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NOTE ON COMPOSITION OF ENTIRE FUNCTIONS AND BOUNDED L-INDEX IN DIRECTION

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We study the following question: "Let $f\colon \mathbb{C}\to\mathbb{C}$ be an entire function of bounded l-index, $\Phi\colon \mathbb{C}^n\to\mathbb{C}$ an be entire function, $n\geq 2$, $l\colon \mathbb{C}\to\mathbb{R}_+$ be a continuous function. What is a positive continuous function $L\colon \mathbb{C}^n\to\mathbb{R}_+$ and a direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ such that the composite function $f(\Phi(z))$ has bounded L-index in the direction $\mathbf{b}$?" In the present paper, early known result on boundedness of L-index in direction for the composition of entire functions $f(\Phi(z))$ is modified. We remove a condition that a directional derivative of the inner function Φ in a direction $\mathbf{b}$ does not equal zero. We relax also the condition $|\partial_{\mathbf{b}}^k\Phi(z)|\leq K|\partial_{\mathbf{b}}\Phi(z)|^k$ for all $z\in\mathbb{C}^n$, where $K\geq 1$ is a constant and

$$\partial_{\mathbf{b}}F(z):=\sum_{j=1}^n\frac{\partial F(z)}{\partial z_j}b_j,\ \partial_{\mathbf{b}}^kF(z):=\partial_{\mathbf{b}}\big(\partial_{\mathbf{b}}^{k-1}F(z)\big).$$

It is replaced by the condition $|\partial_{\mathbf{b}}^k\Phi(z)|\leq K(l(\Phi(z)))^{1/(N(f,l)+1)}|\partial_{\mathbf{b}}\Phi(z)|^k$, where $N(f,l)$ is the l-index of the function f. Under these conditions, the entire function $f(\Phi(z))$ has bounded L-index in the direction $\mathbf{b}$ with the function $L(z)=\max\{1,|\partial_{\mathbf{b}}\Phi(z)|\}\,l(\Phi(z))$ satisfying a additional condition. The described result is an improvement of the previous one.

1. Introduction. The present paper is devoted to compositions of entire functions and theory of entire functions of bounded L-index in direction. We need some notations and definitions. Let $L\colon \mathbb{C}^n\to\mathbb{R}_+$ be any fixed continuous function. An entire function $F(z)$, $z\in\mathbb{C}^n$, is called [1,7,8] a *function of bounded L-index in a direction* $\mathbf{b}=(b_1,\dots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, if there exists $m_0\in\mathbb{Z}_+$ such that for every $m\in\mathbb{Z}_+$ and every $z\in\mathbb{C}^n$

$$\frac{|\partial_{\mathbf{b}}^mF(z)|}{m!L^m(z)}\leq\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{k!L^k(z)}\colon 0\leq k\leq m_0\right\}, \tag{1}$$

where

$$\partial_{\mathbf{b}}^0F(z):=F(z),\ \partial_{\mathbf{b}}F(z):=\sum_{j=1}^n\frac{\partial F(z)}{\partial z_j}b_j=\langle\mathbf{grad}F,\overline{\mathbf{b}}\rangle,\ \partial_{\mathbf{b}}^kF(z):=\partial_{\mathbf{b}}\big(\partial_{\mathbf{b}}^{k-1}F(z)\big),\ k\geq 2.$$

The least such integer m_0 is called the L-index in the direction $\mathbf{b}$ and is denoted by $N_{\mathbf{b}}(F,L)$. In the case $n=1$ and $\mathbf{b}=1$ we obtain the definition of an entire function of one variable of

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bounded l-index (see [11,14]) and $N(F,L) := N_1(F,L)$; in the case $n=1$, $\mathbf{b}=1$ and $L(z)\equiv 1$ it is reduced to the definition of a function of bounded index, supposed by B. Lepson [12].

A detailed review of papers on compositions of functions and boundedness of index is presented in [2]. There is considered the following question: *Let $f\colon \mathbb{C}\to\mathbb{C}$ be an entire function of bounded l-index, $\Phi\colon \mathbb{C}^n\to\mathbb{C}$ be an entire function, $l\colon \mathbb{C}\to\mathbb{R}_+$ be a continuous function. What are a positive continuous function L and a direction $\mathbf{b}\in\mathbb{C}^n$ such that the composite function $f(\Phi(z))$ has bounded L-index in the direction $\mathbf{b}$?* There is answer to the question as Theorem 1 from [2]. The similar questions are considered for analytic functions in the unit ball [4], for entire functions of bounded $\mathbf{L}$-index in joint variables [3], for entire and analytic functions of bounded l-M-index [5], for analytic functions in $\mathbb{C}\times\mathbb{D}$ [6].

Note that the positivity and continuity of the function L are weak restrictions. Therefore, we impose additional conditions by the function L.

For $\eta>0$, $z\in\mathbb{C}^n$, $\mathbf{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ and a positive continuous function $L\colon\ \mathbb{C}^n\to\mathbb{R}_+$ we define

$$\lambda(\eta)=\sup_{z\in\mathbb{C}^n}\sup_{t}\left\{\frac{L(z+t\mathbf{b})}{L(z)}\colon |t|\le\frac{\eta}{L(z)}\right\}.$$

By $Q^n_{\mathbf{b}}$ we denote the class of functions L such that $\lambda(\eta)$ is finite for any $\eta>0$. We also use notation $Q=Q^1_1$ for the class of positive continuous function $l(z)$, when $z\in\mathbb{C}$, $\mathbf{b}=1$, $n=1$, $L\equiv l$.

To prove the main theorem we need auxiliary proposition.

Lemma 1 ([1,7]). *Let $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ and $L\in Q^n_{\mathbf{b}}$. An entire function $F(z)$ has bounded L-index in the direction $\mathbf{b}$ if and only if there exist numbers $p\in\mathbb{Z}_+$, $R>0$ and $C>0$ such that for each $z\in\mathbb{C}^n$, $|z|\ge R$,*

$$\frac{|\partial^{p+1}_{\mathbf{b}}F(z)|}{L^{p+1}(z)}\le C\max\left\{\frac{|\partial^{k}_{\mathbf{b}}F(z)|}{L^{k}(z)}\colon\ 0\le k\le p\right\}. \tag{2}$$

There was obtained the following result

Proposition 1 ([2]). *Let $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, f be an entire function in $\mathbb{C}$, Φ be an entire function in $\mathbb{C}^n$ such that $\partial_{\mathbf{b}}\Phi(z)\ne 0$ and*

$$\left|\partial^{j}_{\mathbf{b}}\Phi(z)\right|\le K\left|\partial_{\mathbf{b}}\Phi(z)\right|^{j},\quad K\equiv\mathrm{const}>0, \tag{3}$$

for all $z\in\mathbb{C}^n$ and every $j\le p$, where p is defined in (2).

Let $l\in Q$, $l(w)\ge 1$, $w\in\mathbb{C}$ and $L\in Q^n_{\mathbf{b}}$, where $L(z)=\left|\partial_{\mathbf{b}}\Phi(z)\right|l(\Phi(z))$. The entire function f has bounded l-index if and only if $F(z)=f(\Phi(z))$ has bounded L-index in the direction $\mathbf{b}$.

Note that the conditions for every $j\in\{1,\ldots,p\}$ $\partial_{\mathbf{b}}\Phi(z)\ne 0$ and $|\partial^{j}_{\mathbf{b}}\Phi(z)|\le K|\partial_{\mathbf{b}}\Phi(z)|^{j}$, in Proposition 1 are generated by the method of proof. In fact, we can remove it and prove more general proposition with some greater function L.

Our main result is following.

Theorem 1. *Let $l\in Q$, $f\colon\mathbb{C}\to\mathbb{C}$ be an entire function of bounded l-index, $\Phi\colon\mathbb{C}^n\to\mathbb{C}$ be an entire function, $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$. Suppose that $l(w)\ge 1$, $w\in\mathbb{C}$, and $L\in Q^n_{\mathbf{b}}$ with*

$$L(z)=\max\{1,|\partial_{\mathbf{b}}\Phi(z)|\}\,l(\Phi(z)) \tag{4}$$

and for all $z \in \mathbb{C}^n$ *and* $k \in \{1, 2, \dots, N(f, l) + 1\}$ *one has*

$$|\partial_{\mathbf{b}}^k \Phi(z)| \le K(l(\Phi(z)))^{1/(N(f,l)+1)} |\partial_{\mathbf{b}} \Phi(z)|^k, \tag{5}$$

where $K \ge 1$ *is a constant. Then the entire function* $F(z) = f(\Phi(z))$ *has bounded* L*-index in the direction* $\mathbf{b}$.

Theorem 1 is new even in one-dimensional case, i.e. for entire functions of one variable having bounded l-index.

2. Proof of main Theorem. For $n = 1$ Lemma 1 is Sheremeta's result [13]. W. K. Hayman [10] proved Lemma 1 for entire functions of bounded index. Analogs of the Hayman Theorem are also known for other classes of holomorphic functions of bounded index [9, 13, 14].

Proof of Theorem 1. In the proof of Proposition 1 in [2] there was established the following formula by the method of mathematical induction

$$\partial_{\mathbf{b}}^k F(z) = f^{(k)}(\Phi(z)) \left(\partial_{\mathbf{b}} \Phi(z)\right)^k + \sum_{j=1}^{k-1} f^{(j)}(\Phi(z)) Q_{j,k}(z), \tag{6}$$

where

$$Q_{j,k}(z) = \sum_{\substack{n_1+2n_2+\ldots+kn_k=k \\ 0\le n_1\le j-1}} c_{j,k,n_1,\ldots,n_k} \left(\partial_{\mathbf{b}} \Phi(z)\right)^{n_1} \left(\partial_{\mathbf{b}}^2 \Phi(z)\right)^{n_2} \ldots \left(\partial_{\mathbf{b}}^k \Phi(z)\right)^{n_k},$$

and $c_{j,k,n_1,\ldots,n_k}$ are some non-negative integer coefficients. There was also deduced that

$$f^{(k)}(\Phi(z)) = \frac{\partial_{\mathbf{b}}^k F(z)}{\left(\partial_{\mathbf{b}} \Phi(z)\right)^k} + \frac{1}{\left(\partial_{\mathbf{b}} \Phi(z)\right)^{2k}} \sum_{j=1}^{k-1} \partial_{\mathbf{b}}^j \Phi(z) \left(\partial_{\mathbf{b}} \Phi(z)\right)^j Q_{j,k}^*(z), \tag{7}$$

where

$$Q_{j,k}^*(z) = \sum_{m_1+2m_2+\ldots+km_k=2(k-j)} b_{j,k,m_1,\ldots,m_k} \left(\partial_{\mathbf{b}} \Phi(z)\right)^{m_1} \left(\partial_{\mathbf{b}}^2 \Phi(z)\right)^{m_2} \ldots \left(\partial_{\mathbf{b}}^k \Phi(z)\right)^{m_k},$$

and $b_{j,k,m_1,\ldots,m_k}$ are some integer coefficients.

Let f be an entire function of bounded l-index. Denote $L_0(z) = l(\Phi(z)) |\partial_{\mathbf{b}} \Phi(z)|$. Taking into account (6) and (4), for $k = p + 1$ we have

$$\begin{aligned} \frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L_0^{p+1}(z)} &\le \frac{|f^{(p+1)}(\Phi(z))|}{L_0^{p+1}(z)} |\partial_{\mathbf{b}} \Phi(z)|^{p+1} + \sum_{j=1}^{p} \frac{|f^{(j)}(\Phi(z))| |Q_{j,p+1}(z)|}{L_0^{p+1}(z)} \le \\ &\le \frac{|f^{(p+1)}(\Phi(z))|\, |\partial_{\mathbf{b}} \Phi(z)|^{p+1}}{l^{p+1}(\Phi(z))\, |\partial_{\mathbf{b}} \Phi(z)|^{p+1}} + \sum_{j=1}^{p} \frac{|f^{(j)}(\Phi(z))|}{l^j(\Phi(z))} \cdot \frac{|Q_{j,p+1}(z)| l^j(\Phi(z))}{|\partial_{\mathbf{b}} \Phi(z)|^{p+1}\, l^{p+1}(\Phi(z))}. \end{aligned} \tag{8}$$

By Lemma 1 inequality (2) is valid for $n = 1$, $F = f$, $L = l$, $\mathbf{b} = 1$ and $p = N(f, l)$.

$$(\forall \tau \in \mathbb{C}): \quad \frac{|f^{(p+1)}(\tau)|}{l^{p+1}(\tau)} \le C \max\left\{ \frac{|f^{(k)}(\tau)|}{l^k(\tau)} : \ 0 \le k \le p \right\}.$$

Applying to (8) this inequalities with $\tau=\Phi(z)$, we obtain

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)}\le\max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:\ 0\le k\le p\right\}\left(C+\sum_{j=1}^{p}\frac{|Q_{j,p+1}(z)|l^{j-p-1}(\Phi(z))}{|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}\right)\le$$
$$\le\max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:\ 0\le k\le p\right\}\Bigg(C+$$
$$+\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\0\le n_1\le j-1}}c_{j,p+1,n_1,\ldots,n_{p+1}}\frac{|\left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2}\ldots\left(\partial_{\mathbf{b}}^{p+1}\Phi(z)\right)^{n_{p+1}}|}{l^{p+1-j}(\Phi(z))\,|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}\Bigg).\qquad(9)$$

In view of condition (5) inequality (9) yields

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)}\le\max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:0\le k\le p\right\}\times$$
$$\times\left(C+\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\0\le n_1\le j-1}}\frac{c_{j,p+1,n_1,\ldots,n_{p+1}}K^{p+1}l(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}{l^{p+1-j}(\Phi(z))\,|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}\right)\le$$
$$\le\max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:0\le k\le p\right\}\left(C+\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\0\le n_1\le j-1}}\frac{c_{j,p+1,n_1,\ldots,n_{p+1}}K^{p+1}}{l^{p-j}(\Phi(z))}\right).\qquad(10)$$

We will use that $l(\Phi(z))\ge1$. Then from (10) it follows that

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)}\le C_1\max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:0\le k\le p\right\},\qquad(11)$$

where

$$C_1=C+K^{p+1}\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\0\le n_1\le j-1}}c_{j,p+1,n_1,\ldots,n_{p+1}}.$$

Applying equality (7), we can estimate the fraction $\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}$

$$\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}\le\frac{|\partial_{\mathbf{b}}^kF(z)|}{l^k(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^k}+\sum_{j=1}^{k-1}\frac{|\partial_{\mathbf{b}}^jF(z)||Q_{j,k}^*(z)|}{l^k(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2k-j}}\le$$
$$\le\max\left\{\frac{|\partial_{\mathbf{b}}^j\Phi(z)|}{l^j(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j}:1\le j\le k\right\}\left(1+\sum_{j=1}^{k-1}\frac{|Q_{j,k}^*(z)|}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\right)\le$$
$$\le\max\left\{\frac{|\partial_{\mathbf{b}}^j\Phi(z)|}{l^j(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j}:1\le j\le k\right\}\Bigg(1+$$
$$+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\ldots,m_k}|\frac{|(\partial_{\mathbf{b}}\Phi(z))^{m_1}(\partial_{\mathbf{b}}^2\Phi(z))^{m_2}\ldots(\partial_{\mathbf{b}}^k\Phi(z))^{m_k}|}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\Bigg).\qquad(12)$$

From inequalities (5) and $l(w)\ge1$ it follows that $|\partial_{\mathbf{b}}^s\Phi(z)|\le Kl^{s/2}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^s$, because $s/2\ge1/(N(f,l)+1)$ for $s\in\{1,2,\ldots,N(f,l)+1\}$. Applying this inequality to (12), we deduce

$$\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}\leq\max\left\{\frac{|\partial_{\mathbf{b}}^j F(z)|}{l^j(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j}: 1\leq j\leq k\right\}\Bigg(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}\times$$

$$\times|b_{j,k,m_1,\ldots,m_k}|K^{m_1+m_2+\ldots+m_k}\frac{(l(\Phi(z)))^{(m_1+2m_2+\ldots+km_k)/2}|\partial_{\mathbf{b}}\Phi(z)|^{m_1+2m_2+\ldots+km_k}}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\Bigg)\leq$$

$$\leq C_2\max\left\{\frac{|\partial_{\mathbf{b}}^j\Phi(z)|}{l^j(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j}: 1\leq j\leq k\right\},$$

where

$$C=1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\ldots,m_k}|K^{m_1+m_2+\ldots+m_k}.$$

Then from inequality (11) we get

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)}\leq C_1\max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}: 0\leq k\leq p\right\}\leq C_1C_2\max\left\{\frac{|\partial_{\mathbf{b}}^j F(z)|}{L_0^j(z)}: 0\leq j\leq p\right\},\quad (13)$$

$p=N(f,l)$. The last inequality is proved for all z such that $\partial_{\mathbf{b}}\Phi(z)\neq 0$.

Remind that $L(z)=l(\Phi(z))\max\{1,|\partial_{\mathbf{b}}\Phi(z)|\}$. Rewrite inequality (13) in the following form:

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\cdot\frac{L^{p+1}(z)}{L_0^{p+1}(z)}\leq C_1C_2\max\left\{\frac{\left|\partial_{\mathbf{b}}^k F(z)\right|}{L^k(z)}\frac{L^k(z)}{L_0^k(z)}:\ 0\leq k\leq p\right\}.$$

Then

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\leq C_1C_2\frac{L_0^{p+1}(z)}{L^{p+1}(z)}\max\left\{\frac{\left|\partial_{\mathbf{b}}^k F(z)\right|}{L^k(z)}\frac{L^k(z)}{L_0^k(z)}:\ 0\leq k\leq p\right\}\leq$$

$$\leq C_1C_2\frac{L_0^{p+1}(z)}{L^{p+1}(z)}\max\left\{\frac{\left|\partial_{\mathbf{b}}^k F(z)\right|}{L^k(z)}:\ 0\leq k\leq p\right\}\max\left\{\frac{L^k(z)}{L_0^k(z)}:\ 0\leq k\leq p\right\}=$$

$$=C_1C_2\frac{(L_0(z)/L(z))^{p+1}}{\min\limits_{0\leq k\leq p}(L_0(z)/L(z))^k}\max\left\{\frac{\left|\partial_{\mathbf{b}}^k F(z)\right|}{L^k(z)}:\ 0\leq k\leq p\right\}.\quad (14)$$

Let $t_0=t(z)=L_0(z)/L(z)$ and $k_0\leq p$ ($k_0\in\mathbb{Z}_+$) be such that $(t_0)^{k_0}=\min_{0\leq k\leq p}t_0^k$. One should observe that $t_0\in(0,1]$ and $p+1-k_0\geq 1$. Hence, $\frac{t_0^{p+1}}{t_0^{k_0}}=t_0^{p+1-k_0}\leq t_0\leq 1$. Therefore,

$$\frac{(L_0(z)/L(z))^{p+1}}{\min_{0\leq k\leq p}(L_0(z)/L(z))^k}=t_0^{p+1-k_0}\leq t_0\leq 1.$$

Thus, from inequality (14) we get

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\leq C_1C_2\max\left\{\frac{\left|\partial_{\mathbf{b}}^k F(z)\right|}{L^k(z)}:\ 0\leq k\leq p\right\}\quad (15)$$

for all z such that $\partial_{\mathbf{b}}\Phi(z)\neq 0$.

If $\partial_{\mathbf{b}}\Phi(z)=0$ then for any $k\in\{1,\ldots,N(f,\ell)+1\}$ inequality (5) implies $\partial^k_{\mathbf{b}}\Phi(z)=0$. In view of (6) it means that for each $k\in\{1,\ldots,N(f,l)+1\}$ $\partial^k_{\mathbf{b}}F(z)=0$. Thus, for the points z such that $\partial_{\mathbf{b}}\Phi(z)=0$ inequality (15) is also satisfied.

Therefore, by Lemma 1 this inequality means that the function F has bounded L-index in the direction $\mathbf{b}$. □

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