1}(z_p)\big) dz_j \dots dz_1}{1 + \int\limits_0^1 \int\limits_{z_1}^1 \dots \int\limits_{z_{n-1}}^1 K_n^I(\boldsymbol{z}^n) \prod\limits_{p=1}^{n} \big(x(z_p) - x_{p-1}(z_p)\big) dz_1 \dots dz_n} =$$

$$= \frac{P_{n-1}^I\big(x(\cdot)\big)}{1 + \int\limits_0^1 \int\limits_{z_1}^1 \dots \int\limits_{z_{n-1}}^1 K_n^I(\boldsymbol{z}^n) \prod\limits_{p=1}^{n} \big(x(z_p) - x_{p-1}(z_p)\big) dz_1 \dots dz_n},$$

$$K_j^I(\boldsymbol{z}^j) = (-1)^j \prod_{p=1}^{j} \big(x_j(z_p) - x_{p-1}(z_p)\big)^{-1} \cdot \frac{\partial^j F\big(x^j(\cdot; \boldsymbol{z}^j)\big)}{\partial z_1 \dots \partial z_j},$$

$$x^j(t; \boldsymbol{z}^j) = x_0(t) + \sum_{p=1}^{j} H(t - z_p)\big[x_p(t) - x_{p-1}(t)\big], \quad j = 1, 2, \dots, n-1,$$

$$K_n^I(\boldsymbol{z}^n) = (-1)^n \prod_{p=1}^{n} \big(x_n(z_p) - x_{p-1}(z_p)\big)^{-1} \cdot \frac{\partial^n}{\partial z_1 \dots \partial z_n} \frac{P_{n-1}^I\big(x^n(\cdot; \boldsymbol{z}^n)\big)}{F\big(x^n(\cdot; \boldsymbol{z}^n)\big)}.$$

Let $n = 3$, $x(z) \equiv x$, $x_0(z) \equiv x_0$, $x_1(z) \equiv x_1$, $x_2(z) \equiv x_2$, $x_3(z) \equiv x_3$, then

$$Q_3(x) = \frac{\sum\limits_{i=0}^{2} \omega_i(x) f(x_0; x_1; \dots; x_i)}{1 - \frac{\omega_3(x)}{f(x_3)} f(x_0; x_1; x_2; x_3)}, \tag{3}$$

where $\omega_i(x) = \prod_{p=0}^{i-1}(x - x_p)$ and $f(x_0; x_1; \dots; x_i) = \sum_{s=0}^{i} \frac{f(x_s)}{\omega_i'(x_s)}$ is a divided difference of i-th order.

The formula (3) is generalized to an arbitrary number of interpolation nodes, namely

$$Q_n(x) = \frac{\sum\limits_{i=0}^{n-1} \omega_i(x) f(x_0; x_1; \dots; x_i)}{1 - \frac{\omega_n(x)}{f(x_n)} f(x_0; x_1; \dots; x_n)}.$$

The following lemma is proved in the paper [5].

Lemma 2 ([5])**.** *Let $Q_n^{IS}(x)$ be a scalar interpolating continued fraction obtained from an integral continued fraction $Q_n^I\big(x(\cdot)\big)$ under the assumption that all interpolation nodes of the frame $x_i(z)$, $i = 0, 1, \dots, n$, and the argument $x(z)$ are identical constants.*

For the existence of an integral interpolating continued fraction of the Hermite type $Q_n^E\big(x(\cdot)\big)$ with any multiplicity of the interpolation nodes obtained by the limit transition from $Q_n^I\big(x(\cdot)\big)$, it is necessary and sufficient that there exists a continued fraction of the Hermite type $Q_n^E\big(x\big)$ with the same multiplicity of interpolation nodes obtained by the limit transition from $Q_n^{IS}\big(x\big)$.

We find the representation of $Q_3(x)$, when the nodes x_0, x_2 are double. For this purpose, we will put $x_1 = x_0 + \alpha$, $x_3 = x_2 + \alpha$ and let α tends to zero. Then we obtain

$$Q_3^H(x) = \frac{f(x_0) + (x - x_0) f(x_0; x_0) + (x - x_0)^2 f(x_0; x_0; x_2)}{1 - \frac{(x-x_0)^2 (x-x_2)}{f(x_2)} f(x_0; x_0; x_2; x_2)}, \tag{4}$$

where the divided differences with multiple nodes are defined as follows (see for example [10]):

$$f(x_0;x_0)=f'(x_0),\qquad f(x_0;x_0;x_2)=\frac{f'(x_0)-f(x_0;x_2)}{x_0-x_2},$$
$$f(x_0;x_0;x_2;x_2)=\frac{f'(x_0)+f'(x_2)-2f(x_0;x_2)}{(x_0-x_2)^2}. \tag{5}$$

The rational Hermitian interpolant (4), (5), obtained by means of a limit transition from suitable interpolation fraction (3) uses redundant information in comparison with the possibilities of ordinary rational interpolation. Namely, the interpolant $Q_3^H(x)$ is the ratio of the second degree polynomial to the third degree polynomial, while there is a natural rational Hermitian interpolant $R_{1,2}^H(x)$, that is the ratio of the first degree polynomial to the second degree polynomial.

The last interpolant has the form

$$R_{1,2}^H(x)=\frac{f(x_0)+(x-x_0)y}{1+(x-x_0)z+(x-x_0)^2w},$$
$$y=\frac{-\big(f(x_2)\big)^2f'(x_0)-\big(f(x_0)\big)^2f'(x_2)+2f(x_0;x_2)f(x_0)f(x_2)}{d},$$
$$z=\frac{(x_0-x_2)f'(x_0)f'(x_2)-f(x_2)f'(x_0)-f(x_0)f'(x_2)+2f(x_0;x_2)f(x_2)}{d}, \tag{6}$$
$$w=-\frac{\big(f(x_0;x_2)\big)^2-f'(x_0)f'(x_2)}{d},$$
$$d=(x_0-x_2)\big(-f(x_0)f'(x_2)+f(x_2)f(x_0;x_2)\big)$$

and satisfies the same Hermitian conditions as $Q_3^H(x)$.

We construct an integral rational Hermite type interpolant $R_{1,2}^H\big(x(\cdot)\big)$, which is the ratio of a functional polynomial of the first degree to a functional polynomial of the second degree. For this purpose, at first we construct an integral rational interpolant $R_{1,2}^I\big(x(\cdot)\big)$ in the following form

$$R_{1,2}^I\big(x(\cdot)\big)=\frac{F\big(x_0(\cdot)\big)+\int\limits_0^1K_{1,1}(z)\big(x(z)-x_0(z)\big)dz}{1+\int\limits_0^1K_{1,2}(z)\big(x(z)-x_0(z)\big)dz+\int\limits_0^1\int\limits_{z_1}^1K_2(\boldsymbol{z}^2)\prod\limits_{i=1}^2\big(x(z_i)-x_{i-1}(z_i)\big)dz_2dz_1}, \tag{7}$$

where the integral kernels are determined from the corresponding continual conditions.

For the first continual node we take the following

$$x^1(z;\xi_1)=x_0(z)+H(z-\xi_1)\big(x_1(z)-x_0(z)\big).$$

Then the interpolation condition in this node leads to the equation

$$K_{1,2}(\xi_1)=-\frac{1}{x_1(\xi_1)-x_0(\xi_1)}\cdot\frac{d}{d\xi_1}\frac{F\big(x_0(\cdot)\big)+\int\limits_{\xi_1}^1K_{1,1}(z)\big(x_1(z)-x_0(z)\big)dz}{F\big(x^1(\cdot;\xi_1)\big)}. \tag{8}$$

We take the following continuous interpolation node in the form

$$x^2(z;\boldsymbol{\xi}^2)=x_0(z)+H(z-\xi_1)\big(x_1(z)-x_0(z)\big)+H(z-\xi_2)\big(x_2(z)-x_1(z)\big) \tag{9}$$

and interpolation condition $R^I_{1,2}\big(x^2(\cdot;\boldsymbol{\xi}^2)\big)=F\big(x^2(\cdot;\boldsymbol{\xi}^2)\big)$ leads to the relation

$$K_2(\boldsymbol{\xi}^2)=\prod_{i=1}^{2}\big(x_i(\xi_i)-x_{i-1}(\xi_i)\big)^{-1}\cdot\frac{\partial^2}{\partial\xi_1\partial\xi_2}\frac{F\big(x_0(\cdot)\big)+\int\limits_0^1 K_{1,1}(z)\big(x^2(z;\boldsymbol{\xi}^2)-x_0(z)\big)dz}{F\big(x^2(\cdot;\boldsymbol{\xi}^2)\big)}. \quad (10)$$

We obtain the similar relation, if in (9), (10) we replace $x_2(z)$ on $x_3(z)$:

$$\begin{gathered}K_2(\boldsymbol{\xi}^2)=\Bigg[\prod_{i=1}^{2}\big(x_i(\xi_i)-x_{i-1}(\xi_i)\big)^{-1}\times\\ \times\frac{\partial^2}{\partial\xi_1\partial\xi_2}\frac{F\big(x_0(\cdot)\big)+\int\limits_0^1 K_{1,1}(z)\big(x^2(z;\boldsymbol{\xi}^2)-x_0(z)\big)dz}{F\big(x^2(\cdot;\boldsymbol{\xi}^2)\big)}\Bigg]_{x_2(z)=x_3(z)}.\end{gathered} \quad (11)$$

Equating the right-hand sides of relations (10), (11) and assuming $\xi_2=\xi_1$, we obtain an integral equation for determining of the kernel $K_{1,1}(z)$:

$$\begin{gathered}K_{1,1}(\xi_1)a(\xi_1)+b_1(\xi_1)\int\limits_{\xi_1}^{1}K_{1,1}(z)\big(x_2(z)-x_0(z)\big)dz+\\ +b_2(\xi_1)\int\limits_{\xi_1}^{1}K_{1,1}(z)\big(x_3(z)-x_0(z)\big)dz=F\big(x_0(\cdot)\big)g(\xi_1),\\ a(\xi_1)=a_1(\xi_1)-a_2(\xi_1),\quad a_2(\xi_1)=a_1(\xi_1)\Big|_{x_2(z)=x_3(z)},\quad b_2(\xi_1)=b_1(\xi_1)\Big|_{x_2(z)=x_3(z)},\end{gathered}$$

$$a_1(\xi_1)=\left[\frac{\frac{\partial}{\partial\xi_2}F^{-1}\big(x^2(\cdot;\boldsymbol{\xi}^2)\big)}{x_2(\xi_2)-x_1(\xi_2)}+\frac{2x_1(\xi_2)-x_0(\xi_2)-x_2(\xi_2)}{\prod\limits_{i=1}^{2}\big(x_i(\xi_i)-x_{i-1}(\xi_i)\big)}\cdot\frac{\partial}{\partial\xi_1}F^{-1}\big(x^2(\cdot;\boldsymbol{\xi}^2)\big)\right]_{\xi_2=\xi_1}, \quad (12)$$

$$b_1(\xi_1)=\left[\frac{\frac{\partial^2}{\partial\xi_1\partial\xi_2}F^{-1}\big(x^2(\cdot;\boldsymbol{\xi}^2)\big)}{\prod\limits_{i=1}^{2}\big(x_i(\xi_i)-x_{i-1}(\xi_i)\big)}\right]_{\xi_2=\xi_1},\quad g(\xi_1)=-b_1(\xi_1)+b_2(\xi_1).$$

Using elementary transformations we convert the integral equation (12) into the standard form of the integral Volterra equation of the second kind. Then, the following lemma is valid (see, for example, [10]).

Lemma 3. *Let* $a(\xi_1)$, $b_1(\xi_1)$, $b_2(\xi_1)$, $g(\xi_1)$ *be continuous functions on the segment* $[0,1]$ *and* $a(\xi_1)\ge\alpha>0$ *on the same segment. Then the integral equation* (12) *has a unique continuous solution* $K_{1,1}(\xi_1)$.

Substituting $K_{1,1}(\xi_1)$ in (8), (11), we obtain the expressions for all kernels, which are included in the integral rational interpolant $R^I_{1,2}\big(x(\cdot)\big)$.

We should observe that the interpolant (7), (8), (11) is the one that retains any rational functional of the form

$$R_{1,2}\big(x(\cdot)\big) = \frac{K_0 + \int\limits_0^1 K_{1,1}(z)x(z)dz}{1 + \int\limits_0^1 K_{1,2}(z)x(z)dz + \int\limits_0^1\int\limits_0^1 K_2(\boldsymbol{z}^2)\prod\limits_{i=1}^2 x(z_i)dz_2dz_1}. \tag{13}$$

In this case the following theorem holds.

Theorem 1. *Let the conditions of Lemma 2 be satisfied. Then in order that rational functional* (8), (11), (12) *admit interpolation on the continual nodes*

$$x^2(z;\boldsymbol{\xi}^2) = x_0(z) + \sum_{i=1}^2 H(z-\xi_i)\big(x_i(z) - x_{i-1}(z)\big), \quad x^2(z;\boldsymbol{\xi}^2)\Big|_{x_2(z)=x_3(z)}$$

it is sufficient that functional $F\big(x(\cdot)\big)$ satisfies the substitution rule

$$\frac{\partial^p}{\partial z_1\partial z_2\ldots\partial z_p}\left[F\Big(x^{p+1}\big(\cdot;\boldsymbol{z}^{p+1}\big)\Big)\Big|_{z_{p+1}=z_p}\right] =$$

$$= \left[\frac{\partial^p}{\partial z_1\partial z_2\ldots\partial z_p}F\Big(x^{p+1}\big(\cdot;\boldsymbol{z}^{p+1}\big)\Big)\right]\Bigg|_{z_{p+1}=z_p}\frac{x_{p+1}(z_p)-x_{p-1}(z_p)}{x_p(z_p)-x_{p-1}(z_p)}, \quad p=\overline{1,n}.$$

Example 1. For functional $F\big(x(\cdot)\big) = \Big(1+\Big(\int_0^1 x(s)ds\Big)^2\Big)^{-1}$ we obtain

$$K_{1,1}(z)\equiv 0,\ K_2(\boldsymbol{z}^2) = F\big(x_0(\cdot)\big),$$

$$K_{1,2}(z) = 2F\big(x_0(\cdot)\big)\left[\int\limits_z^1 x_1(s)ds + \int\limits_0^z x_0(z)dz\right],\ R^I_{1,2}\big(x(\cdot)\big) = F\big(x(\cdot)\big).$$

To obtain a functional interpolation rational interpolant with two double interpolation nodes it is not possible to use the above technique through the limit transition.

Let us define continual interpolation conditions of the Hermite type

$$\begin{gathered}
R^{H'}_{1,2}\big(x_0(\cdot)\big)H(\cdot-\xi_1) = F'\big(x_0(\cdot)\big)H(\cdot-\xi_1),\\
R^H_{1,2}\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big) = F\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big),\\
R^{H'}_{1,2}\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big)H(\cdot-\xi_2) =\\
= F'\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big)H(\cdot-\xi_2).
\end{gathered} \tag{14}$$

We find the functional interpolation rational approximation to the functional $F\big(x(\cdot)\big)$ in the form

$$R^H_{1,2}\big(x(\cdot)\big) = \frac{F\big(x_0(\cdot)\big) + \int\limits_0^1 K_{1,1}(z)\big(x(z)-x_0(z)\big)dz}{1+\int\limits_0^1 K_{1,2}(z)\big(x(z)-x_0(z)\big)dz + \int\limits_0^1\int\limits_{z_1}^1 K_2(\boldsymbol{z}^2)\prod\limits_{i=1}^2\big(x(z_i)-x_0(z_i)\big)dz_2dz_1}. \tag{15}$$

From conditions (14) we obtain the system of equations

$$-F\big(x_0(\cdot)\big)\int\limits_{\xi_1}^{1} K_{1,2}(z)dz + \int\limits_{\xi_1}^{1} K_{1,1}(z)dz = F'\big(x_0(\cdot)\big)H(\cdot-\xi_2),$$

$$K_2(\boldsymbol{\xi}^2) = \frac{1}{x_2(\xi_1)-x_0(\xi_1)}\times$$

$$\times\frac{\partial^2}{\partial\xi_1\partial\xi_2}\frac{\Big[F\big(x_0(\cdot)\big)+\int\limits_{\xi_1}^{1}K_{1,1}(s)\big(x_2(s)-x_0(s)\big)ds\Big]F'\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big)H(\cdot-\xi_2)}{F\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big)^2},$$

$$K_{1,2}(\xi_1)\big(x_2(\xi_1)-x_0(\xi_1)\big)+\big(x_2(\xi_1)-x_0(\xi_2)\big)\int\limits_{\xi_1}^{1}K_2(\xi_1,z_2)\big(x_2(z_2)-x_0(z_2)\big)dz_2+$$

$$+\frac{d}{d\xi_1}\frac{F\big(x_0(\cdot)\big)+\int\limits_{\xi_1}^{1}K_{1,1}(s)\big(x_2(s)-x_0(s)\big)ds}{F\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big)}=0, \tag{16}$$

relative to the kernels (15).

We denote

$$M(s,\xi_1)=\frac{x_2(s)-x_0(s)}{x_2(\xi_1)-x_0(\xi_1)}\cdot\frac{\partial}{\partial s}\frac{F'\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big)H(\cdot-s)}{F\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big)^2},$$

then from the second equation (16) we obtain the formula

$$\int\limits_{\xi_1}^{1}K_2(\xi_1,s)\big(x_2(s)-x_0(s)\big)ds=\int\limits_{\xi_1}^{1}\frac{\partial M(s,\xi_1)}{\partial\xi_1}ds\cdot F\big(x_0(\cdot)\big)+$$

$$+\int\limits_{\xi_1}^{1}\frac{\partial M(s,\xi_1)}{\partial\xi_1}ds\int\limits_{\xi_1}^{1}K_{1,1}(t)\big(x_2(t)-x_0(t)\big)dt-\int\limits_{\xi_1}^{1}M(s,\xi_1)ds\cdot\big(x_2(\xi_1)-x_0(\xi_1)\big)K_{1,1}(\xi_1). \tag{17}$$

From formula (17) and expressions (16) we obtain the integral equation

$$a(\xi_1)K_{1,1}(\xi_1)+b(\xi_1)\int\limits_{\xi_1}^{1}K_{1,1}(t)\big(x_2(t)-x_0(t)\big)dt+g(\xi_1)=0, \tag{18}$$

where

$$a(\xi_1)=\frac{1}{F\big(x_0(\cdot)\big)}-\int\limits_{\xi_1}^{1}M(s,\xi_1)ds\cdot\big(x_2(\xi_1)-x_0(\xi_1)\big)-\frac{1}{F\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big)},$$

$$b(\xi_1)=\int\limits_{\xi_1}^{1}\frac{\partial M(s,\xi_1)}{\partial\xi_1}ds-\frac{\frac{d}{d\xi_1}F\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big)}{\big(x_2(\xi_1)-x_0(\xi_1)\big)F\big(x_0(\cdot)+H(\cdot-\xi_1)\big(x_2(\cdot)-x_0(\cdot)\big)\big)^2},$$

$$g(\xi_1)=\frac{1}{F\big(x_0(\cdot)\big)}+F\big(x_0(\cdot)\big)b(\xi_1).$$

Lemma 4. *Let $a(\xi_1)$, $b_1(\xi_1)$, $b_2(\xi_1)$, $g(\xi_1)$ be continuous functions on the segment $[0,1]$ and $a(\xi_1) \geq \alpha > 0$ on the same segment. Then the integral equation (18) has a unique continuous solution $K_{1,1}(\xi_1)$ in the following form*

$$K_{1,1}(\xi_1) = \frac{b(\xi_1)}{a(\xi_1)} \int\limits_{\xi_1}^{1} \frac{x_2(s) - x_0(s)}{a(s)} g(s) \cdot \exp\Big(\int\limits_{s}^{\xi_1} \frac{x_2(t) - x_0(t)}{a(t)} b(t) dt \Big) ds - \frac{g(\xi_1)}{a(\xi_1)}. \qquad (19)$$

Substituting $K_{1,1}(\xi_1)$ in (16) we obtain the expressions for all kernels, which are included in the integral rational interpolant $R_{1,2}^{H}\big(x(\cdot)\big)$.

We note that the interpolant (15), (16) is the one that retains any rational functional of the form (13).

In this case the following theorem holds.

Theorem 2. *Let the conditions of Lemma 2 be satisfied. Then, in order that rational functional (15), (16) satisfies the interpolation conditions (14), it is sufficient that functional $F\big(x(\cdot)\big)$ satisfies the substitution rule.*

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