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INVERSOR OF DIGITS OF Q_2^*-REPRESENTATION OF NUMBERS

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We consider structural, integral, differential properties of function defined by equality

$$I(\Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{Q_2^*}_{[1-\alpha_1][1-\alpha_2]\ldots[1-\alpha_n]\ldots}, \quad \alpha_n \in A \equiv \{0,1\}$$

for two-symbol polybasic non-self-similar representation of numbers of closed interval $[0;1]$ that is a generalization of classic binary representation and self-similar two-base Q_2-representation. For additional conditions on the sequence of bases, singularity of the function and self-affinity of the graph are proved. Namely, the derivative is equal to zero almost everywhere in the sense of Lebesgue measure. The integral of the function is calculated.

Introduction. According to the well-known Lebesgue theorem every function of bounded variation is either purely discrete, or absolutely continuous, or singular, or is a mixture (a linear combination of two or three functions of these types). Now the class of the singular functions is studied insufficiently [10, 13]. Today an arsenal of effective tools to define and study of the singular functions is relatively poor. The search for its expansion continues [1, 10, 13]. Each new case of their natural representation has some scientific interest. Recently a new way of defining them by inverting the digits of the representation in one or another system of encoding (representation) of the number is initiated. The inversors of digits of Q_2-representation [9], Q_3-representation [6], A_2-continued fraction representation [2], G_2-representation [7] and others are studied. Almost all of them are singular functions (except for the last one).

The singular functions [5], continuous nowhere monotonic and non-differentiable functions [3] are bright representatives of functions with locally complicated topological, metric and fractal properties. The various systems of encoding of numbers are widely used for their construction and description of properties [1, 4, 10].

In the paper, we consider inversor of digits of polybasic Q_2^*-representation that is a generalization of Q_2-representation and study its differential, integral and others properties. Thus we introduce a new continuous class of functions with locally complicated differential properties.

1. Basic objects and facts. Let $A = \{0,1\}$ be an alphabet, let $L = A \times A \times \ldots$ be a space of sequences of zeros and ones, and let $\|q_{ik}\|$ be an infinite stochastic matrix with two rows and an infinite number of columns, which has the properties $q_{ik} > 0$, $q_{0k} + q_{1k} = 1$, $\prod\limits_{k=1}^{\infty} \max\{q_{0k}, q_{1k}\} = 0$. Put $\beta_{0k} \equiv 0$, $\beta_{1k} \equiv q_{0k}$, $\beta_{2k} \equiv 1$ for any $k \in \mathbb{N}$.

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Theorem 1 ([12]). *For any number $x \in [0;1]$, there exists a sequence $(\alpha_n) \in L$ of zeros and ones such that*

$$x = \beta_{\alpha_1 1} + \sum_{k=2}^{\infty} (\beta_{\alpha_k k} \prod_{j=1}^{k-1} q_{\alpha_j j}) \equiv \Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_k...}. \quad (1)$$

Definition 1. An expansion of number x in series (1) is called its Q_2^**-expansion* and abbreviated (symbolic) notation $\Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_k...}$ is called its Q_2^**-representation*. At the same time $\alpha_k = \alpha_k(x)$ is called *k-th digit* of this representation.

Remark 1. Q_2^*-representation is a two-symbol encoding of numbers of the unit interval, in particular the fractional part of a real number.

If $q_{ik} = q_i$ for any $k \in N$ then Q_2^*-representation is Q_2-representation. Thus, Q_2^*-representation is a generalization of Q_2-representation, and therefore of classic binary representation. For any $m \in N$, the matrix $\|q_{in}\|_{n=m}^{\infty}$ that formed from a matrix $\|q_{in}\|$ satisfies the above-mentioned requirements and generates its own $Q_2^*(m)$-representation.

One of the simplest problem which leads to the concept of Q_2^*-representation of numbers is a problem about analytical expression of the probability distribution function of a random variable with independent (but, generally speaking, not identically distributed) digits of its binary representation [13].

The following statements provide a comprehensive answer to the question of identification and comparison of numbers.

Theorem 2 ([12]). *For any set $(\alpha_1, ..., \alpha_m)$ of zeros and ones the following equality is satisfied:*

$$\Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_m 1(0)} = \Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_m 0(1)}. \quad (2)$$

Numbers $x_1 = \Delta^{Q_2^*}_{c_1...c_m 0 d_1 d_2...}$ and $x_2 = \Delta^{Q_2^*}_{c_1...c_m 1 d'_1 d'_2...}$ satisfy the non-strict inequality $x_1 \leq x_2$ and the strict inequality $x_1 < x_2$ if there exist d_n, d'_n such that $d_n - d'_n \neq 1$.

Corollary 1. *If $\Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_n...} = x_1 < x_2 = \Delta^{Q_2^*}_{\alpha'_1\alpha'_2...\alpha'_n...}$, then there exists m such that $0 = \alpha_m \neq \alpha'_m = 1$ and $\alpha_i = \alpha'_i$ for $i < m$.*

Corollary 2. *There is a countable set of numbers that have two Q_2^*-representations (these are the numbers of the form (2)). All other numbers of the closed interval $[0;1]$ have the unique Q_2^*-representation.*

Definition 2. Numbers of the interval $[0;1]$ having two Q_2^*-representations ($\Delta^{Q_2^*}_{c_1...c_m 1(0)} = \Delta^{Q_2^*}_{c_1...c_m 0(1)}$) are called Q_2^**-binary* numbers. The rest of the numbers in this interval having only one Q_2^*-representation are called Q_2^**-unary* numbers.

The set of Q_2^*-binary numbers is countable.

Definition 3. A Q_2^**-cylinder* of rank m with base $c_1c_2...c_m$ is a set $\Delta^{Q_2^*}_{c_1c_2...c_m}$ of numbers x from $[0;1]$ such that they have Q_2^*-representation $x = \Delta^{Q_2^*}_{c_1c_2...c_m\alpha_1\alpha_2...}$ where $(\alpha_n) \in L$.

The Q_2^*-cylinders have the following properties:

1) $\Delta^{Q_2^*}_{c_1...c_m} = \Delta^{Q_2^*}_{c_1...c_m 0} \cup \Delta^{Q_2^*}_{c_1...c_m 1}$; $[0;1] = \bigcup\limits_{c_1 \in A} ... \bigcup\limits_{c_m \in A} \Delta^{Q_2^*}_{c_1...c_m}$;

2) $\max \Delta^{Q_2^*}_{c_1c_2\ldots c_m0} = \min \Delta^{Q_2^*}_{c_1c_2\ldots c_m1}$;

3) The cylinder $\Delta^{Q_2^*}_{c_1c_2\ldots c_m}$ is a closed interval $[a;b]$: $a = \sum\limits_{k=1}^{m} \beta_{\alpha_k k}\Big(\prod\limits_{j=1}^{k-1} q_{\alpha_j j}\Big)$, $b = a + \prod\limits_{j=1}^{m} q_{\alpha_j j}$;

4) $|\Delta^{Q_2^*}_{c_1c_2\ldots c_m}| = \prod\limits_{i=1}^{m} q_{c_i i}$; $\frac{|\Delta^{Q_2^*}_{c_1c_2\ldots c_m i}|}{|\Delta^{Q_2^*}_{c_1c_2\ldots c_m}|} = q_{i,m+1}$;

5) For any (c_m), $\bigcap\limits_{m=1}^{\infty} \Delta^{Q_2^*}_{c_1\ldots c_m} = \Delta^{Q_2^*}_{c_1\ldots c_m\ldots}$.

The following statement is obvious: *every Q_2^*-binary number is the endpoint of an infinite number of cylinders of different ranks, starting with someone.* The least rank of such a cylinder is called the rank of Q_2^*-binary number (point). Remark that there are two sequences of cylinders such that one of its endpoints is a given Q_2^*-binary number.

Definition 4. The $\overline{Q}_2^*$-representation ($\overline{Q}_2^* = \|\{\overline{q}_{in}\|$) is called *symmetric* to Q_2^*-representation ($Q_2^* = \|q_{in}\|$) if $\overline{q}_{0n} = q_{1n}$.

Theorem 3. *For any number $x \in [0;1]$, the following equality is true:*

$$\Delta^{Q_2^*}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots} + \Delta^{\overline{Q}_2^*}_{[1-\alpha_1(x)][1-\alpha_2(x)]\ldots[1-\alpha_n(x)]\ldots} = 1. \tag{3}$$

Proof. For numbers 0 and 1, the statement is obvious. Its general truth is proved by examples for Q_2^*-binary numbers (of 1st and 2nd rank).

In fact, for numbers $x = \Delta^{Q_2^*}_{1(0)} = q_{01}$ and $x' = \Delta^{\overline{Q}_2^*}_{0(1)} = \Delta^{\overline{Q}_2^*}_{1(0)} = \overline{q}_{01} = q_{11}$, we have $x + x' = q_{01} + q_{11} = 1$.

Similarly, for $x = \Delta^{Q_2^*}_{01(0)}$ and $x' = \Delta^{\overline{Q}_2^*}_{10(1)} = \Delta^{\overline{Q}_2^*}_{11(0)}$,
$\Delta^{Q_2^*}_{01(0)} + \Delta^{\overline{Q}_2^*}_{11(0)} = q_{01}q_{02} + (\overline{q}_{01} + \overline{q}_{11}\overline{q}_{02}) = q_{01}q_{02} + q_{11} + q_{01}q_{12} = q_{01}(q_{02} + q_{12}) + q_{11} = 1.$

For numbers $x = \Delta^{Q_2^*}_{11(0)}$ and $x' = \Delta^{\overline{Q}_2^*}_{00(1)} = \Delta^{\overline{Q}_2^*}_{01(0)}$,
$\Delta^{Q_2^*}_{11(0)} + \Delta^{\overline{Q}_2^*}_{01(0)} = (q_{01} + q_{11}q_{02}) + \overline{q}_{01}\overline{q}_{02} = q_{01} + q_{11}q_{02} + q_{11}q_{12} = 1.$

For Q_2^*-binary numbers of the 3rd rank, we have
$\Delta^{Q_2^*}_{001(0)} + \Delta^{\overline{Q}_2^*}_{111(0)} = q_{01}q_{02}q_{03} + (q_{11} + q_{01}q_{12} + q_{01}q_{02}q_{13}) = q_{11} + q_{01}q_{12} + q_{01}q_{02} = 1;$
$\Delta^{Q_2^*}_{011(0)} + \Delta^{\overline{Q}_2^*}_{101(0)} = (q_{01}q_{02} + q_{01}q_{12}q_{03}) + (q_{11} + q_{01}q_{12}q_{13}) = 1;$
$\Delta^{Q_2^*}_{101(0)} + \Delta^{\overline{Q}_2^*}_{011(0)} = (q_{01} + q_{11}q_{02}q_{03}) + (q_{11}q_{12} + q_{11}q_{02}q_{13}) = 1;$
$\Delta^{Q_2^*}_{111(0)} + \Delta^{\overline{Q}_2^*}_{001(0)} = (q_{01} + q_{11}q_{02} + q_{11}q_{12}q_{03}) + q_{11}q_{12}q_{13} = 1.$

Similarly, we can check the equality (3) for any Q_2^*-binary number.

For a general case, we recode the cylinders $\Delta^{Q_2^*}_{c_1c_2\ldots c_m} = \Delta_{[1-c_1][1-c_2]\ldots[1-c_m]}$ and number $x = \Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} = \Delta_{[1-\alpha_1][1-\alpha_2]\ldots[1-\alpha_n]\ldots}$. Let us pose the question: how a number

$$x' = \Delta^{\overline{Q}_2^*}_{[1-\alpha_1][1-\alpha_2]\ldots[1-\alpha_n]\ldots}$$

is related to a number x? It is clear that $x' = 1 - x$. In fact, by giving the number x in the form $x = \Delta^{Q_2^*}_{\underbrace{0\ldots0}_{a_1}1\underbrace{0\ldots0}_{a_2}1\ldots\underbrace{0\ldots0}_{a_m}1\ldots}$, we have $x' = \Delta^{\overline{Q}_2^*}_{\underbrace{1\ldots1}_{a_1}0\underbrace{1\ldots1}_{a_2}0\ldots\underbrace{1\ldots1}_{a_m}0\ldots}$. Whence it is obvious that $x' + x = 1 = \Delta^{\overline{Q}_2^*}_{(1)}$, since $|\Delta^{Q_2^*}_{c_1\ldots c_m}| = \prod\limits_{i=1}^{m} q_{c_i,i} = |\Delta^{\overline{Q}_2^*}_{[1-c_1][1-c_2]\ldots[1-c_m]}|$.

Thus, $x + x' = 1$, which is equivalent to the equality (3). □

2. Object of study: inversor of digits.

Definition 5. An *inversor* of digits of Q_2^*-representation of numbers of closed interval $[0;1]$ is a function $y = I(x)$ defined by equality

$$y = I(x = \Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_n...}) = \Delta^{Q_2^*}_{[1-\alpha_1][1-\alpha_2]...[1-\alpha_n]...}.$$

Since $I(\Delta^{Q_2^*}_{c_1c_2...c_n0(1)}) = I(\Delta^{Q_2^*}_{c_1c_2...c_n1(0)})$, this function is well-defined.

If $q_{0k} = \frac{1}{2}$ for any $k \in N$, i.e., Q_2^*-representation is a classic binary representation then $I(x) = 1 - x$.

Theorem 4. *A function $I(x)$ is a continuous, strictly decreasing function with $I(0) = 1$, $I(1) = 0$.*

Proof. The following equalities are obvious: $I(0) = I(\Delta^{Q_2^*}_{(0)}) = \Delta^{Q_2^*}_{(1)} = 1$, $I(1) = I(\Delta^{Q_2^*}_{(1)}) = \Delta^{Q_2^*}_{(0)} = 0$.

Let $x_1 < x_2$. Then according to Corollary 1 $x_1 = \Delta^{Q_2^*}_{c_1...c_{m-1}0d_1d_2...}$, $x_2 = \Delta^{Q_2^*}_{c_1...c_{m-1}1d'_1d'_2...}$ and there is $m \in N$ such that $d_m - d'_m \neq 1$. According to Theorem 2 we have

$$I(x_1) - I(x_2) = \Delta^{Q_2^*}_{[1-c_1]...[1-c_m]1[1-d_1]...} - \Delta^{Q_2^*}_{[1-c_1]...[1-c_m]0[1-d'_1]...} > 0.$$

Therefore, the function $I(x)$ is strictly decreasing.

Let $x_0 = \Delta^{Q_2^*}_{c_1...c_n...}$ be any Q_2^*-unary point from $(0;1)$, and let $x_0 \neq x = \Delta^{Q_2^*}_{\alpha_1...\alpha_n...}$. Then there is m such that $\alpha_i = c_i$ for $i < m$ and $\alpha_m \neq c_m$. The condition $x \to x_0$ is equivalent to $m \to \infty$.

We consider an expression

$$|I(x) - I(x_0)| = (\prod_{i=1}^{m-1} q_{c_i i})|(\beta_{1-\alpha_m,m} + \beta_{1-\alpha_{m+1},m+1}q_{1-\alpha_m,m} + ...)-$$
$$(-\beta_{1-c_m,m} + \beta_{1-c_{m+1},m+1}q_{1-c_m,m} + ...)|.$$

Since the absolute value in the right-hand side of the equation does not exceed 1 as the difference of two numbers from an interval $[0;1]$, we have $|I(x) - I(x_0)| \to 0$ as $n \to \infty$. Therefore, the function $I(x)$ is continuous at the point x_0.

For Q_2^*-binary point $x_0 = \Delta^{Q_2^*}_{c_1...c_m0(1)} = \Delta^{Q_2^*}_{c_1...c_m1(0)}$, the continuity of the function is proved similarly, but for the proof of the left-hand continuity of the function one need to use the first representation. It is also necessary to use the second representation for right-hand continuity of function. Therefore, $I(x)$ is continuous at each point of the interval $[0;1]$. □

Definition 6. If a matrix $\|q_{ik}\|$ satisfies the condition $\lim\limits_{k\to\infty} q_{0k} = q_0$ then we say that it *has an asymptotic property.*

Theorem 5. *If for almost all numbers $x = \Delta^{Q_2^*}_{\alpha_1(x)\alpha_2(x)...\alpha_n(x)...}$ of closed interval $[0;1]$ the sequence $\frac{q_{1-\alpha_n(x)n}}{q_{\alpha_n(x)}}$ is either divergent or its limit is not equal to 1 then the function $I(x)$ is a singular function, i.e., it is a continuous function such that its derivative is equal to 0 almost everywhere (in the sense of Lebesgue measure).*

Proof. Since the function I is continuous and monotonic, we see that according to the well-known Lebesgue Theorem it has a finite derivative almost everywhere (in the sense of Lebesgue measure). We consider Q_2^*-unary point $x_0 = \Delta^{Q_2^*}_{c_1\ldots c_n\ldots}$ such that there exists a finite derivative $I'(x_0)$. Then the derivative is calculated by the formula

$$-I'(x_0) = \lim_{n\to\infty} \frac{I(\Delta^{Q_2^*}_{c_1\ldots c_n(0)}) - I(\Delta^{Q_2^*}_{c_1\ldots c_n(1)})}{|\Delta^{Q_2^*}_{[1-c_1]\ldots[1-c_n]}|} = \prod_{n=1}^{\infty} \frac{q_{1-c_n,n}}{q_{c_n n}}. \tag{4}$$

Under the conditions of the theorem, the last product is divergent on the set of full Lebesgue measure. Being finite, it diverges to zero. Therefore, $I(x)$ is a singular function.

When the conditions of the theorem are satisfied, the multipliers of the last infinite product are separated from zero, and therefore the necessary condition of its convergence is not satisfied. Therefore, according to the Lebesgue Theorem, the derivative of the function I is zero at the point x_0. This proves the singularity of the inversor. □

Corollary 3. *If a matrix $\|q_{ik}\|$ has an asymptotic property with $0 < \lim\limits_{k\to\infty} q_{0k} = q_0 < 1$ and $q_0 \neq \frac{1}{2}$, then function $I(x)$ is a singular function.*

Proof. If the matrix $\|q_{ik}\|$ has an asymptotic property and $q_0 \neq \frac{1}{2}$ then multipliers of product (4) are separated from 1. Therefore, the necessary condition for its convergence is not fulfilled. Thus, the infinite product diverges to zero. And this proves the singularity of the function $I(x)$. □

Lemma 1. *If a matrix $\|q_{ik}\|$ has an asymptotic property and $q_0 = \frac{1}{2}$ then the function I has not derivative (neither finite nor infinite) at every Q_2-binary point.*

Proof. Let $x_0 = \Delta^{Q_2^*}_{c_1c_2\ldots c_m0(1)} = \Delta^{Q_2^*}_{c_1c_2\ldots c_m1(0)}$ be any Q_2^*-binary point of interval $(0;1)$. The proof is by reductio ad absurdum. Suppose that there exists a derivative $I'(x_0)$. Then it is equal to the cylindrical derivative, namely, it can be calculated by formulas:

$$-I'(x_0) = \Big(\prod_{i=1}^{m} \frac{q_{1-c_i,i}}{q_{c_i}}\Big) \frac{q_{1,m+1}}{q_{0,m+1}} \prod_{i=m+2}^{\infty} \frac{q_{1i}}{q_{0i}}; \quad I'(x_0) = \Big(\prod_{i=1}^{m} \frac{q_{1-c_i,i}}{q_{c_i}}\Big) \frac{q_{0,m+1}}{q_{1,m+1}} \prod_{i=m+2}^{\infty} \frac{q_{0i}}{q_{1i}}.$$

If $q_0 \neq \frac{1}{2}$ then members of sequences $((\frac{q_{0i}}{q_{1i}})$ and $(\frac{q_{1i}}{q_{0i}}))$ are separated from 1. Therefore, one of the last multipliers in the different expressions of the derivative is zero, and the other is infinity. That is why these expressions of derivative do not acquire a common value. The resulting contradiction proves the statement. □

3. Self-affine and integral properties of inversor of digit of Q_2-representation of numbers.

Lemma 2. *A graph Γ of function I that is the following union $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\Gamma_i = \{M(x;y)\colon x = \Delta^{Q_2}_{i\alpha_2\alpha_3\ldots}, y = I(x)\}$ is a self-affine set of space R^2. Moreover, $\Gamma_i = \varphi_i(\Gamma)$, where* $\varphi_i\colon \begin{cases} x' = \Delta^{Q_2}_{i\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots} = \beta_i + q_i x, \\ y' = \Delta^{Q_2}_{[1-i][1-\alpha_1]\ldots[1-\alpha_n]\ldots} = \beta_{1-i} + q_{1-i} I(x). \end{cases}$

Proof. 1. First of all we prove the inclusion $\Gamma_i \subset \varphi_i(\Gamma)$. Let $M*(x*;y*) \in \Gamma_i$, i.e., $x* = \Delta^{Q_2}_{i\alpha_2\ldots\alpha_n\ldots}$ and $y* = \Delta^{Q_2}_{[1-\alpha_2]\ldots[1-\alpha_n]\ldots}$, $y* = I(x*)$. We consider a point $M(x;y)$, where $x = \Delta^{Q_2}_{\alpha_2\ldots\alpha_n\ldots}$, $y = \Delta^{Q_2}_{[1-\alpha_2][1-\alpha_3]\ldots[1-\alpha_n]\ldots}$. It is clear that $I(x) = y$. And so $M* \in \Gamma$ and $M* \in \varphi_i(\Gamma)$, i.e., $\Gamma_i \subset \varpi_i(\Gamma)$.

2. To prove inclusion $\varpi_i(\Gamma) = \Gamma_i$ we consider a point $M(x;y) \in \Gamma$ and its image $M'(x';y')$ under affine transformation φ_i. Obviously that $I(x') = y'$ and since the first digit of Q_2-representative of number x' is a digit i, we have $M' \in \Gamma_i$. So $\varphi_i(\Gamma) \subset \Gamma_i$ and $\varphi_i(\Gamma) = \Gamma_i$. This completes the proof of the lemma. □

Corollary 4. *The self-affine dimension of the graph* Γ *of the function* I *is equal to the number* $\dfrac{-2}{\log_2(q_0q_1)}$, *which is a fractional number if* $q_0 \neq \frac{1}{2}$.

Indeed, given the self-affinity structure of the graph (expressions φ_0 and φ_1), the equation for determining the self-affine dimension has the form

$$\begin{vmatrix} q_0 & 0 \\ 0 & q_1 \end{vmatrix}^{\frac{x}{2}} + \begin{vmatrix} q_1 & 0 \\ 0 & q_0 \end{vmatrix}^{\frac{x}{2}} = 1.$$

Its solution is the dimension of the graph Γ.

Theorem 6. *For the inversor* I *of digits of* Q_2*-representation of numbers,*

$$\int_0^1 I(x)dx = \frac{q_0^2}{1 - 2q_0q_1}. \tag{5}$$

The following equality is true:

$$\int_{\Delta^{Q_2}_{c_1\ldots c_m}}^1 I(x)dx = AP + (q_0q_1)^m \int_0^1 I(x)dx = AP + \frac{q_0^{m+2}q_1^m}{1 - 2q_0q_1}, \tag{6}$$

where $P = |\Delta^{Q_2}_{c_1\ldots c_m}| = \prod_{i=1}^m q_{c_i}$, $A = I(\Delta^{Q_2}_{c_1\ldots c_m(1)}) = \Delta^{Q_2}_{[1-c_1]\ldots[1-c_m](0)}$.

Proof. Since $\int_0^1 I(x)dx = \int_0^{q_0} I(x)dx + \int_{q_0}^1 I(x)dx$, taking into account

$$I(x = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_k\ldots}) = \beta_{1-\alpha_1} + \sum_{k=2}^{\infty} (\beta_{1-\alpha_k} \prod_{i=1}^{k-1} q_{1-\alpha_i}),$$

$$I(x = \Delta^{Q_2}_{0\alpha_2\ldots\alpha_k\ldots}) = q_0 + q_1(\beta_{1-\alpha_2} + \sum_{k=3}^{\infty} (\beta_{1-\alpha_k} \prod_{i=1}^{k-1} q_{1-\alpha_i})),$$

$$I(x = \Delta^{Q_2}_{1\alpha_2\ldots\alpha_k\ldots}) = 0 + q_0(\beta_{1-\alpha_2} + \sum_{k=3}^{\infty} (\beta_{1-\alpha_k} \prod_{i=1}^{k-1} q_{1-\alpha_i}))$$

and self-affine property of graph of function (see previous lemma) we obtain

$$\int_0^{q_0} I(x)dx = \int_{\Delta_0^{Q_2}} I(x)dx = \int_0^1 (q_0 + q_1y)d(q_0x) = q_0^2 + q_0q_1 \int_0^1 I(x)dx,$$

$$\int_{q_0}^1 I(x)dx = \int_{\Delta_1^{Q_2}} I(x)dx = \int_0^1 (q_0y)d(q_0 + q_1x) = q_0q_1 \int_0^1 I(x)dx.$$

Hence, $\int_0^1 I(x)dx = q_0^2 + 2q_0q_1\int_0^1 I(x)dx$, and therefore we have equality (5). Taking into account that 1) the integral expresses the area of the corresponding curvilinear trapezoid; 2) the area has additive property; 3) the graph is a self-affine figure; 4) the function I is strictly decreasing, we have

$$\int_{\Delta^{Q_2}_{c_1\ldots c_m}}^{1} I(x)d = |\Delta^{Q_2}_{c_1\ldots c_m}|I(\Delta^{Q_2}_{c_1\ldots c_m(1)}) + |\Delta^{Q_2}_{c_1\ldots c_m}||\Delta^{Q_2}_{[1-c_1]\ldots[1-c_m]}|\int_0^1 I(x)dx.$$

However, $|\Delta^{Q_2}_{c_1\ldots c_m}||\Delta^{Q_2}_{[1-c_1]\ldots[1-c_m]}| = (q_0q_1)^m$. Then, according to the previous lemma, we obtain equality (6). □

Remark 2. Theorem 6 allow us to calculate the integral $f(t) = \int_0^t I(x)dx$ as the sum of the integrals over all the cylinders preceding t.

REFERENCES

1. O.M. Baranovskyi, M.V. Pratsiovytyi, G.M. Torbin, Ostrogradsky–Sierpiński–Pierce series and their applications. Kyiv: Nauk. Dumka, 2013, 288 p.
2. M. Pratsiovytyi, A.Chuikov, *Continuous distributions whose functions preserve tails of an A_2-continued fraction representation of numbers*, Random Operators and Stochastic Equations, **27** (2019), №3, 199–206.
3. Peter R. Massopust, Fractal functions, fractal surfaces, and wavelets. Academic Press; 1 edition (January 18, 1995), 383 p.
4. F. Schweiger, Ergodic theory of fibred systems and metric number theory. Oxford: Clarendon Press, 1995, 320 p.
5. Ya.V. Goncharenko, N.V. Dyvliash, *Testing of the statistical hypothesis about the parameter of the Salem-type probability distribution*, Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Dragomanova. Ser 1. Fiz.-Mat. Nauky, **16** (2014), №2, 100–108. (in Ukrainian)
6. I.V. Zamriy, M.V. Pratsiovytyi, *Singularity of inversor of the digit for the Q_3-representation of the fractional part of a real number, its fractal and integral properties*, J. Math. Sci. (N.Y.), **215** (2016), №3, 323–340.
7. I.M. Lysenko, Yu.P. Maslova, M.V. Pratsiovytyi, *Two-symbol numerical system with two bases having different signs and related functions*, Proc. Inst. Math. Nat. Acad. Sc. Ukraine, **16** (2019), №2, 50–62.
8. M.V. Pratsiovytyi, *Random variables with independent Q_2-symbols (in Russian)*, in: Asymptotic Methods in the Study of Stochastic Models, Inst. Math. Natl. Acad. Sci. Ukraine, Kyiv, 1987, 92–102.
9. M.V. Pratsiovytyi, S.V. Skrypnyk, *Q_2-representation of the fractional part of a real number and the inversor of its digits*, Nauk. Chasop. Nats. Pedagog. Univ. M. Dragomanov. Ser 1. Fiz.-Mat. Nauky, **15** (2013), 134–143. (in Ukrainian)
10. M.V. Pratsiovytyi, Fractal approach in investigation of singular probability distributions, Natl. Pedagog. Dragomanov Univ. Publ., Kyiv, 1998. (in Ukrainian)
11. M.V. Pratsiovytyi, *Fractal properties of distributions of random variables whose Q-digits form a homogeneous Markov chain*, in: Asymptotic Methods in the Study of Stochastic Models, Inst. Math. Natl. Acad. Sci. Ukraine, Kyiv, 1994, 245–254. (in Russian)
12. G.M. Torbin, M.V. Pratsiovytyi, *Random variables with independent Q_2^*-digits*, in: Random Evolutions: Theoretical and Applied Problems, Inst. Math. Natl. Acad. Sci. Ukraine, Kyiv, 1992, 95–104. (in Russian)
13. A.F. Turbin, M.V. Pratsiovytyi, Fractal Sets, Functions, and Probability Distributions, Nauk. Dumka, Kyiv, 1992. (in Russian)

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