УДК 517.53

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STRUCTURE OF THE SET OF BOREL EXCEPTIONAL VECTORS FOR ENTIRE CURVES. II

Ya. I. Savchuk[1], A. I. Bandura[2]. *Structure of the set of Borel exceptional vectors for entire curves. II*, Mat. Stud. **55** (2021), 137–145.

We have obtained a description of structure of the sets of Picard and Borel exceptional vectors for transcendental entire curve in some sense. We consider only p-dimensional entire curves with linearly independent components without common zeros. In particular, the set of Borel exceptional vectors together with the zero vector is a finite union of subspaces in $\mathbb{C}^p$ of dimension at most $p-1$. Moreover, the sum of their dimensions does not exceed p if any pairwise intersection of the subspaces contains only the zero vector. A similar result is also valid for the set of Picard exceptional vectors. Another result shows that the structure of the set of Borel exceptional vectors for an entire curve of integer order differs somewhat from the structure of such a set for an entire curve of non-integer order. For a transcendental entire curve $\vec{G}:\mathbb{C}\to\mathbb{C}^p$ with linearly independent components and without common zeros having non-integer or zero order the set of Borel exceptional vectors together with the zero vector is a subspace in $\mathbb{C}^p$ of dimension at most $p-1$.

However, the set of Picard exceptional vectors does not possess this property. We propose two examples of entire curves. The first example shows the set of Borel exceptional vectors together with the zero vector for p-dimensional entire curve of integer order is union of subspaces of dimension at most $p-1$ such that the total sum of these dimensions does not exceed p and intersection of any pair of these subspaces contains only zero vector. The set of Picard exceptional vectors for the curve has the same property. In the second example, we construct a q-dimensional entire curve of non-integer order for which the set of Borel exceptional vectors together with the zero vector is a subspace in $\mathbb{C}^q$ of dimension at most $q-1$ and the set of Picard exceptional vectors together with the zero vector do not have the property. This set is a union of some subspaces.

Recently, we introduced [1] the concept of a Borel exceptional vector for an entire curve $\vec{G}\colon\mathbb{C}\to\mathbb{C}^p$. In a discussion on this paper A. Eremenko (Purdue University, USA) suggested that the structure of the set of Picard exceptional vectors is similar to the structure of the set of Borel exceptional vectors as well as the possibility to solve an inverse problem. The present paper is devoted to this problem posed by Prof. A.E. Eremenko.

In this paper, we use main results of the theory of entire curves and notation from [3] and [7]. Let us remind this notation.

Denote an entire curve $\vec{G}\colon\mathbb{C}\to\mathbb{C}^p$ as $\vec{G}(z)=(g_1(z),\, g_2(z),\, \ldots,\, g_p(z))$, where $g_k(z)$ are entire functions, $k\in\{1,2,\ldots,p\}$. Let us consider entire curves with linearly independent

2010 *Mathematics Subject Classification:* 30D35.
Keywords: entire curve; integral curve; Borel exceptional vector; Nevanlinna deficient vector; meromorphic function; admissible system of vectors; Picard exceptional vector; Picard exceptional value.
doi:10.30970/ms.55.2.137-145

components and has no common zeros. In other words, we assume that an entire curve $\vec{G}\colon \mathbb{C} \to \mathbb{C}^p$ has linearly independent components $g_k(z)$ and has not common zeros for all $g_k(z)$, $k \in \{1, 2, \dots, p\}$.

For $\vec{a} = (a_1, a_2, \dots, a_p) \in \mathbb{C}^p$ and $\vec{b} = (b_1, b_2, \dots, b_p) \in \mathbb{C}^p$ the notation $\vec{a}\vec{b}$ means the dot product of these vectors, that is $\vec{a}\vec{b} = \sum_{j=1}^{p} a_j \bar{b}_j$, where $\bar{b}_j$ is complex conjugate to b_j.

For every p-dimensional vector $\vec{a} = (a_1, a_2, \dots, a_p) \neq \vec{0}$ the dot product $\vec{G}(z)\vec{a} = \sum_{k=1}^{p} g_k(z)\overline{a}_k$ is an entire function. Denote by $n(t, \vec{a}, \vec{G})$ a number of zeros of the dot product $\vec{G}(z)\vec{a}$ in the disc $\{z\colon |z| \le t\}$, where each zero is counted according to its multiplicity. Every zero of the function $\vec{G}(z)\vec{a}$ is called a-point of entire curve $\vec{G}(z)$. Let us denote

$$N(r, \vec{a}, \vec{G}) = \int_0^r \frac{n(t, \vec{a}, \vec{G}) - n(0, \vec{a}, \vec{G})}{t} dt + n(0, \vec{a}, \vec{G}) \ln r,$$

where $n(0, \vec{a}, \vec{G})$ stands for the multiplicity of zero of the dot product $\vec{G}(z)\vec{a}$ at the point $z = 0$.

The growth characteristic $T(r, \vec{G})$ is defined as following

$$T(r, \vec{G}) = \frac{1}{2\pi} \int_0^{2\pi} \ln \|\vec{G}(re^{i\varphi})\| d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \ln \sqrt{\sum_{k=1}^{p} |g_k(re^{i\varphi})|^2} d\varphi.$$

We will use the definition of the growth category from [4, p.44]. Let $\alpha(r)$ be a function defined for $r > 0$, which is non-negative and non-decreasing for sufficiently large r (if $\alpha(r)$ satisfies this condition, we write $\alpha(r) \in \Lambda$).

The number $\rho = \rho[\alpha] = \overline{\lim\limits_{r\to+\infty}} \frac{\ln^+ \alpha(r)}{\ln r}$ is called the order of $\alpha(r)$. The number $\sigma = \sigma[\alpha] = \overline{\lim\limits_{r\to+\infty}} \frac{\alpha(r)}{r^\rho}$ is called the magnitude of type of the function $\alpha(r)$. If $\sigma = 0$, we say that $\alpha(r)$ has minimal type; if $0 < \sigma < \infty$, we say that $\alpha(r)$ has normal (or mean) type; if $\sigma = \infty$, we say that $\alpha(r)$ has maximal type.

Let $\alpha(r)$ be a function of finite order ρ. We say that $\alpha(r)$ belongs to the convergence class or to the divergence class depending on whether the integral $\int_1^\infty \frac{\alpha(r)}{r^{\rho+1}} dr$ converges or diverges.

We say that functions $\alpha_1(r)$, $\alpha_2(r) \in \Lambda$ are of the same growth category if they have the same order, and, if the order is finite, have the same type and either both belong to the convergence class, or both belong to the divergence class. We say that $\alpha_2(r)$ is of higher growth category than $\alpha_1(r)$ if one of the following conditions is satisfied:

1. $\rho[\alpha_2] > \rho[\alpha_1]$.
2. $\rho[\alpha_1] = \rho[\alpha_2] < \infty$, $\alpha_1(r)$ is of minimal type, and $\alpha_2(r)$ is of normal or maximal type.
3. $\rho[\alpha_1] = \rho[\alpha_2] < \infty$, $\alpha_1(r)$ is of normal type, and $\alpha_2(r)$ is of maximal type.
4. $\rho[\alpha_1] = \rho[\alpha_2] < \infty$, $\alpha_1(r)$ and $\alpha_2(r)$ are of minimal type, $\alpha_1(r)$ belongs to the convergence class, and $\alpha_2(r)$ belongs to the divergence class.

By analogy with the definition of Picard exceptional value of a meromorphic function (see [4, p. 49]) a vector $\vec{a} \in \mathbb{C}^p \setminus \{\vec{0}\}$ is called a *Picard exceptional vector of an entire curve* $\vec{G}\colon \mathbb{C} \to \mathbb{C}^p$, if the function $\vec{G}(z)\,\vec{a}$ has a finite number of zeros.

Let us remind a definition of a Borel exceptional vector of entire curve from [1]. A vector $\vec{a} \in \mathbb{C}^p \setminus \{\vec{0}\}$ is called a *Borel exceptional vector of entire curve* $\vec{G}\colon \mathbb{C} \to \mathbb{C}^p$, if the growth category of $N(r, \vec{a}, \vec{G})$ is lower than the growth category of $T(r, \vec{G})$.

Other kinds of exceptional and deficient vectors for entire curves were considered in [11,12] (Valiron deficient vectors), [8, 9] (Nevanlinna deficient vectors), [5] (averaged deficiency). Moreover, there is a recent paper [2] on Picard values of p-adic meromorphic functions, with investigations of Picard-Hayman behavior of derivatives of meromorphic functions on an algebraically closed field K, complete with respect to a non-trivial ultrametric absolute value. More modern bibliography on this topic is listed in a review paper of S. Mori [6].

The set of Picard exceptional vectors for an entire curve $\vec{G}$ we denote by $\mathbf{P}(\vec{G})$. Obviously $\mathbf{P}(\vec{G}) \subset \mathbf{B}(\vec{G})$ for any transcendetal entire curve $\vec{G}$, where $\mathbf{B}(\vec{G})$ is the set of Borel exceptional vectors of the entire curve $\vec{G}$ (see [1]). Therefore, Theorems 1 and 3 from [1] are valid for Picard exceptional vectors.

In [1] we described the structure of the set of Borel exceptional vectors for a transcendental entire curve $\vec{G}$. The following results were proved

Theorem 1 ([1]). *For any transcendental entire curve $\vec{G}\colon \mathbb{C} \to \mathbb{C}^p$ with linearly independent components and without common zeros any admissible system of Borel exceptional vectors cannot have more than p vectors.*

Theorem 2 ([1]). *For any transcendental entire curve $\vec{G}\colon \mathbb{C} \to \mathbb{C}^p$ with linearly independent components and without common zeros the set $\mathbf{B}(\vec{G}) \cup \{\vec{0}\}$ is a finite union of subspaces $A_j \subset \mathbb{C}^p$ of dimension not greater than $p-1$. There exist at most p linearly independent vectors such that every A_j is spanned by some of these vectors.*

Theorem 3 ([1]). *Any transcendental entire curve $\vec{G}\colon \mathbb{C} \to \mathbb{C}^p$ of non-integer or zero order with linearly independent components and without common zeros has at most $(p-1)$ linearly independent Borel exceptional vectors.*

Now we improve these results.

Theorem 4. *For any transcendental entire curve $\vec{G}\colon \mathbb{C} \to \mathbb{C}^p$ with linearly independent components and without common zeros the set $\mathbf{B}(\vec{G}) \cup \{\vec{0}\}$ is a finite union of subspaces $A_1, A_2, \ldots, A_m$ of $\mathbb{C}^p$ of dimension at most $p-1$. Moreover, $\dim A_1 + \dim A_2 + \ldots + \dim A_m \le p$ and $A_i \cap A_j = \{\vec{0}\}$ for any $i, j = \overline{1, m}$, $i \ne j$.*

The theorem easily follows from the next lemma.

Lemma 1. *Let $\vec{G}\colon \mathbb{C} \to \mathbb{C}^p$ be a transcendental entire curve with linearly independent components and without common zeros, B_1 and B_2 be subspaces in $\mathbb{C}^p$ such that $B_1 \subset \mathbf{B}(\vec{G}) \cup \{\vec{0}\}$ and $B_2 \subset \mathbf{B}(\vec{G}) \cup \{\vec{0}\}$. If $B_1 \cap B_2$ contains a non-zero vector. Then the linear span B of B_1 and B_2 is contained in $\mathbf{B}(G) \cup \{0\}$.*

Proof. Let us consider the case when $B_1 \not\subset B_2$ and $B_2 \not\subset B_1$, otherwise the statement is obvious. Clearly, $B_1 \cap B_2$ is a subspace in $\mathbb{C}^p$ of dimension at least 1. Denote $\dim (B_1 \cap B_2) = q_0$, $\dim B_1 = q_1$, $\dim B_2 = q_2$, $\dim B = q$. Obviously, that $q_1 + q_2 - q_0 = q$.

Let $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_{q_0}$ be a basis in $B_1 \cap B_2$. We supplement these vectors in the subspace B_1 by the vectors $\vec{b}^{(1)}_{q_0+1}, \vec{b}^{(1)}_{q_0+2}, \ldots, \vec{b}^{(1)}_{q_1}$ such that the set $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_{q_0}, \vec{b}^{(1)}_{q_0+1}, \vec{b}^{(1)}_{q_0+2}, \ldots, \vec{b}^{(1)}_{q_1}$ is a basis in B_1. Also we supplement the vectors $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_{q_0}$ in the subspace B_2 by the vectors $\vec{b}^{(2)}_{q_0+1}, \vec{b}^{(2)}_{q_0+2}, \ldots, \vec{b}^{(2)}_{q_2}$ such that a set $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_{q_0}, \vec{b}^{(2)}_{q_0+1}, \vec{b}^{(2)}_{q_0+2}, \ldots, \vec{b}^{(2)}_{q_2}$ is a basis in B_2. Clearly, $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_{q_0}, \vec{b}^{(1)}_{q_0+1}, \vec{b}^{(1)}_{q_0+2}, \ldots, \vec{b}^{(1)}_{q_1}, \vec{b}^{(2)}_{q_0+1}, \vec{b}^{(2)}_{q_0+2}, \ldots, \vec{b}^{(2)}_{q_2}$ is a basis in B. Let us construct a vector-valued function

$$\vec{\vec{G}}(z) = \left(\vec{G}(z)\vec{b}_1, \ldots, \vec{G}(z)\vec{b}_{q_0}, \vec{G}(z)\vec{b}^{(1)}_{q_0+1}, \ldots, \vec{G}(z)\vec{b}^{(1)}_{q_1}, \vec{G}(z)\vec{b}^{(2)}_{q_0+1}, \ldots, \vec{G}(z)\vec{b}^{(2)}_{q_2}\right) \cdot \Phi(z),$$

where $\Phi(z)$ is a some meromorphic function in $\mathbb{C}$ without zeros and for which its poles are common zeros of the functions

$$\vec{G}(z)\vec{b}_1, \ldots, \vec{G}(z)\vec{b}_{q_0}, \vec{G}(z)\vec{b}^{(1)}_{q_0+1}, \ldots, \vec{G}(z)\vec{b}^{(1)}_{q_1}, \vec{G}(z)\vec{b}^{(2)}_{q_0+1}, \ldots, \vec{G}(z)\vec{b}^{(2)}_{q_2}.$$

Then the function $\vec{\vec{G}}(z)$ is q-dimensional entire curve.

Let us consider a q_1-dimensional entire curve

$$\vec{G}_1(z) = \left(\vec{G}(z)\vec{b}_1, \ldots, \vec{G}(z)\vec{b}_{q_0}, \vec{G}(z)\vec{b}^{(1)}_{q_0+1}, \ldots, \vec{G}(z)\vec{b}^{(1)}_{q_1}\right) \cdot \Phi_1(z),$$

where $\Phi_1(z)$ is a meromorphic function in $\mathbb{C}$ without zeros and for which its poles are common zeros of the functions

$$\vec{G}(z)\vec{b}_1, \ldots, \vec{G}(z)\vec{b}_{q_0}, \vec{G}(z)\vec{b}^{(1)}_{q_0+1}, \ldots, \vec{G}(z)\vec{b}^{(1)}_{q_1}.$$

For any vector $\vec{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{q_1}) \in \mathbb{C}^{q_1} \setminus \{\vec{0}\}$ and the vector $\vec{b} = \lambda_1\vec{b}_1 + \ldots + \lambda_{q_1}\vec{b}_{q_1} \in B_1 \subset \mathbf{B}(\vec{G}) \cup \{\vec{0}\}$ corresponding to it one has $\vec{G}(z)\vec{b} = \vec{G}_1(z)\vec{\lambda}/\Phi_1(z)$. Therefore,

$$N\left(r, \vec{b}, \vec{G}\right) = N\left(r, \vec{\lambda}, \vec{G}_1\right) + N(r, \Phi_1).$$

Thus, the growth category of $N(r, \vec{\lambda}, \vec{G}_1)$ is lower that the growth category of $T(r, \vec{G})$. Therefore, the growth category of $T(r, \vec{G}_1)$ is also lower than that of $T(r, \vec{G})$. Hence, (see [7, p.7]) all functions

$$T\left(r, \frac{\vec{G}(z)\vec{b}_2}{\vec{G}(z)\vec{b}_1}\right), T\left(r, \frac{\vec{G}(z)\vec{b}_3}{\vec{G}(z)\vec{b}_1}\right), \ldots, T\left(r, \frac{\vec{G}(z)\vec{b}_{q_0}}{\vec{G}(z)\vec{b}_1}\right),$$
$$T\left(r, \frac{\vec{G}(z)\vec{b}^{(1)}_{q_0+1}}{\vec{G}(z)\vec{b}_1}\right), \ldots, T\left(r, \frac{\vec{G}(z)\vec{b}^{(1)}_{q_1}}{\vec{G}(z)\vec{b}_1}\right)$$

also have lower growth category than the growth category of $T\left(r, \vec{G}\right)$.

Let us denote by $\Phi_2(z)$ a meromorphic function in $\mathbb{C}$ without zeros and for which its poles are common zeros of the functions $\vec{G}(z)\vec{b}_1, \ldots, \vec{G}(z)\vec{b}_{q_0}, \vec{G}(z)\vec{b}^{(2)}_{q_0+1}, \ldots, \vec{G}(z)\vec{b}^{(2)}_{q_2}$. Similarly we obtain that the q_2-dimensional entire curve

$$\vec{G}_2(z) = \left(\vec{G}(z)\vec{b}_1, \ldots, \vec{G}(z)\vec{b}_{q_0}, \vec{G}(z)\vec{b}^{(2)}_{q_0+1}, \ldots, \vec{G}(z)\vec{b}^{(2)}_{q_2}\right) \cdot \Phi_2(z)$$

has lower growth category than the growth category of $T(r, \vec{G})$. Therefore, the functions $T\left(r, \frac{\vec{G}(z)\vec{b}^{(2)}_{q_0+1}}{\vec{G}(z)\vec{b}_1}\right), \ldots, T\left(r, \frac{\vec{G}(z)\vec{b}^{(2)}_{q_2}}{\vec{G}(z)\vec{b}_1}\right)$ have lower growth category than that of $T(r, \vec{G})$.

Thus, we show that the quotient of every component of the entire curve $\vec{\tilde{G}}(z)$ by $\vec{G}(z)\vec{b}_1$ has lower growth category than the lower growth category of $T(r,\vec{G})$. Therefore, (see proof of Theorem 3 in [1]) the growth category of $T(r,\vec{\tilde{G}})$ is lower than that of $T(r,\vec{G})$. Then $B \subset \mathbf{B}(\vec{G}) \cup \{\vec{0}\}$ (see the considerations in the proof of lemma in [1]). Obviously, $B_1 \subset B$ and $B_2 \subset B$. □

Let L be some q-dimensional subspace in $\mathbb{C}^p$. We call [8] the system of vectors M of L *admissible* in L, if for $card\, M \leq q$ all vectors of M are linearly independent and if for $card\, M > q$ any q vectors of M are linearly independent.

Let a set S lay in a q-dimensional subspace of the space $\mathbb{C}^p$ and contain q linearly independent vectors. We call [8] the subset $M \subset S$ *maximally admissible* in S, if: a) any q vectors of M are linearly independent; b) any vector of $S \setminus M$ is a linear combination of some $q-1$ vectors of M.

We need the following lemma from [8].

Lemma 2 ([8], Lemma 1)**.** *For any set $S \subset \mathbb{C}^p$ there exists a maximally admissible subset M of S.*

Proof of Theorem 4. In view of Lemma 2, we choose a maximally admissible in $\mathbb{C}^p$ system of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$ from $\mathbf{B}(\vec{G})$. By Theorem 2 the set $\mathbf{B}(\vec{G}) \cup \{\vec{0}\}$ is a finite union of the subspaces $A_j \subset \mathbb{C}^p$ of dimension $\leq p-1$. Every such a subspace has a basis generated by some vectors from $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$. Obviously, the set $\bigcup_j A_j$ is not changed if we remove all sets A_j, for which there exist $A_s \neq A_j$ such that $A_j \subset A_s$. After this removal and relettering we obtain $\mathbf{B}(\vec{G}) \cup \{\vec{0}\} = \bigcup_j A_j$, where $A_j \not\subset A_s$, if $j \neq s$. Then $A_j \cap A_s = \{\vec{0}\}$ for $j \neq s$, otherwise by Lemma 1 there exists a subspace $A \subset \mathbf{B}(\vec{G}) \cup \{\vec{0}\}$ such that $A_j \subset A$, $A_s \subset A$, $A_j \neq A$ and $A_s \neq A$. By Theorem 1 one has $k \leq p$. Clearly, every vector $\vec{a}_i$ belongs only to one subspace from A_j. Hence, $\dim A_1 + \dim A_2 + \ldots + \dim A_m \leq p$. □

Note that the structure of the set of Nevanlinna deficient vectors is similar to the structure $\mathbf{B}(\vec{G})$ (see also [8], [9]), though the condition $A_j \cap A_s = \{\vec{0}\}$ for $j \neq s$ does not hold in general.

An analog of Theorem 4 can be proved for Picard exceptional vectors.

Theorem 5. *For any transcendental entire curve $\vec{G}\colon \mathbb{C} \to \mathbb{C}^p$ with linearly independent components and without common zeros the set $\mathbf{P}(\vec{G}) \cup \{\vec{0}\}$ is a finite union of subspaces $A_1, A_2, \ldots, A_m$ of dimension $\leq p-1$ from $\mathbb{C}^p$, where $\dim A_1 + \dim A_2 + \ldots + \dim A_m \leq p$ and $A_i \cap A_j = \{\vec{0}\}$ for all $i, j \in \{1, \ldots, m\}$, $i \neq j$.*

To prove this theorem, it suffices to repeat arguments similar to the arguments in the proof of Lemma 1 and Theorem 2 in [1], and also in the proof of Theorem 4 in the present paper. We formulate and prove, for example, a lemma which is similar to Lemma 1 from [1].

Lemma 3. *Let $\vec{G}\colon \mathbb{C} \to \mathbb{C}^p$ be a transcendental entire curve with linearly independent components and without common zeros, $\vec{b}_1$, $\vec{b}_2, \ldots, \vec{b}_q$ be a system of linearly independent vectors of $\mathbf{P}(\vec{G})$, B be a linear span of vectors $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_q$ and $B_1, B_2, \ldots, B_q$ be linear spans of vector systems $\vec{b}_2, \vec{b}_3, \ldots, \vec{b}_q$, $\vec{b}_1, \vec{b}_3, \vec{b}_4, \ldots, \vec{b}_q$, ... , $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_{q-1}$. Then one has one of the following cases:*

1. *$B \subset \mathbf{P}(\vec{G}) \cup \{\vec{0}\}$;*
2. *$B \cap \mathbf{P}(\vec{G}) \subset \bigcup_{j=1}^{q} B_j$.*

Proof. Let us consider a vector-valued function in $\mathbb{C}^q$

$$\vec{G}_1(z) = (\vec{G}(z)\vec{b}_1\,,\ \vec{G}(z)\vec{b}_2\,,\ \ldots\,, \vec{G}(z)\vec{b}_q) \cdot \Phi(z),$$

where $\Phi(z)$ is some meromorphic function in $\mathbb{C}$ without zeros and whose poles are common zeros of these functions $\vec{G}(z)\vec{b}_1\,,\ \vec{G}(z)\vec{b}_2\,,\ \ldots\,, \vec{G}(z)\vec{b}_q$. Since the vectors $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_q$ are linearly independent, the function $\vec{G}_1(z)$ is q-dimensional entire curve.

Clearly, that $n(r, \Phi) = O(1)$, that is $N(r, \Phi) = O(\ln r)$.

Obviously, for any vector $\vec{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_q) \in \mathbb{C}^q \backslash \{\vec{0}\}$ and the vector $\vec{b} = \lambda_1\vec{b}_1 + \ldots + \lambda_q\vec{b}_q$ corresponding to it one has $\vec{G}(z)\vec{b} = \vec{G}_1(z)\vec{\lambda}/\Phi(z)$. It follows that

$$N(r, \vec{b}, \vec{G}) = N(r, \vec{\lambda}, \vec{G}_1) + N(r, \Phi). \tag{1}$$

Obviously, two cases are possible

1. $T(r, \vec{G}_1) = O(\ln r)$;
2. $\vec{G}_1(z)$ is a transcendental entire curve, i.e. $\ln r = o(T(r, \vec{G}_1))$.

We will consider each case separately.

1) Obviously, $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_q$ is a basis in B. Therefore any vector $\vec{b} \in B\backslash\{\vec{0}\}$ can be represented as $\vec{b} = \lambda_1\vec{b}_1 + \ldots + \lambda_q\vec{b}_q$. We choose $\vec{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_q)$. Then from (1) it follows that $N(r, \vec{b}, \vec{G}) = O(\ln r)$, because $N(r, \vec{\lambda}, \vec{G}_1) \le T(r, \vec{G}_1) + O(1)$. Hence, $B \subset \mathbf{P}(\vec{G}) \cup \{\vec{0}\}$.

2) Suppose that there exists $\vec{b}_0 = \lambda_{01}\vec{b}_1 + \ldots + \lambda_{0q}\vec{b}_q \in B \cap \mathbf{P}(\vec{G})$, $\vec{b}_0 \notin \bigcup_{j=1}^{q} B_j$. it is obvious that the system of vectors $\vec{\lambda}^{(0)} = (\bar{\lambda}_{01}, \ldots, \bar{\lambda}_{0q})$, $\vec{\lambda}^{(1)} = (1, 0, \ldots, 0)$, $\vec{\lambda}^{(2)} = (0, 1, 0, \ldots, 0)$, $\ldots$, $\vec{\lambda}^{(q)} = (0, \ldots, 0, 1)$ is admissible in $\mathbb{C}^q$. Taking into account (1), we have $N(r, \vec{b}_j, \vec{G}) = N(r, \vec{\lambda}^{(j)}, \vec{G}_1) + N(r, \Phi)$, $j = 0, 1, \ldots q$. Hence, $N(r, \vec{\lambda}^{(j)}, \vec{G}_1) = O(\ln r)$. Since $\vec{G}_1(z)$ is the transcendental entire curve, all vectors $\vec{\lambda}^{(0)}$, $\vec{\lambda}^{(1)}$,..., $\vec{\lambda}^{(q)}$ are Picard exceptional for q-dimensional entire curve $\vec{G}_1$, but it is impossible. The obtained contradiction proves Lemma 3. □

In view of Theorem 3 from [1], a stronger version of Theorem 5 holds in the case of non-integer or zero order. If a transcendental entire curve has non-integer or zero order then the inequality $\dim A_1 + \dim A_2 + \ldots + \dim A_m \le p$ can be replaced by the inequality $\dim A_1 + \dim A_2 + \ldots + \dim A_m \le p - 1$. Theorem 4 is clarified more significantly.

Theorem 6. *If a transcendental entire curve $\vec{G} \colon \mathbb{C} \to \mathbb{C}^p$ with linearly independent components and without common zeros has non-integer or zero order then $\mathbf{B}(\vec{G}) \cup \{\vec{0}\}$ is a subspace in $\mathbb{C}^p$ of dimension at most $p - 1$.*

Proof. Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$ be a maximally admissible in $\mathbb{C}^p$ system of vectors from $\mathbf{B}(\vec{G})$ and L be the linear span of these vectors. Obviously, L is a subspace of dimension k in $\mathbb{C}^p$ and

$$\mathbf{B}(\vec{G}) \cup \{\vec{0}\} \subset L. \tag{2}$$

By Theorem 3 we have $k \le p - 1$. Let us consider the entire functions $f_1(z) = \vec{G}(z)\vec{a}_1$, $f_2(z) = \vec{G}(z)\vec{a}_2$, $\ldots$, $f_k(z) = \vec{G}(z)\vec{a}_k$. Clearly, the growth category of each function does not exceed that of $T(r, \vec{G})$. We will show that this growth category cannot be the same as the growth category of $T(r, \vec{G})$. Suppose that for some j the function $f_j(z)$ has the same growth category as $T(r, \vec{G})$. Thus, f_j is a transcendental entire function and it has non-integer or zero order. Therefore, the function has no finite Borel exceptional values (see [4, p. 114, Th.1.1]) because any transcendental meromorphic function of non-integer or zero order can have at most one Borel exceptional value.

By hypothesis of the theorem the vector $\vec{a}_j$ is Borel exceptional vector for $\vec{G}(z)$, i.e. $N(r, \vec{a}_j, \vec{G}) = N(r, 0, f_j)$ has lower growth category than that of $T(r, \vec{G})$, and of $f_j(z)$. It means that the number 0 is a Borel exceptional value for the function $f_j(z)$, but it is impossible. Thus, all functions $f_1(z)$, $f_2(z)$, ..., $f_k(z)$ have lower growth categories than the growth category of $T(r, \vec{G})$.

Let us consider an arbitrary non-zero vector $\vec{a} \in L$. Obviously, we can represent the vector as $\vec{a} = \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \ldots + \alpha_k\vec{a}_k$. Then the entire function $f(z) = \vec{G}(z)\vec{a} = \alpha_1 f_1(z) + \alpha_2 f_2(z) + \ldots + \alpha_k f_k(z)$ has lower growth category than that of $T(r, \vec{G})$ because it is a linear combination of functions of lower growth category. From the inequality $N(r, \vec{a}, \vec{G}) = N(r, 0, f) \le T(r, f)$ we conclude that the vector $\vec{a}$ is a Borel exceptional for the curve $\vec{G}(z)$.

Thus, in view of (2) we proved that $\mathbf{B}(\vec{G}) \cup \{\vec{0}\} = L$. □

Theorem 6 is not valid for the set $\mathbf{P}(\vec{G}) \cup \{\vec{0}\}$. Below we will demonstrate this fact. Example 1 also shows that Theorem 6 does not hold for functions of integral order.

Example 1. Let us consider an entire curve ($n \in \mathbb{N}$)

$$\vec{G}(z) = (1, z, ..., z^{p_1-1}, e^{z^n}, ze^{z^n}, ..., z^{p_2-1}e^{z^n}, ..., e^{(m-1)z^n}, ze^{(m-1)z^n}, ..., z^{p_m-1}e^{(m-1)z^n}), \quad (3)$$
$$p_1 + p_2 + \ldots + p_m = p, \ m \ge 2.$$

Obviously, the curve $\vec{G}(z)$ has no common zeros, its components are linearly independent and has order n, normal type.

Denote by $\vec{e}_{js}$ a vector from $\mathbb{C}^p$, whose component with number $p_1 + p_2 + \ldots + p_{j-1} + s$ equals 1, and all other components equal zero. Clearly, $\vec{G}(z)\vec{e}_{js} = g_{js}(z)$. Let A_j be the subspace from $\mathbb{C}^p$, which is the linear span of the vectors $\vec{e}_{j1}, \vec{e}_{j2}, \ldots, \vec{e}_{jp_j}$. Obviously, $\dim A_j = p_j$ and

$$\dim A_1 + \dim A_2 + \ldots + \dim A_m = p.$$

For any non-zero vector $\vec{a}_j = \overline{\alpha}_1\vec{e}_{j1} + \overline{\alpha}_2\vec{e}_{j2} + \ldots + \overline{\alpha}_{p_j}\vec{e}_{jp_j} \in A_j$ one has

$$\vec{G}(z)\vec{a}_j = \alpha_1 g_{j1}(z) + \alpha_2 g_{j2}(z) + \ldots + \alpha_{p_j} g_{jp_j} = (\alpha_1 + \alpha_2 z + \ldots + \alpha_{p_j} z^{p_j - 1})e^{jz^n} \not\equiv 0.$$

Hence, $n(r, \vec{a}_j, \vec{G}) \le p_j$, and $\vec{a}_j$ is a Picard and, moreover, a Borel exceptional vector for the considered entire curve.

Thus, we show that

$$\bigcup_{j=1}^{m} A_j \subset \mathbf{P}(\vec{G}) \cup \{\vec{0}\}. \quad (4)$$

Let us consider an arbitrary vector $\vec{a} \in \mathbb{C}^p$, $\vec{a} \notin \bigcup_{j=1}^m A_j$. All collection of vectors $\vec{e}_{js}$ is a basis in $\mathbb{C}^p$. Therefore, $\vec{a}$ can be represented as a linear combination of these vectors. In this linear combination we only leave non-zero summands. We obtain $\vec{a} = \beta_1\vec{e}_{j_1 s_1} + \beta_2\vec{e}_{j_2 s_2} + \ldots + \beta_k\vec{e}_{j_k s_k}$, where all $\beta_l \ne 0$, $k \le p$, $1 \le j_1 \le j_2 \le \ldots \le j_k \le m$, and $j_1 < j_k$, otherwise $\vec{a} \in A_{j_1}$.

Obviously, for linearly independent functions $g_{j_1 s_1}(z), g_{j_2 s_2}(z), \ldots, g_{j_k s_k}(z)$ only 0 can be a common zero. Denote by r its multiplicity. Then the vector-valued function $\vec{G}_k(z) = (g_{j_1 s_1}(z), g_{j_2 s_2}(z), \ldots, g_{j_k s_k}(z))z^{-r}$ is a k-dimensional entire curve without common zeros.

Clearly its order and type do not exceed order and type of the entire curve $\vec{G}(z)$. Also, they are not lower than the order and type of the meromorphic function

$$\frac{g_{j_k s_k}(z)}{g_{j_1 s_1}(z)} = \frac{z^{s_k-1}e^{(j_k-1)z^n}}{z^{s_1-1}e^{(j_1-1)z^n}} = z^{s_k-s_1}e^{(j_k-j_1)z^n}.$$

This meromorphic function is of order n and of normal type. Thus, $\vec{G}_k(z)$ is an entire curve of order n and of normal type.

The vectors $\vec{e}_1 = (1,0,\dots,0)$, $\vec{e}_2 = (0,1,0,\dots,0)$, $\dots$, $\vec{e}_k = (0,0,\dots,0,1)$ and $\vec{b} = (\beta_1,\beta_2,\dots,\beta_k)$ form an admissible system of vectors in $\mathbb{C}^k$, because all $\beta_l \neq 0$. Clearly every vector $\vec{e}_l$ is the Picard and, moreover, Borel exceptional vector for $\vec{G}_k(z)$, because $\vec{G}_k(z)\vec{e}_l = z^{-r}g_{j_l s_l}(z) = z^{s_l-r-1}e^{(j_l-1)z^n}$. According to Theorem 1 the vector $\vec{b}$ cannot be Borel exceptional for $\vec{G}_k$. Thus, $N(r,\vec{b},\vec{G}_k)$ has the same growth category as $T(r,\vec{G}_k)$, i.e. it has order n of normal type. Since $\vec{G}_k(z)\vec{b} = z^{-r}\vec{G}(z)\vec{a}$ we deduce that the vector $\vec{a} \notin \bigcup_{j=1}^m A_j$ cannot be Borel and, moreover, Picard exceptional for $\vec{G}$.

Taking into account (4), we have proved that for the entire curve of form (3) the following equality holds

$$\mathbf{P}(\vec{G}) \cup \{\vec{0}\} = \mathbf{B}(\vec{G}) \cup \{\vec{0}\} = \bigcup_{j=1}^{m} A_j.$$

Example 2. Let us consider an entire curve $\vec{G}_q\colon \mathbb{C} \to \mathbb{C}^q$, $q > p$, in which the first p components are same as in the entire curve of form (3), and the next $q-p = p_{m+1}$ components have the form:

$$g_{m+1,s}(z) = z^{s-1}\varphi(z), s = 1,2,\dots,p_{m+1},$$

where $\varphi(z)$ is an entire function of non-integer order $\rho > n$. Thus,

$$\vec{G}_q(z) = (1, z, \dots, z^{p_1-1}, \dots, e^{(m-1)z^n}, ze^{(m-1)z^n}, \dots, z^{p_m-1}e^{(m-1)z^n}, \varphi(z), z\varphi(z), \dots, z^{p_{m+1}-1}\varphi(z)), \tag{5}$$

Let us consider the vectors

$$\vec{e}_{js}, j = 1,2,\dots,m+1, s = 1,2,\dots,p_j. \tag{6}$$

Let A_j be the linear span of the vectors $\vec{e}_{j1}, \vec{e}_{j2}, \dots, \vec{e}_{jp_j}$.

It is easy to check that the entire curve has order ρ and the same growth category as $\varphi(z)$. Any linear combination of the first p components of this curve is a polynomial or an entire function of order n. Hence, the combination has lower growth category than the growth category $\vec{G}_q$. Therefore, $A \subset \mathbf{B}(\vec{G}_q) \cup \{\vec{0}\}$, where A is a linear combination of the vectors $\vec{e}_{js}$, $j = 1,2,\dots,m$, $s = 1,2,\dots,p_j$. Arbitrary vector $\vec{a} \in \mathbb{C}^q \backslash A$ can be expanded by orthonormal basis (6): $\vec{a} = \sum_{j=1}^{m+1}\sum_{s=1}^{p_j} \overline{\alpha}_{js}\vec{e}_{js}$. In this expansion one of the coefficients $\overline{\alpha}_{m+1,1}$, $\overline{\alpha}_{m+1,2}$, $\dots$, $\overline{\alpha}_{m+1,p_{m+1}}$ must be non-zero, otherwise $\vec{a} \in A$. Then

$$\vec{G}_q(z)\vec{a} = \sum_{j=1}^{m} e^{(j-1)z^n} \sum_{s=1}^{p_j} \alpha_{js}z^{s-1} + \varphi(z) \sum_{s=1}^{p_{m+1}} \alpha_{m+1,s}z^{s-1} = h(z)$$

is an entire function of order ρ. The function has the same growth category as $\vec{G}_q$, because $\sum_{s=1}^{p_{m+1}} \alpha_{m+1,s}z^{s-1} \not\equiv 0$ is a polynomial. We have mentioned above that an entire function of non-integer order cannot have finite Borel exceptional values. Therefore, $N(r, \vec{a}, \vec{G}_q) = N(r, 0, h)$ has the same growth category as h and $\vec{G}_q$. Hence, $\vec{a} \notin \mathbf{B}(\vec{G}_q)$.

We proved that $\mathbf{B}(\vec{G}_q) \cup \{\vec{0}\} = A$. Using arguments from Example 1, it is easy to check that $\mathbf{P}(\vec{G}_q) \cup \{\vec{0}\} = \bigcup_{j=1}^{m} A_j$.

Examples 1 and 2 confirm the sufficiency of Theorems 4-6 in some sense.

Acknowledgments. The research of the second author was funded by the National Research Foundation of Ukraine, 2020.02/0025, 0120U103996.

The authors cordially thank the Reviewer for all valuable remarks and propositions improving the text.

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Received 12.01.2021
Revised 12.04.2021