ISOMORPHISMS OF SOME ALGEBRAS OF ANALYTIC FUNCTIONS OF BOUNDED TYPE ON BANACH SPACES

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1. Introduction and preliminaries. Let $X$ be a complex locally convex topological vector space. A function $P: X \to \mathbb{C}$ is an $n$-homogeneous polynomial if there exists a symmetric $n$-linear map $B_P$ defined on the Cartesian power $X^n$ to $\mathbb{C}$ such that $P(x) = B_P(x, \ldots, x)$. The space of all $n$-homogeneous polynomials on $X$ is denoted by $\mathcal{P}(n, X)$. The direct sum of spaces $\mathcal{P}(n, X)$, $n = 0, 1, 2, \ldots$ forms a unital algebra of continuous polynomials $\mathcal{P}(X)$.

A continuous function $f: X \to \mathbb{C}$ is said to be an entire analytic function (or just entire function) if its restriction on any finite dimensional subspace is analytic. The algebra of all entire functions on $X$ is denoted by $H(X)$. There are a lot of various topologies on $H(X)$. In the paper we assume that $H(X)$ is endowed with the topology of the uniform convergence on compact subsets of $X$. If $f$ is bounded on all bounded subsets of $X$, then $f$ is called an entire function of bounded type. It is well-known that every function $f$ of bounded type can be represented as a series of homogeneous polynomials $f_n$, so-called Taylor polynomials, such that $f(x) = \sum_{n=0}^{\infty} f_n(x)$, and the series uniformly converges on any bounded subset of $X$.

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The algebra of all entire functions of bounded type on $X$ is denoted by $H_b(X)$. It is known that if $X$ is a Banach space or a (DF)-space (see [12]), then $H_b(X)$ is a Fréchet algebra. In particular, if $X$ is a Banach space, then the metrizable topology on $H_b(X)$ can be generated by norms

$$
\|f\|_r = \sup \{|f(x)| : \|x\| \leq r\}, \quad r \in \mathbb{Q}_+.
$$

If $X$ and $Y$ are locally convex topological vector spaces, then a mapping $F: X \to Y$ is analytic if $\phi \circ F$ is an analytic function for every continuous linear functional $\phi$ on $Y$. A mapping $F: X \to X$ is called an analytic automorphism if $F$ is analytic, bijective and $F^{-1}$ is analytic. For more detailed information about analytic mappings on locally convex spaces we refer the reader to [10, 17].

For a given Banach space $X$ we denote by $M_b(X)$ the spectrum of $H_b(X)$. In other words, $M_b(X)$ consists of all nonzero continuous complex valued homomorphisms (characters) of $H_b(X)$. A point evaluation functional $\delta_x: f \mapsto f(x)$ for a fixed $x \in X$ is a typical example of a character of $H_b(X)$. A radius function $R(\varphi)$ of a character $\varphi$ is defined as the infimum of all $r > 0$ such that $\varphi$ is continuous on the normed space $(H_b(X), \|\cdot\|, r)$ and can be computed (see [2]) by

$$
R(\varphi) = \limsup_{n \to \infty} \|\varphi_n\|^{1/n} < \infty, \quad (1)
$$

where $\varphi_n$ is the restriction of $\varphi$ to the Banach space $(P^n(X), \|\cdot\|_1)$. According to [2], $R(\delta_x) = \|x\|$. Algebras of entire functions of bounded type on Banach spaces and their spectra were studied by many authors (see e.g. [2, 5, 23]).

Let $\mathbb{P} = \{P_1, P_2, \ldots, P_n, \ldots\}$ be a sequence of polynomials on a Banach space $X$. We denote by $P_{\mathbb{P}}(X)$ the minimal unital algebra containing polynomials in $\mathbb{P}$. Let $H_{b\mathbb{P}}(X)$ be the closure of $P_{\mathbb{P}}(X)$ in $H_b(X)$. Throughout in the paper we assume that the sequence $\mathbb{P}$ is algebraically independent and $\|P_n\|_1 = 1$, $n \in \mathbb{N}$. Recall that a sequence of elements in an algebra is algebraically independent if every nontrivial algebraic combination of element of this sequence is not equal to zero. Clearly that $\mathbb{P}$ forms an algebraic basis in $P_{\mathbb{P}}(X)$, that is, every polynomial in $P_{\mathbb{P}}(X)$ can be uniquely represented as an algebraic combination of elements in $P_{\mathbb{P}}(X)$. It is easy to see that the basis $\mathbb{P}$ is not unique.

Algebras $H_{b\mathbb{P}}(X)$ for various sequences of polynomials were considered in [9, 15, 22, 18]. We denote by $M_{b\mathbb{P}} = M_{b\mathbb{P}}(X)$ the spectrum of $H_{b\mathbb{P}}(X)$. It is known (see [9, 15]) that the spectrum $M_{b\mathbb{P}}$ can be described as the set of sequences

$$
\{(\varphi(P_1), \varphi(P_2), \ldots, \varphi(P_n), \ldots) : \varphi \in M_{b\mathbb{P}}\}.
$$

In [15] it is proved that the radius function of any character $\varphi \in M_{b\mathbb{P}}$ can be computed by the same formula (1), where $\varphi_n$ is the restriction of $\varphi$ to $P^n(X)$. Typical examples of $H_{b\mathbb{P}}(X)$ can be obtained as algebras of symmetric analytic functions with respect to appropriate symmetry groups of isometric operators on $X$. Algebras of symmetric analytic functions on Banach spaces were studied in [1, 3, 4, 6, 7, 8, 11, 13, 14, 20, 21].

In Section 2 we consider conditions under which two algebras $H_{b\mathbb{A}}(X)$ and $H_{b\mathbb{P}}(Y)$ are isomorphic via a mapping $\Theta$ such that $\Theta(A_n) = P_n$, $n \in \mathbb{N}$. In Section 3 we propose some applications for algebras of symmetric analytic functions of bounded type.

2. Conditions of continuity.

**Proposition 1.** Let $X$ be a complex Banach space. Then the radius function of any point evaluation functional $\delta_x$ on $H_{b\mathbb{P}}(X)$ is less or equal than $\|x\|$.  

Proof. Since the space of homogeneous polynomials $\mathcal{P}_x(^nX)$ is a subspace in the space of all $n$-homogeneous continuous polynomials $\mathcal{P}(^nX)$, the norm of the restriction of $\delta_x$ to $\mathcal{P}_x(^nX)$ is less or equal than the norm of the restriction of $\delta_x$ to $\mathcal{P}(^nX)$. Thus, the radius function of $\delta_x$ on $H_{b\mathcal{P}}(X)$ is less or equal than the norm of the restriction of $\delta_x$ considered as a functional on $H_b(X)$. But due to [2] we know that the radius function of $\delta_x$ on $H_b(X)$ is equal to $\|x\|$.

**Proposition 2.** Let $Z$ be a locally convex topological vector space and $H_0(Z)$ a subalgebra of $H(Z)$ which separates points of $Z$. Suppose that the both spectra of $H(Z)$ and $H_0(Z)$ consist of point evaluation functionals $\delta_z, z \in Z$. Let $A: H(Z) \to H_0(Z)$ be a surjective continuous homomorphism. Then there exists an analytic injective mapping $\Phi : Z \to Z$ such that $A(f)(z) = f \circ \Phi(z)$ for every $f \in H(Z)$ and $z \in Z$.

**Proof.** Let $A'$ be the adjoint operator

$$A' : H_0'(Z) \to H'(Z), \quad A' (\varphi)(f) = \varphi(A(f)),$$

where $\varphi \in H_0'(Z)$, $f \in H(Z)$. Denote by $\tilde{A}'$ the restriction of $A'$ onto the subset of multiplicative functionals in $H'(Z)$ of the form $\delta_z, z \in Z$, that is, onto the spectrum of $H_0(Z)$. Since $A$ is an algebra homomorphism, $\tilde{A}'$ maps the spectrum of $H_0(Z)$ to the spectrum of $H(Z)$.

Set $\Phi(z) = y$, where $\delta_y = \tilde{A}'(\delta_z)$. Then for every $f \in H(Z)$, $f \circ \Phi = A(f) \in H_0(Z)$, that is, $\Phi$ is an analytic map by definition. To show that $\Phi$ is injective, let us suppose that $\Phi(z_1) = \Phi(z_2)$. Then $A(f)(z_1) = A(f)(z_2)$ for every $f \in H(Z)$. Since $A$ is surjective, $g(z_1) = g(z_2)$ for every $g \in H_0$. Hence $z_1 = z_2$.

**Corollary 1.** Let $H(Z)$ and $H_0(Z)$ be as in Proposition 2 and $A$ be a topological isomorphism of algebras. Then $\Phi$ is an analytic automorphism.

Let $\mathbb{A} = \{A_1, A_2, \ldots, A_n, \ldots\}$ and $\mathbb{P} = \{P_1, P_2, \ldots, P_n, \ldots\}$ be sequences of algebraically independent polynomials on Banach spaces $X$ and $Y$ respectively, $\|A_n\|_1 = \|P_n\|_1 = 1$, $\text{deg } A_n = \text{deg } P_n = n, n \in \mathbb{N}$. Let us consider the following algebraic isomorphism of the algebras of polynomials $\Theta : \mathcal{P}_x(\mathbb{A}) \to \mathcal{P}_x(\mathbb{P})$ defined on the algebraic basis of $\mathcal{P}_x(\mathbb{A})$ by $\Theta : A_n \mapsto P_n$. Then the algebraically adjoint operator

$$\Theta^* : \mathcal{P}_x(\mathbb{P}) \to \mathcal{P}_x(\mathbb{A})$$

is defined by

$$\Theta^*(\psi)(P) = (\psi \circ \Theta)(P), \quad \psi \in \mathcal{P}_x(\mathbb{P}), \quad P \in \mathcal{P}_x(\mathbb{A}).$$

Here $\mathcal{P}_x(\mathbb{A})$ is the space of all (not necessary continuous) linear functionals on $\mathcal{P}_x(\mathbb{A})$. Let us denote by $\tilde{\Theta}^*$ the restriction of $\Theta^*$ to the spectrum $M_{b\mathcal{P}}$ of $H_{b\mathcal{P}}(Y)$.

**Theorem 1.** Suppose that $\tilde{\Theta}^*$ maps $M_{b\mathcal{P}}$ to $M_{bh}$ and there is a function $K : [0, +\infty) \to [0, +\infty)$, bounded on every segment in $[0, +\infty)$ such that

$$R(\tilde{\Theta}^*(\psi)) \leq K(R(\psi)), \quad \psi \in M_{b\mathcal{P}}.$$

Then $\Theta$ is a continuous homomorphism which can be extended to a continuous homomorphism (which we denote by the same symbol $\Theta$) from $H_{bh}(X)$ to $H_{b\mathcal{P}}(Y)$.

**Proof.** For every $y \in Y$ let $\psi_y = \tilde{\Theta}^*(\delta_y) \in M_{bh}$. If $a \in H_{bh}(X)$, then $a(x)$ can be written as

$$a(x) = \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1k_2\ldots k_n} A_1^{k_1}(x) A_2^{k_2}(x) \ldots A_n^{k_n}(x),$$

where $k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}$ and $\alpha_{k_1\ldots k_n} \in \mathbb{C}$. So we can formally extend $\Theta$ to $H_{bh}(X)$ by
Corollary 2. If the mapping $\Theta$ is surjective, then under conditions of Theorem 1, $\Theta$ is a topological isomorphism.
3. Applications for algebras of symmetric analytic functions. Let $S$ be a group of isometries on a Banach space $X$. A function $f: X \to \mathbb{C}$ is said to be $S$-symmetric (or just symmetric) if $f(\sigma(x)) = f(x)$ for all $\sigma \in S$ and $x \in X$. We denote by $\mathcal{P}_s(X)$ the algebra of all symmetric polynomials on $X$ and by $H_{bs}(X)$ its completion in $H_b(X)$. For many cases $\mathcal{P}_s(X)$ has an algebraic basis $\mathbb{P}$ and so $H_{bs}(X) = H_{bs}(X)$. In [14] it is proved that if $S$ is the group of all measurable automorphisms of $[0; 1]$ which preserve the Lebesgue measure, then polynomials

$$R_n(x) = \int_{[0;1]} (x(t))^n dt, \quad x \in L_\infty[0;1]$$

form an algebraic basis in the algebra of symmetric polynomials $\mathcal{P}_s(L_\infty[0;1])$. The spectrum $M_{bs}(L_\infty[0;1])$ of $H_{bs}(L_\infty[0;1])$ coincides with the set of point evaluation functionals and can be described as the set of sequences

$$\Lambda^s = \{\xi_n: \xi_n = R_n(x), \ x \in L_\infty[0;1], \ n \in \mathbb{N}\} = \{\xi_n \in \mathbb{C}: \sup_n |\xi_n|^{1/n} < \infty\}.$$ 

The set $\Lambda^s$ can be naturally identified with the (DF)-space $H'(\mathbb{C})_\beta$, the strong dual to the Fréchet space $H(\mathbb{C})$ of entire functions on $\mathbb{C}$. According to [13], algebra $H_{bs}(L_\infty[0;1])$ is isomorphic to the algebra $H(H'(\mathbb{C})_\beta)$ of all entire functions on $H'(\mathbb{C})_\beta$. Similar results can be obtained for some other algebras of symmetric analytic functions of bounded type [20].

In [15] (see also [11]) was considered algebra $H_{bl}(\ell_\infty)$ generated by polynomials $I_n(y) = y_n^n, \ y = (y_1, y_2, \ldots) \in \ell_\infty$, and proved that the set of sequences $\{\xi_n: \xi_n = I_n(y), \ y \in \ell_\infty, \ n \in \mathbb{N}\}$ coincides with the set $\Lambda^s$ defined above. So the spectrum of $H_{bs}(L_\infty[0;1])$ coincides with the spectrum of $H_{bl}(\ell_\infty)$ as a point set and if $\Theta: R_n \mapsto I_n$, then $\Theta^*$ is a bijection from $M_{bs}$ onto $M_{bs}(L_\infty[0;1])$. Thus we have the following result.

**Theorem 2.** There exists a topological isomorphism $\Theta: H_{bs}(L_\infty[0;1]) \to H_{bl}(\ell_\infty)$ such that $\Theta: R_n \mapsto I_n$.

**Proof.** Note first that according to [15, 14], both $M_{bl}$ and $M_{bs}(L_\infty[0;1])$ consists of point evaluation functionals. Also, for every $\delta_y \in M_{bl}, y \in \ell_\infty$ we have $R(\delta_y) = ||y||$. Indeed, let $y_n$ be such that $||y|| - ||y_n|| \leq \varepsilon$. Then

$$R(\delta_y) = \sup_{Q \in \mathcal{P}_s(\ell_\infty), ||Q|| \leq 1} |Q(y)|^{1/n} \geq |I_n(y)|^{1/n} = |y_n| \geq ||y|| - \varepsilon.$$

Since it is true for every $\varepsilon > 0, R(\delta_y) \geq ||y||$. But from Proposition 1 we have the inverse inequality.

Let $y = (y_1, y_2, \ldots, y_n, \ldots) \in \ell_\infty$ be an arbitrary vector. Then the sequence of complex numbers $\xi = (\xi_1, \xi_2, \ldots, \xi_n, \ldots) = (y_1, y_2, \ldots, y_n, \ldots)$ satisfies the condition $\sup_n \sqrt[n]{|\xi_n|} < \infty$. According to [14] there exists $x_\xi \in L_\infty[0,1]$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$ and $||x_\xi|| \leq \frac{2}{M} \sup_{n \in \mathbb{N}} \sqrt{|\xi_n|}$, where $M = \prod_{n=1}^\infty \cos \left(\frac{\pi}{2} \frac{1}{n+1}\right)$. Note that $0 < M < 1$. Thus $\tilde{\Theta}(\delta_y) = x_\xi$ and using Proposition 1 we have $R(\delta_{x_\xi}) \leq ||x_\xi|| \leq \frac{2}{M} \sup_{n \in \mathbb{N}} \sqrt{|y_n|} = K(\frac{||y||}{M}) = K(R(\delta_y))$, where $K(t) = 2t/M$. Thus we can apply Corollary 2.
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