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THE RENEWAL EQUATION IN NONLINEAR APPROXIMATION

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The family of Markov processes are considered. We study the multidimensional renewal equation in nonlinear approximation. The purpose of the paper is to find the limit of renewal function.

Let $X^\varepsilon(t)$ be a family of Markov processes with continuous time and a finite number of states $1, 2, .., n$, and $X^\varepsilon(t) \to X(t)$, where $\varepsilon \to 0$.

Suppose that $\xi_i^\varepsilon(t)$ is a process with independent increments, $t \geq 0$, $\xi_i^\varepsilon(t) > 0$, $i = 1, 2, .., n$. We consider the following process

$$\zeta^\varepsilon(t) = \begin{cases} \xi^\varepsilon_{i(1)}(t), & \text{if } t < \tau, X^\varepsilon(0) = i \\ \xi^\varepsilon_{i(1)}(\tau) + \xi^\varepsilon_{j(2)}(t-\tau), & \text{if } \tau \leq t < \tau_1, X^\varepsilon(\tau) = j \\ \xi^\varepsilon_{i(1)}(\tau) + \xi^\varepsilon_{j(2)}(\tau_1-\tau) + \xi^\varepsilon_{s(3)}(t-\tau_1), & \text{if } \tau_1 \leq t < \tau_2, X^\varepsilon(\tau_2) = s \\ \vdots \end{cases}$$

In this representation $\xi^\varepsilon_{i(p)}(t)$ are independent copies of $\xi^\varepsilon_i(t)$.

Consider the next conditional expectation $E(e^{-\zeta^\varepsilon(t)} | X^\varepsilon(0) = i) = E_i(e^{-\zeta^\varepsilon(t)})$.

This implies

$$E_i(e^{-\zeta^\varepsilon(t)}) = E_i(e^{-\zeta^\varepsilon(t)}, \tau > t) + \sum_{j=1}^{n} \int_0^t E_i(e^{-\lambda\zeta^\varepsilon(t)}, \tau \in du, X^\varepsilon(\tau) = j) \cdot E_j(e^{-\lambda\zeta^\varepsilon(t-u)}).$$

Then

$$E_i(e^{-\lambda\zeta^\varepsilon(u)}, \tau \in du, X(\tau) = j | X^\varepsilon(0) = i) = E(e^{-\lambda\xi_i^\varepsilon(t)}) \cdot P\{\tau \in du, X(\tau) = j | X^\varepsilon(0) = i\}.$$

Denote $P\{\tau \in du, X(\tau) = j | X^\varepsilon(0) = i\} = p^\varepsilon_{ij}(du)$. As result, we obtain multidimensional renewal equation

$$E_i(e^{-\lambda\zeta^\varepsilon(t)}) = E_i(e^{-\lambda\zeta_i^\varepsilon(t)}) \cdot P\{\tau < t | X^\varepsilon(0) = i\} + \sum_{j=1}^{m} \int_0^t (E_i(e^{-\lambda\zeta^\varepsilon(u)})) p^\varepsilon_{ij}(du) \cdot E_j(e^{-\lambda\zeta^\varepsilon(t-u)}).$$

The following conditions are true for this renewal equation.

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1. $E_i(e^{-\lambda\zeta^\varepsilon(u)})p^\varepsilon_{ij}(du) \geq 0$, $E_i(e^{-\lambda\zeta^\varepsilon(u)})p^\varepsilon_{ij}(du) < \infty$.
2. $E_i(e^{-\lambda\zeta^\varepsilon(u)})p^\varepsilon_{ij}(du)$ is an indecomposable matrix.
3. There is a weak convergence $E_i(e^{-\lambda\zeta^\varepsilon(u)})p^\varepsilon_{ij}(du) \to E_i(e^{-\lambda\zeta(u)})p_{ij}(du)$, when $\varepsilon \to 0$.
4. $E_i(e^{-\lambda\zeta(u)})p_{ij}(du)$ is a block folding matrix.
5. $E_i(e^{-\lambda\zeta(u)})p_{ij}(du)$ is uniformly integrable.

Suppose $w_1 \in E_1, w_2 \in E_2, ..., w_r \in E_r$, $D = \{w_1, ..., w_r\}$.

Consider the next renewal functions

$$H^{\varepsilon(0)}_{ij}(t) = E_i(e^{-\lambda\xi^\varepsilon_i(t)}) \cdot P\{\tau < t | X^\varepsilon(0) = i\} + \sum_{s=1}^{r} E_i(e^{-\lambda\zeta^\varepsilon(u)}) \cdot p^\varepsilon_{iw_s}(du) * E_{w_s j}(e^{-\lambda\zeta^\varepsilon(t-u)})$$

and

$$H^{\varepsilon(n)}_{ij}(t) = H^{\varepsilon(0)}_{ij}(t) + \sum_{m\notin D} E_i(e^{-\lambda\zeta^\varepsilon(u)}) \cdot p^\varepsilon_{im}(du) * E^{\varepsilon(n-1)}_{mj}(t), \varepsilon \to 0.$$

Let us introduce the sequence $L^{\varepsilon(0)}_{ij} = E_i(e^{-\lambda\zeta^\varepsilon(u)}) \cdot p^\varepsilon_{ij}(du)$. Then

$$L^{\varepsilon(n)}_{ij}(t) = E_i(e^{-\lambda\zeta^\varepsilon(u)}) \cdot p^\varepsilon_{ij}(du) + \sum_{m\notin D} E_i(e^{-\lambda\zeta^\varepsilon(u)}) \cdot p^\varepsilon_{im}(du) \cdot L^{\varepsilon(n-1)}_{mj}(t).$$

Hence

$$H^{\varepsilon(n)}_{ij}(t) = H^{\varepsilon(0)}_{ij}(t) + \sum_{m\notin D} L^{\varepsilon(n-1)}_{im}(t) \cdot H^{\varepsilon(0)}_{mj}(t) + \sum_{i=1}^{r} L^{\varepsilon(n)}_{iw_s} \cdot E_i(e^{-\lambda\zeta^\varepsilon(u)})p^\varepsilon_{iw_s}(du)E_{w_s j}(e^{-\lambda\zeta^\varepsilon(t-u)}).$$

Therefore,

$$H^\varepsilon_{w_s j}(t) = L^\varepsilon_{w_s j}(t) + \sum_{n=1}^{r} L^\varepsilon_{w_s w_n} * H^\varepsilon_{w_n j}(t).$$

The solution of this equation can be represented in the next form

$$E_i(e^{-\lambda\zeta^\varepsilon(t)}) = \sum_{n=1}^{k} \int_0^t E_i(e^{-\lambda\zeta^\varepsilon_n(t)})p_{ij}(du)L^\varepsilon_{w_n j}(t-u).$$

Theorem 1. *If $E(e^{-\lambda\zeta^\varepsilon(u)})p_{ij}(du)$ is a non-lattice matrix and conditions 1–5 hold, then there exist a matrix C and a normalizing factor $g(\varepsilon) \to 0$ $(\varepsilon \to 0)$ such that for $i \in E_s$, $j \in E_k$*

$$\lim_{\varepsilon\to 0} H^\varepsilon_{ij}\Big[\frac{t}{g(\varepsilon)}, \frac{t}{g(\varepsilon)} + u\Big] = u \cdot q_{sk}(t) \cdot \frac{p^{(k)}_j}{\pi_k}$$

where $q_{sk} = [e^{tC}]_{sk}$, $\pi_k = \sum_{i,j\in E_k} p^{(k)}_i \cdot \int_0^\infty tF_{ij}(dt)$, $H^\varepsilon_{ij}[\frac{t}{g(\varepsilon)}, \frac{t}{g(\varepsilon)} + u]$ is a renewal function on the interval $[\frac{t}{g(\varepsilon)}, \frac{t}{g(\varepsilon)} + u]$.

Proof. Suppose $w_1 \in E_1, w_2 \in E_2, ..., w_r \in E_r$, $D = \{w_1, ..., w_r\}$ and introduce the next functions

$$H^{\varepsilon(0)}_{ij}(t) = \delta_{ij} + \sum_{s=1}^{r} E_{iw_s}(e^{-\lambda\zeta^\varepsilon(t)}) * H^\varepsilon_{w_s j}(t),$$

$$H_{ij}^{\varepsilon(n)}(t) = H_{ij}^{\varepsilon(0)}(t) + \sum_{m\notin D} E_{im}(e^{-\lambda\zeta^{\varepsilon}(t)}) * H_{mj}^{\varepsilon(n-1)(t)}.$$

We consider

$$H_{ij}^{\varepsilon}(t) - H_{ij}^{\varepsilon(n)}(t) = \sum_{m_1\notin D} E_{im_1}(e^{-\lambda\zeta^{\varepsilon}(t)}) * (H_{m_1 j}^{\varepsilon}(t) - H_{m_1 j}^{\varepsilon(n-1)}(t)) = ... =$$

$$= \sum_{m_1\notin D,\ldots,m_n\notin D} E_{im_1}(e^{-\lambda\zeta^{\varepsilon}(t)}) * .. * E_{m_{n-1}m_n}(e^{-\lambda\zeta^{\varepsilon}(t)}) * (H_{m_n j}^{\varepsilon}(t) - H_{m_n j}^{\varepsilon(0)}(t)) =$$

$$= \sum_{m_1\notin D,\ldots,l\notin D} E_{im_1}(e^{-\lambda\zeta^{\varepsilon}(t)}) * \cdots * E_{m_n l}(e^{-\lambda\zeta^{\varepsilon}(t)}) * H_{lj}^{\varepsilon}(t) \le E(e^{-\lambda\zeta^{\varepsilon(*n)}(t)}) * H^{\varepsilon}(t) \to 0,$$

when $\varepsilon \to 0$.

So, $H_{ij}^{\varepsilon}(t) \to H_{ij}^{\varepsilon(n)}(t)$, when $n \to \infty$, $\varepsilon \to 0$.

Arguing as in [1] we deduce

$$H_{w_s w_k}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} + y\Big) - H_{w_s w_k}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)}\Big) \to y \cdot q_{sk}(t) \cdot \frac{1}{m_k},$$

where $q_{sk}(t)$ is the (s,k)-th matrix element.

Then,

$$H_{w_s j}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} + u\Big) - H_{w_s j}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)}\Big) = \sum_{m=1}^{r} \int_0^{\frac{t}{g(\varepsilon)}+y} H_{w_s w_m (du) L_{w_m j}^{\varepsilon}}\Big(\frac{t}{g(\varepsilon)} + y - u\Big) -$$

$$-\sum_{m=1}^{r} \int_0^{\frac{t}{g(\varepsilon)}} H_{w_s w_m}^{\varepsilon}(du) L_{w_m j}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} - u\Big).$$

Integrating by parts we obtain

$$H_{w_s j}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} + u\Big) - H_{w_s j}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)}\Big) =$$

$$= \sum_{m=1}^{r} \int_0^{\frac{t}{g(\varepsilon)}} [H_{w_s w_m}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} + y - u\Big) - H_{w_s w_m}\Big(\frac{t}{g(\varepsilon) - u}\Big)] L_{w_m j}^{\varepsilon}(du) +$$

$$+ \sum_{m=1}^{r} \int_0^{\frac{t}{g(\varepsilon)}+y} H_{w_s w_m}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} + y - u\Big) L_{w_m j}(du) - \sum_{m=1}^{r} \Big[L_{w_m j}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} + y\Big) - L_{w_m j}\Big(\frac{t}{g(\varepsilon)}\Big)\Big].$$

We then find the limit

$$\int_{\frac{t}{g(\varepsilon)}}^{\frac{t}{g(\varepsilon)}+y} H_{w_s w_m}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} + y - u\Big) L_{w_m j}^{\varepsilon}(du) - \Big[L_{w_m j}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} + y\Big) - L_{w_m j}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)}\Big)\Big] \le$$

$$\le (H_{w_s w_m}(y) - I)\Big[L_{w_m j}\Big(\frac{t}{g(\varepsilon)} + y\Big) - L_{w_m j}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)}\Big)\Big] \longrightarrow 0, \quad \varepsilon \longrightarrow 0.$$

Then, for a fixed $T > 0$ and we evaluate the next integral

$$\sup_{\varepsilon} \int_T^{\frac{t}{g(\varepsilon)}} \Big[H_{w_s w_m}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} + y - u\Big) - H_{w_s w_m}^{\varepsilon}\Big(\frac{t}{g(\varepsilon)} - u\Big)\Big] L_{w_m j}^{\varepsilon}(du) \le$$

$$\leq \sup_{\varepsilon} \sum_{r\geq[T/h]} \int_{rh}^{(r+1)h} \Big[H^{\varepsilon}_{w_s w_m}\Big(\frac{t}{g(\varepsilon)}+y-u\Big) - H_{w_s w_m}\Big(\frac{t}{g(\varepsilon)}-u\Big)\Big] L^{\varepsilon}_{w_m j}(du) \leq$$

$$\leq \sup_{\varepsilon} \sum_{r\geq[T/h]} \Big[H^{\varepsilon}_{w_s w_m}\Big(\frac{t}{g(\varepsilon)}+y-rh\Big) - H_{w_s w_m}\Big(\frac{t}{g(\varepsilon)}-(rh+h)\Big)\Big]\times$$

$$\times \int_{rh}^{(r+1)h} L^{\varepsilon}_{w_m j}(du) \leq [A(y+h)+2B] \sup_{\varepsilon} \int_{rh}^{(r+1)h} L^{\varepsilon}_{w_m j}(du) \leq$$

$$\leq [Ay+2B]\cdot \sup_{\varepsilon}[L^{\varepsilon}_{w_m j}(\infty) - L^{\varepsilon}_{w_m j}(T)] + A\cdot \sup_{\varepsilon}\int_{T}^{\infty} u L^{\varepsilon}_{w_m j}(du).$$

Hence,

$$\overline{\lim_{\varepsilon\to 0}} \sup_{0\leq u\leq T} \Big[H^{\varepsilon}_{w_s w_k}\Big(\frac{t}{g(\varepsilon)}+y-u\Big) - H_{w_s w_k}\Big(\frac{t}{g(\varepsilon)}-u\Big)\Big] \leq Ay+2B < \infty$$

and from [2] we obtain

$$\int_0^T \Big[H^{\varepsilon}_{w_s w_k}\Big(\frac{t}{g(\varepsilon)}+y\Big) - H^{\varepsilon}_{w_s w_k}\Big(\frac{t}{g(\varepsilon)}\Big)\Big] L^{\varepsilon}_{w_k j}(du) \longrightarrow y\cdot q_{sk}(t)\cdot \frac{1}{m_k} L_{w_k j}(t).$$

Then

$$H^{\varepsilon}_{w_s j}\Big(\frac{t}{g(\varepsilon)}+y\Big) - H^{\varepsilon}_{w_s j}\Big(\frac{t}{g(\varepsilon)}\Big) \longrightarrow y\cdot q_{sk}\cdot \frac{1}{m_k} L_{w_k j}.$$

As result,

$$H^{\varepsilon}_{ij}\Big(\frac{t}{g(\varepsilon)}+y\Big) - H^{\varepsilon}_{ij}\Big(\frac{t}{g(\varepsilon)}\Big) \longrightarrow y\cdot L_{i w_s}\cdot q_{sk}(t)\cdot \frac{1}{m_k} L_{w_k j}.$$

□

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