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THE LAGUERRE TRANSFORM OF A CONVOLUTION PRODUCT OF VECTOR-VALUED FUNCTIONS

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The Laguerre transform is applied to the convolution product of functions of a real argument (over the time axis) with values in Hilbert spaces. The main results have been obtained by establishing a relationship between the Laguerre and Laplace transforms over the time variable with respect to the elements of Lebesgue weight spaces. This relationship is built using a special generating function. The obtained dependence makes it possible to extend the known properties of the Laplace transform to the case of the Laguerre transform. In particular, this approach concerns the transform of a convolution of functions.

The Laguerre transform is determined by a system of Laguerre functions, which forms an orthonormal basis in the weighted Lebesgue space. The inverse Laguerre transform is constructed as a Laguerre series. It is proven that the direct and the inverse Laguerre transforms are mutually inverse operators that implement an isomorphism of square-integrable functions and infinite squares-summable sequences.

The concept of a q-convolution in spaces of sequences is introduced as a discrete analogue of the convolution products of functions. Sufficient conditions for the existence of convolutions in the weighted Lebesgue spaces and in the corresponding spaces of sequences are investigated. For this purpose, analogues of Young's inequality for such spaces are proven. The obtained results can be used to construct solutions of evolutionary problems and time-dependent boundary integral equations.

1. Introduction. Convolution products are widely used in representation formulas for solutions of problems for differential equations (see, for example, [4, 21, 24]). Convolutions are also dealt with in integral equations of various kinds. Therefore, the development of efficient approaches to performing operations on the convolution products is an actual problem in applied researches. In particular, this concerns of the convolution products of vector-valued functions of a real variable in the evolution problems.

Recall that the convolution product of integrable scalar functions $f\colon \mathbb{R} \to \mathbb{C}$ and $g\colon \mathbb{R} \to \mathbb{C}$ is defined as follows:

$$u(t) = (f * g)(t) := \int\limits_{\mathbb{R}} f(t-s)g(s)ds, \quad t \in \mathbb{R}. \tag{1}$$

If $f(t) = 0$ and $g(t) = 0$ for $t < 0$, then we have $(f * g)(t) = 0$ for $t \leq 0$ and

$$(f * g)(t) = \int\limits_0^t f(t-s)g(s)ds, \quad t > 0. \tag{2}$$

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In practical computations, especially in the case of vector-valued functions, the integral (2) requires special methods (for comprehensive reviews of this issue, see [9, 17, 21]). One of such approaches is based on integral transforms over the time variable and leads to expressions with algebraic operations on the corresponding images of functions. In particular, in the case of the Laplace transform, we deal with the product of images. However, the use of the Laplace transform is limited due to the computational complexity of the inverse transform. In this respect, the Laguerre transform [11, 12] is seen more constructive, since its inverse transform consists in summing a series with orthonormal functions. As a result, this transform may serve as the foundation of efficient algorithms for calculating convolutions.

The first examples [3, 19] (known to the author), where the Laguerre transform was applied to the convolutions of the functions of several variables, were related to the retarded single layer potentials. In the works [13, 14] this approach was also extended to the retarded double layer potentials, and the convergence of the Laguerre series, related to the Laguerre transform, was investigated in the corresponding spaces of vector-valued functions. We also note here other applications of the Laguerre transform that are not related to the direct use of the convolution product of functions. In particular, it was used to reduce evolution problems to boundary value problems for infinite systems of elliptic equations (for details, see [2, 6, 7, 10, 15, 22] and references therein).

This paper is concerned with new properties of the Laguerre transform. We generalize the Laguerre transform using an orthonormal base composed of the Laguerre functions, and establish a relation between this transform and the Laplace transform. As a result, some known properties of the Laplace transform are extended to the Laguerre transform.

The aim of this paper is to obtain and to prove the representation formula for the coefficients of the Laguerre series, which is an expansion of the convolution product of the vector-valued functions. The definitions of required functional spaces and the integral transforms of their elements are given in Section 2. In Section 3 the relation between the Laguerre transform and the Laplace transform is established. In Section 4 the sufficient conditions for existence of the convolution in the Lebesgue spaces are considered and here our main results are stated and proved about the Laguerre transform of the convolution of the vector-valued functions.

2. Definitions of Laguerre and Laplace transforms of elements of the space $L^2_\alpha(\mathbb{R}_+;X)$. Let X be a complex Hilbert space equipped with the inner product $(\cdot,\cdot)_X$ and the induced norm $\|\cdot\|_X := \sqrt{(\cdot,\cdot)_X}$, and let $\alpha \geq 0$ be an arbitrary fixed number, $\mathbb{R}_+ := (0,+\infty)$.

2.1. Definitions of the space $L^2_\alpha(\mathbb{R}_+;X)$ and Laplace transform of its elements. By $L^2_\alpha(\mathbb{R}_+;X)$ we mean the Hilbert space of measurable functions $f\colon \mathbb{R} \to X$ such that $f(t) = 0$ for $t < 0$ and

$$\int\limits_{\mathbb{R}_+} \|f(t)\|^2_X\, e^{-\alpha t} dt < \infty,$$

with the inner product

$$(f,g)_{L^2_\alpha(\mathbb{R}_+;X)} = \int\limits_{\mathbb{R}_+} \big(f(t), g(t)\big)_X\, e^{-\alpha t} dt, \quad f,g \in L^2_\alpha(\mathbb{R}_+;X), \tag{3}$$

and the induced norm

$$\|f\|_{L^2_\alpha(\mathbb{R}_+;X)} = \sqrt{(f,f)_{L^2_\alpha(\mathbb{R}_+;X)}}, \quad f \in L^2_\alpha(\mathbb{R}_+;X). \tag{4}$$

The *Laplace transform* of $f \in L^2_\alpha(\mathbb{R}_+; X)$ is a function of a complex variable

$$(\mathfrak{L}f)(p) \equiv \widehat{f}(p) := \int\limits_{\mathbb{R}_+} e^{-i\eta t}\big(e^{-\xi t} f(t)\big)dt = \int\limits_{\mathbb{R}_+} e^{-pt} f(t)dt, \quad p = \xi + i\eta \in \mathbb{C}, \ \operatorname{Re} p = \xi \geqslant \alpha/2, \tag{5}$$

obtained as a result of the composition of operations of multiplication of this function by $e^{-\xi \cdot}$ and the Fourier transform (with η variable).

Let $\Pi := \{p \in \mathbb{C} \,|\, \operatorname{Re} p > \alpha/2\}$ be an open complex half-plane and $\overline{\Pi} := \{p \in \mathbb{C} \,|\, \operatorname{Re} p \geq \alpha/2\}$ be the closure of Π. By $\mathcal{L}^2_\alpha(\overline{\Pi}; X) := \mathfrak{L}\big(L^2_\alpha(\mathbb{R}_+; X)\big)$ we denote the image of the Laplace transform of the space $L^2_\alpha(\mathbb{R}_+; X)$. As known, the set $\mathcal{L}^2_\alpha(\overline{\Pi}; X)$ consists of functions $p \to \widehat{f}(p)$, which are holomorphic in Π and are continuous on $\overline{\Pi}$, and, moreover, satisfy some growth conditions at infinity.

For any function $f \in L^2_\alpha(\mathbb{R}_+; X)$ and for each $\xi \geqslant \alpha/2$ the function $\eta \mapsto \widehat{f}(\xi + i\eta)$ belongs to the space $L^2(\mathbb{R}; X)$, so the inverse Fourier transform is available

$$e^{-\xi t} f(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} e^{i\eta t} \Bigg[\int\limits_0^{+\infty} e^{-i\eta s} e^{-\xi s} f(s) ds \Bigg] d\eta, \quad t \in \mathbb{R}_+.$$

Hence

$$f(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} e^{(\xi+i\eta)t} \Bigg[\int\limits_0^{+\infty} e^{-(\xi+i\eta)s} f(s) ds \Bigg] d\eta = \frac{1}{2\pi i} \int\limits_{\xi+i\mathbb{R}} e^{pt}\, \widehat{f}(p)\, dp, \quad t \in \mathbb{R}_+.$$

It follows that the inverse Laplace transform can be defined on the set $\mathcal{L}^2_\alpha(\Pi; X)$:

$$f(t) \equiv (\mathfrak{L}^{-1}\widehat{f})(t) := \frac{1}{2\pi i} \int\limits_{\xi+i\mathbb{R}} e^{pt}\, \widehat{f}(p)\, dp, \quad t \in \mathbb{R}_+, \quad \xi \geqslant \alpha/2. \tag{6}$$

Note that the inverse Laplace transform does not depend on the choice of the value of ξ, which determines the line along which integrate by the formula (6).

It is known (see, for example [8, formula (49)]), that the Parseval-Plancherel equality holds for arbitrary $f, g \in L^2_\alpha(\mathbb{R}_+; X)$:

$$\frac{1}{2\pi} \int\limits_{\gamma/2+i\mathbb{R}} \big(\widehat{f}(p), \widehat{g}(p)\big)_X dp = \int\limits_{\mathbb{R}_+} e^{-\gamma t} \big(f(t), g(t)\big)_X dt, \quad \gamma \geqslant \alpha. \tag{7}$$

In particular, from this we have

$$\frac{1}{2\pi} \int\limits_{\gamma/2+i\mathbb{R}} \|\widehat{f}(p)\|^2_X\, dp = \int\limits_{\mathbb{R}_+} e^{-\gamma t} \|f(t)\|^2_X dt, \quad \gamma \geqslant \alpha. \tag{8}$$

Hence it follows that for each $f, g \in L^2_\alpha(\mathbb{R}_+; X)$ we have

$$(f, g)_{L^2_\alpha(\mathbb{R}_+;X)} = \frac{1}{2\pi} \int\limits_{\alpha/2+i\mathbb{R}} \big(\widehat{f}(p), \widehat{g}(p)\big)_X\, dp, \quad \|f\|_{L^2_\alpha(\mathbb{R}_+;X)} = \Bigg[\frac{1}{2\pi} \int\limits_{\alpha/2+i\mathbb{R}} \|\widehat{f}(p)\|^2_X\, dp\Bigg]^{1/2}.$$

For any $m \in \mathbb{N}$ we define the weighted Sobolev space:

$$H^m_\alpha(\mathbb{R}_+;X) := \{f \in L^2_\alpha(\mathbb{R}_+;X) |\ f^{(k)} \in L^2_\alpha(\mathbb{R}_+;X),\ k = \overline{1,m}\}, \tag{9}$$

with standard norm

$$\|f\|_{H^m_\alpha(\mathbb{R}_+;X)} = \left[\sum_{k=0}^{m} ||f^{(k)}||^2_{L^2_\alpha(\mathbb{R}_+;X)}\right]^{1/2}, \quad f \in H^m_\alpha(\mathbb{R}_+;X). \tag{10}$$

Here and in what follows, $f^{(k)}$ denotes the k-order derivative of the function $f \in L^2_\alpha(\mathbb{R}_+;X)$ in the sense of the space $D'(\mathbb{R};X)$ of X-values distributions over $\mathbb{R}$. It is known, that for any function $f \in H^m_\alpha(\mathbb{R}_+;X)$ and any point $t_0 \in [0,+\infty)$ there exist the traces $f(t_0) \in X, ...,\ f^{(m-1)}(t_0) \in X$, moreover $f(0) = 0, ...,\ f^{(m-1)}(0) = 0$. In the space $H^m_\alpha(\mathbb{R}_+;X)$ we consider, in addition to the standard norm, another one

$$|f|_{\alpha,m,X} = \left[\frac{1}{2\pi}\int\limits_{\alpha/2+i\mathbb{R}} (1+|p|^2)^m \|\widehat{f}(p)\|^2_X\, dp\right]^{1/2}, \quad f \in H^m_\alpha(\mathbb{R}_+;X). \tag{11}$$

Lemma 1. *For any $m \in \mathbb{N}$ the norms $|\cdot|_{\alpha,m,X}$ and $\|\cdot\|_{H^m_\alpha(\mathbb{R}_+;X)}$ are equivalent in the space $H^m_\alpha(\mathbb{R}_+;X)$, that is, the following inequality holds*

$$2^{-m/2}|u|_{\alpha,m,X} \le \|u\|_{H^m_\alpha(\mathbb{R}_+;X)} \le |u|_{\alpha,m,X}, \quad u \in H^m_\alpha(\mathbb{R}_+;X). \tag{12}$$

Proof. Let $k \in \{1,...,m\}$. Taking into account the well-known property of the Laplace transform $\widehat{u^{(k)}}(p) = p^k\widehat{u}(p)$, we can write

$$\|u^{(k)}\|_{L^2_\alpha(\mathbb{R}_+;X)} = \left[\frac{1}{2\pi}\int\limits_{\alpha/2+i\mathbb{R}} \|p^k\widehat{u}(p)\|^2_X\, dp\right]^{1/2} = \left[\frac{1}{2\pi}\int\limits_{\alpha/2+i\mathbb{R}} |p|^{2k}\|\widehat{u}(p)\|^2_X\, dp\right]^{1/2}.$$

This leads to the new representation of the standard norm

$$\|u\|^2_{H^m_\alpha(\mathbb{R}_+;X)} = \frac{1}{2\pi}\int\limits_{\alpha/2+i\mathbb{R}} \left[1+\sum_{k=1}^{m}|p|^{2k}\right]\|\widehat{u}(p)\|^2_X\, dp. \tag{13}$$

Using the binomial formula and the equality $\sum_{j=0}^m C^j_m = 2^m$ it is easy to check that the following inequality holds for an arbitrary $a \ge 0$

$$2^{-m}(1+a)^m \le \sum_{k=0}^{m} a^k \le (1+a)^m. \tag{14}$$

Provided $a = |p|^2$ in (14), we obtain

$$\frac{2^{-m}}{2\pi}\int\limits_{\alpha/2+i\mathbb{R}} (1+|p|^2)^m\|\widehat{u}(p)\|^2_X dp \le \frac{1}{2\pi}\int\limits_{\alpha/2+i\mathbb{R}} \left[\sum_{k=0}^{m}|p|^{2k}\right]\|\widehat{u}(p)\|^2_X dp \le \tag{15}$$
$$\le \frac{1}{2\pi}\int\limits_{\alpha/2+i\mathbb{R}} (1+|p|^2)^m\|\widehat{u}(p)\|^2_X dp,$$

whence the inequality (12) follows directly. □

Lemma 2. *For $f \in L^2_\alpha(\mathbb{R}_+;X)$ and $\delta > 0$ the following inequality holds*

$$\|\widehat{f}(p)\|_X \leq \frac{1}{\sqrt{2\delta}}\|f\|_{L^2_\alpha(\mathbb{R}_+;X)}, \quad \operatorname{Re} p \geq \alpha/2 + \delta. \tag{16}$$

Proof. Let $f \in L^2_\alpha(\mathbb{R}_+;X)$ and $p = \xi + i\eta$, where $\xi, \eta \in \mathbb{R}$, $\xi \geqslant \alpha/2 + \delta$. Using the Cauchy–Bunyakovsky inequality and the equality $|e^{-i\eta}| = 1$ leads to the required estimate:

$$\|\widehat{f}(p)\|_X = \left\| \int\limits_{\mathbb{R}_+} e^{-i\eta t} e^{-\xi t} f(t) dt \right\|_X \leqslant \int\limits_{\mathbb{R}_+} |e^{-i\eta t}|\, e^{-(\xi - \frac{\alpha}{2})t} e^{-\frac{\alpha}{2}t} \|f(t)\|_X \, dt \leqslant$$

$$\leq \left[\int\limits_{\mathbb{R}_+} e^{-2\delta t} dt \right]^{1/2} \left[\int\limits_{\mathbb{R}_+} e^{-\alpha t} \|f(t)\|_X^2 \, dt \right]^{1/2} = \frac{1}{\sqrt{2\delta}} \|f\|_{L^2_\alpha(\mathbb{R}_+;X)}.$$

□

2.2. Vector sequences and their convolutions. Let X, Y, Z be complex Hilbert spaces and let $q\colon X \times Y \to Z$ be a continuous sesquilinear map, namely, there exists constant $K > 0$ such that

$$\|q(u,v)\|_Z \leq K\|u\|_X\|v\|_Y \quad \forall u \in X, \quad \forall v \in Y. \tag{17}$$

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Consider a space of the vector sequences

$$l^2(X) := \left\{ \mathbf{f} = (f_0,\, f_1, ...,\, f_k, ...\,)^\top \in X^{\mathbb{N}_0} \quad | \quad \sum_{k=0}^{\infty} \|f_k\|_X^2 < \infty \right\},$$

with the inner product and the induced norm

$$(\mathbf{f}, \mathbf{g}) = \sum_{k=0}^{\infty} (f_k, g_k)_X, \quad \|\mathbf{f}\|_{l^2(X)} := \left[\sum_{k=0}^{\infty} \|f_k\|_X^2 \right]^{1/2}, \quad \mathbf{f}, \mathbf{g} \in l^2(X).$$

Definition 1 ([16])**.** By a *q-convolution* of the sequences $\mathbf{f} \in X^{\mathbb{N}_0}$ and $\mathbf{g} \in Y^{\mathbb{N}_0}$ we mean a sequence $\mathbf{h} := (h_0, h_1, ..., h_j, ...)^\top \in Z^{\mathbb{N}_0}$, where

$$h_j := \sum_{i=0}^{j} q\left(f_{j-i}, g_i\right) \equiv \sum_{i=0}^{j} q\left(f_i, g_{j-i}\right), \quad j \in \mathbb{N}_0. \tag{18}$$

To shorten notation, the q-convolution of the sequences $\mathbf{f}$ and $\mathbf{g}$ is written in the form

$$\mathbf{h} = \mathbf{f} \underset{q}{\circ} \mathbf{g}.$$

Lemma 3. *Let $\mathbf{f} \in X^{\mathbb{N}_0}$ and $\mathbf{g} \in Y^{\mathbb{N}_0}$. Then following equalities hold*

$$\Theta_n := \sum_{j=0}^{n} \sum_{i=0}^{j} q(f_{j-i}, g_i) = \sum_{j=0}^{n} q\Big(f_{n-j}, \sum_{i=0}^{j} g_i\Big) = \sum_{j=0}^{n} q\Big(\sum_{i=0}^{j} f_i,\ g_{n-j}\Big), \quad n \in \mathbb{N}_0. \tag{19}$$

Proof. We begin by proving the first equality. By making the substitution $m = j - i$ and then reversing the order of summation, one finds that

$$\Theta_n = \sum_{j=0}^{n} \sum_{m=0}^{j} q(f_m, g_{j-m}) = \sum_{m=0}^{n} \sum_{j=m}^{n} q(f_m, g_{j-m}) = \sum_{m=0}^{n} q\biggl(f_m, \sum_{j=m}^{n} g_{j-m}\biggr), \quad n \in \mathbb{N}_0.$$

Sequential substituting $i = j - m$ and $j = n - m$ yields

$$\Theta_n = \sum_{m=0}^{n} q\biggl(f_m, \sum_{i=0}^{n-m} g_i\biggr) = \sum_{j=0}^{n} q\biggl(f_{n-j}, \sum_{i=0}^{j} g_i\biggr), \quad n \in \mathbb{N}_0.$$

The proof of the second equality

$$\Theta_n = \sum_{j=0}^{n} q\biggl(\sum_{i=0}^{j} f_i,\ g_{n-j}\biggr), \quad n \in \mathbb{N}_0,$$

is similar. □

2.3. Definition of Laguerre transform based on Laguerre functions. We recall that Laguerre polynomials can be written in the form [11, 23]

$$L_n(t) = \sum_{k=0}^{n} (-1)^k \frac{n!\, t^k}{(k!)^2 (n-k)!}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}_0. \tag{20}$$

This polynomials form an orthonormal basis in $L^2_1(\mathbb{R}_+;\mathbb{C})$.

Let us define the Laguerre functions through the Laguerre polynomials by the formula

$$l_n(t) = \sqrt{\sigma} L_n(\sigma t)\, e^{-\frac{\beta}{2}t}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}_0, \tag{21}$$

where $\beta > 0$ is an arbitrary constant and $\sigma := \alpha + \beta$.

Proposition 1 ([18])**.** *The system of Laguerre functions forms an orthonormal basis in the space $L^2_\alpha(\mathbb{R}_+;\mathbb{C})$.*

Consider the Laguerre transform of vector-valued functions, based on the Laguerre functions.

Theorem 1. *The following assertions are true:*

1°. *A mapping $\mathcal{L}\colon L^2_\alpha(\mathbb{R}_+;X) \to X^{\mathbb{N}_0}$, that matches any function $f \in L^2_\alpha(\mathbb{R}_+;X)$ to the sequence $\mathbf{f} = (f_0,\ f_1, \ldots,\ f_k, \ldots)^\top \in X^{\mathbb{N}_0}$ by the formula*

$$f_k \equiv \bigl(\mathcal{L}f\bigr)_k := \int\limits_{\mathbb{R}_+} f(t)\, l_k(t)\, e^{-\alpha t} dt, \quad k \in \mathbb{N}_0, \tag{22}$$

is injective and has the space $l^2(X)$ as an image. Moreover,

$$\|f\|_{L^2_\alpha(\mathbb{R}_+;X)} = \|\mathbf{f}\|_{l^2(X)}, \tag{23}$$

namely, the mapping $\mathcal{L}\colon L^2_\alpha(\mathbb{R}_+;X) \to l^2(X)$ is isometric.

2°. *A mapping* $\mathcal{L}^{-1}\colon l^2(X) \to L^2_\alpha(\mathbb{R}_+; X)$, *that matches any sequence* $\mathbf{h} = (h_0, h_1, ..., h_k, ...)^\top \in l^2(X)$ *to the function* $h \in L^2_\alpha(\mathbb{R}_+; X)$ *by the formula*

$$h(t) \equiv (\mathcal{L}^{-1}\mathbf{h})(t) := \sum_{k=0}^{\infty} h_k\, l_k(t), \quad t \in \mathbb{R}_+, \tag{24}$$

is injective and has the space $L^2_\alpha(\mathbb{R}_+; X)$ *as an image. Moreover,*

$$\|\mathbf{h}\|_{l^2(X)} = \|h\|_{L^2_\alpha(\mathbb{R}_+;X)}, \tag{25}$$

namely, the mapping $\mathcal{L}^{-1}\colon L^2_\alpha(\mathbb{R}_+; X) \to l^2(X)$ *is isometric.*

3°. *Let* $f \in L^2_\alpha(\mathbb{R}_+; X)$. *Then*

$$\mathcal{L}^{-1}\mathcal{L}f = f, \tag{26}$$

namely, mappings $\mathcal{L}$ *and* $\mathcal{L}^{-1}$ *are mutually inverse operators.*

Proof. The proof of this theorem completely repeats the path of the proof of Theorem 2 in [13]. In this case the only difference is in the use of an orthonormal basis formed not from the Laguerre polynomials, but from the Laguerre functions. □

Definition 2. The mappings

$$\mathcal{L}\colon L^2_\alpha(\mathbb{R}_+; X) \to l^2(X) \quad \text{and} \quad \mathcal{L}^{-1}\colon l^2(X) \to L^2_\alpha(\mathbb{R}_+; X),$$

referred to in the Theorem 1, we call the *direct* and *inverse Laguerre transform* respectively, and (25) is referred to as the *Parseval equality.*

3. The relationship between the Laguerre and Laplace transforms.

Let $f \in L^2_\alpha(\mathbb{R}_+; X)$ and $\mathbf{f} := \mathcal{L}f$. Then we define a sequence $\mathbf{f}^*$ by the rule

$$f_0^* = \frac{f_0}{\sqrt{\sigma}}, \quad f_n^* = \frac{f_n - f_{n-1}}{\sqrt{\sigma}}, \quad n \in \mathbb{N}. \tag{27}$$

Since $\mathbf{f} \in l^2(X)$, we have $\mathbf{f}^* \in l^2(X)$.

Consider the vector-valued function of a complex variable

$$G_f(z) := \sum_{n=0}^{\infty} f_n^* z^n, \quad z \in \mathbb{C}, \quad |z| < 1. \tag{28}$$

Note that it is holomorphic in the unit disk $\mathbb{D} := \{z\colon |z| < 1\}$, because the sequence $(\|f_0\|_X, \|f_1\|_X, ..., \|f_k\|_X, ...)^\top$ is bounded. The following lemma is a standard exercise.

Lemma 4. *The linear fractional transformation*

$$p = \frac{\alpha + \frac{\beta}{2}(1+z)}{1-z}\,, \tag{29}$$

maps the disk $\mathbb{D}$ *on to the half-plane* $\operatorname{Re} p > \alpha/2$ *conformally.*

Theorem 2. *For any* $f \in L^2_\alpha(\mathbb{R}_+; X)$ *the following equality holds*

$$G_f(z) = \widehat{f}(p), \quad |z| < 1, \quad \operatorname{Re} p > \alpha/2, \tag{30}$$

where variables p *and* z *are related by the formula* (29).

Proof. Here we generalize the method of proof that was used in [11] in the case of simpler basic functions. Consider the function

$$d(z,t) := e^{\zeta(z)t}, \quad |z| < 1,\ t \geq 0, \tag{31}$$

where

$$\zeta(z) := -\frac{\gamma z + \frac{\beta}{2}}{1-z}, \quad \gamma := \alpha + \beta/2. \tag{32}$$

Notice that

$$\operatorname{Re}\zeta(z) = \frac{\alpha(|z|^2 - \operatorname{Re} z) + \frac{\beta}{2}(|z|^2 - 1)}{|z|^2 - 2\operatorname{Re} z + 1}.$$

We first prove that the following equality holds in $L^2_\alpha(\mathbb{R}_+;\mathbb{C})$

$$\sum_{n=0}^{\infty} l_n(\cdot) z^n = \frac{\sqrt{\sigma}}{1-z} d(z,\cdot), \quad |z| < 1. \tag{33}$$

It is easy to compute that

$$2\operatorname{Re}\zeta(z) - \alpha = \frac{(\alpha+\beta)(|z|^2-1)}{|z|^2 - 2\operatorname{Re} z + 1} < 0, \quad |z| < 1.$$

So that, taking into account the equality $|e^z|^2 = e^{2\operatorname{Re} z}$, for any z, $|z| < 1$, we obtain

$$\int_{\mathbb{R}_+} |d(z,t)|^2 \, e^{-\alpha t} dt = \int_{\mathbb{R}_+} e^{(2\operatorname{Re}\zeta(z) - \alpha)t} dt < \infty, \quad |z| < 1. \tag{34}$$

That means $d(z,\cdot) \in L^2_\alpha(\mathbb{R}_+;\mathbb{C})$, therefore the Laguerre transform is applicable to this function. Since $\zeta(z) - \frac{\beta}{2} - \alpha = -\frac{\sigma}{1-z}$, we obtain

$$\big(\mathcal{L} d(z,\cdot)\big)_n := \int_{\mathbb{R}_+} d(z,t)\, l_n(t)\, e^{-\alpha t} dt = \sqrt{\sigma} \int_{\mathbb{R}_+} e^{-\frac{\sigma t}{1-z}} L_n(\sigma t) dt, \quad n \in \mathbb{N}_0.$$

By making the substitution $\tau = \sigma t$ and then using the formula [1, 4.11.25]

$$\int_{\mathbb{R}_+} e^{-b\tau} L_n(\tau) d\tau = \frac{(b-1)^n}{b^{n+1}}, \ \operatorname{Re} b > 0, \quad n \in \mathbb{N}_0,$$

with $b = \frac{1}{1-z}$ (note that $\operatorname{Re} b = \frac{1 - \operatorname{Re} z}{|z|^2 - 2\operatorname{Re} z + 1} > 0$) we deduce that

$$\big(\mathcal{L} d(z,\cdot)\big)_n = \frac{1}{\sqrt{\sigma}} \int_{\mathbb{R}_+} e^{-\frac{\tau}{1-z}} L_n(\tau) d\tau = \frac{1}{\sqrt{\sigma}} \frac{z^n (1-z)^{n+1}}{(1-z)^n} = \frac{1-z}{\sqrt{\sigma}} z^n, \quad |z| < 1, \quad n \in \mathbb{N}_0.$$

So, for an arbitrary point z from the disk $\mathbb{D}$ the expression in the left hand side of equation (33) is the Laguerre expansion of the function $\frac{\sqrt{\sigma}}{1-z} d(z,\cdot)$.

Let G^N_f denote the particular sun of expansion (28) with $N+1$ terms. Taking into account (27) we can write

$$G^N_f(z) = \sum_{n=0}^{N} f^*_n z^n = \frac{1-z}{\sqrt{\sigma}} \sum_{n=0}^{N-1} f_n z^n + \frac{1}{\sqrt{\sigma}} f_N z^N, \quad |z| < 1. \tag{35}$$

Let as substitute the representation (22) of Laguerre coefficients $f_n,\ n\in\{0,1,...,N-1\}$, into (35) and swap the operations of summation and integration:

$$G_f^N(z)=\frac{1-z}{\sqrt{\sigma}}\sum_{n=0}^{N-1}z^n\int\limits_{\mathbb{R}_+}f(t)\,l_n(t)\,e^{-\alpha t}dt+\frac{1}{\sqrt{\sigma}}f_N z^N=$$

$$=\frac{1-z}{\sqrt{\sigma}}\int\limits_{\mathbb{R}_+}f(t)\Bigg[\sum_{n=0}^{N-1}z^n\,l_n(t)\Bigg]e^{-\alpha t}\,dt+\frac{1}{\sqrt{\sigma}}f_N z^N,\quad |z|<1. \tag{36}$$

Let $N\to\infty$. Due to the limit $\|f_N\|_X \underset{N\to\infty}{\to} 0$, using of formula (33) gives

$$G_f^N(z)\underset{N\to\infty}{\to}F(z):=\int\limits_{\mathbb{R}_+}f(t)e^{(\zeta(z)-\alpha)t}\,dt,\quad |z|<1, \tag{37}$$

whence, since $\zeta(z)-\alpha=-\frac{\alpha+\frac{\beta}{2}(1+z)}{1-z}$, the equality (30) follows at once. $\square$

4. Laguerre transform of convolution of vector-valued functions. Consider the convolution of the functions $f\in L^2_\alpha(\mathbb{R}_+;X)$ and $g\in L^2_\alpha(\mathbb{R}_+;Y)$ in the form

$$(f\underset{q}{*}g)(t):=\int\limits_0^t q\big(f(t-s),g(s)\big)\,ds,\quad t\in\mathbb{R}_+. \tag{38}$$

Theorem 3. *Suppose that*

$$f\in L^2_\alpha(\mathbb{R}_+;X),\quad g\in L^2_\alpha(\mathbb{R}_+;Y),\quad \textit{and}\quad f\underset{q}{*}g\in L^2_\alpha(\mathbb{R}_+;Z).$$

Then the following equality holds

$$\mathcal{L}(f\underset{q}{*}g)=\mathbf{f}^*\underset{q}{\circ}\mathbf{g}\ \in l^2(Z), \tag{39}$$

where $\boldsymbol{f}:=\mathcal{L}f$ *and* $\mathbf{g}:=\mathcal{L}g$*, and the sequence* $\boldsymbol{f}^*$ *is created from elements of the sequence* $\boldsymbol{f}$ *by the rule* (27).

Proof. By the formula [5, (2.25)] the following equality holds

$$\widehat{(f\underset{q}{*}g)}(p)=q\big(\widehat{f}(p),\ \widehat{g}(p)\big),\quad \mathrm{Re}\,p>\alpha/2. \tag{40}$$

Hence, using (30), we get

$$G_{f\underset{q}{*}g}(z)=q\big(G_f(z),G_g(z)\big),\quad |z|<1, \tag{41}$$

therefore

$$\sum_{m=0}^{\infty}(f\underset{q}{*}g)^*_m\,z^m=q\Bigg(\sum_{i=0}^{\infty}f_i^*z^i\,,\ \sum_{j=0}^{\infty}g_j^*z^j\Bigg),\quad |z|<1.$$

Let us rewrite the expression in the right-hand side of this equality in the form of a power series and equate the coefficients with equal powers of variable z in the left and right parts of the obtained equality. In result we come to the sequence of such equations

$$(f \underset{q}{*} g)^*_m = \sum_{j=0}^{m} q(f^*_{m-j}, g^*_j), \quad m \in \mathbb{N}_0. \tag{42}$$

From the system of equations (27), rewritten for this convolution, we have the expression

$$(f \underset{q}{*} g)_n = \sqrt{\sigma} \sum_{m=0}^{n} (f \underset{q}{*} g)^*_m, \quad n \in \mathbb{N}_0, \tag{43}$$

which may be transformed by substitution (42) to the following form

$$(f \underset{q}{*} g)_n = \sqrt{\sigma} \sum_{m=0}^{n} \sum_{k=0}^{m} q(f^*_{m-k}, g^*_k), \quad n \in \mathbb{N}_0. \tag{44}$$

Applying the formula (19) to the expression in the right-hand side of (44), it takes the form

$$(f \underset{q}{*} g)_n = \sqrt{\sigma} \sum_{k=0}^{n} q\left(f^*_{n-k}, \sum_{m=0}^{k} g^*_m \right), \quad n \in \mathbb{N}_0. \tag{45}$$

From (27), we deduce

$$g_k = \sqrt{\sigma} \sum_{m=0}^{k} g^*_m, \quad k \in \mathbb{N}_0, \tag{46}$$

therefore reducing the sum in (45) by (46) leads to the representation

$$(f \underset{q}{*} g)_n = \sum_{k=0}^{n} q(f^*_{n-k}, g_k), \quad n \in \mathbb{N}_0, \tag{47}$$

which completes the proof. □

Further we show that the condition $f \underset{q}{*} g \in L^2_\alpha(\mathbb{R}_+; Z)$ in Theorem 3 will be fulfilled if the convolution components have some additional properties, namely, belonging this functions to certain Lebesgue spaces.

For $p \in [1, \infty]$ we will consider Lebesgue spaces $L^p(\mathbb{R}_+; X)$ of measurable functions $v\colon \mathbb{R} \to X$ such that $v(t) = 0$ when $t < 0$ and we assume $\int_{\mathbb{R}_+} \|v(t)\|^p_X dt < \infty$ if $1 \le p < \infty$ and $\|v\|^p_X$ are essentially bounded on $\mathbb{R}$ in the case $p = \infty$. The norms are given by

$$\|v\|_{L^p(\mathbb{R}_+;X)} = \left[\int_{\mathbb{R}_+} \|v(t)\|^p_X \, dt \right]^{1/p} \quad \text{for } p \in [1, \infty), \quad \|v\|_{L^\infty(\mathbb{R}_+;X)} = \operatorname*{ess\,sup}_{t \in \mathbb{R}} \big(\|v(t)\|_X \big).$$

Hence $L^p(\mathbb{R}_+; X)$ is the Banach space for any $p \in [1, \infty]$.

Lemma 5. *Given $p, c, d \in [1, \infty]$ such that*

$$\frac{1}{c} + \frac{1}{d} = 1 + \frac{1}{p}\,. \tag{48}$$

Suppose, that $e^{-\frac{\alpha}{p}\cdot} f \in L^c(\mathbb{R}_+; X)$ and $e^{-\frac{\alpha}{p}\cdot} g \in L^d(\mathbb{R}_+; Y)$. Then $e^{-\frac{\alpha}{p}\cdot}(f \underset{q}{*} g) \in L^p(\mathbb{R}_+; Z)$ and the inequality holds:

$$\|e^{-\frac{\alpha}{p}\cdot}(f \underset{q}{*} g)\|_{L^p(\mathbb{R}_+;Z)} \leq K\, \|e^{-\frac{\alpha}{p}\cdot} f\|_{L^c(\mathbb{R}_+;X)} \cdot \|e^{-\frac{\alpha}{p}\cdot} g\|_{L^d(\mathbb{R}_+;Y)}. \tag{49}$$

Proof. By Young's inequality (see, for example, [20, IX.4]), using (48), we have

$$\big\|\, \big(e^{-\frac{\alpha}{p}\cdot}\, \|f(\cdot)\|_X\,\big) * \big(\, e^{-\frac{\alpha}{p}\cdot}\, \|g(\cdot)\|_Y\,\big)\, \big\|_{L^p(\mathbb{R}_+)} \leq \big\|e^{-\frac{\alpha}{p}\cdot}\, \|f(\cdot)\|_X\big\|_{L^c(\mathbb{R}_+)} \cdot \big\|e^{-\frac{\alpha}{p}\cdot}\, \|g(\cdot)\|_Y\big\|_{L^d(\mathbb{R}_+)}.$$

From here, taking into account following relations

$$\|\, e^{-\frac{\alpha}{p}\cdot}\, (f \underset{q}{*} g)\, \|^p_{L^p(\mathbb{R}_+;Z)} = \int\limits_{\mathbb{R}_+} \Big\| \int\limits_0^t q\big(f(t-s), g(s)\big)\, ds \Big\|^p_Z e^{-\alpha t} dt \leq$$

$$\leq \int\limits_{\mathbb{R}_+} \Big[\int\limits_0^t \big\|q\big(f(t-s), g(s)\big)\big\|_Z ds \Big]^p e^{-\alpha t} dt \leq K^p \int\limits_{\mathbb{R}_+} \Big[\int\limits_0^t \big\|f(t-s)\big\|_X \big\|g(s)\big\|_Y\, ds \Big]^p e^{-\alpha t} dt \leq$$

$$\leq K^p \int\limits_{\mathbb{R}_+} \Big[\int\limits_0^t e^{-\frac{\alpha}{p}(t-s)} \big\|f(t-s)\big\|_X \cdot e^{-\frac{\alpha}{p}s} \big\|g(s)\big\|_Y\, ds \Big]^p dt =$$

$$= K^p\, \big\|\, \big(\, e^{-\frac{\alpha}{p}\cdot}\, \|f(\cdot)\|_X\,\big) * \big(\, e^{-\frac{\alpha}{p}\cdot}\, \|g(\cdot)\|_Y\,\big)\, \big\|^p_{L^p(\mathbb{R}_+;\mathbb{R})} \leq$$

$$\leq K^p\, \big\|\, e^{-\frac{\alpha}{p}\cdot}\, \|f(\cdot)\|_X\, \big\|^p_{L^c(\mathbb{R}_+)} \cdot \big\|\, e^{-\frac{\alpha}{p}\cdot}\, \|g(\cdot)\|_Y\, \big\|^p_{L^d(\mathbb{R}_+)} =$$

$$= K^p\, \big\|\, e^{-\frac{\alpha}{p}\cdot}\, f\, \big\|^p_{L^c(\mathbb{R}_+;X)} \cdot \big\|\, e^{-\frac{\alpha}{p}\cdot}\, g\, \big\|^p_{L^d(\mathbb{R}_+;Y)},$$

we get desired inequality (49), which can be considered as an analogue of Young's inequality in the corresponding Lebesgue spaces. □

Theorem 4. *Suppose $f \in L^2_\alpha(\mathbb{R}_+; X)$ and $g \in L^2_\alpha(\mathbb{R}_+; Y)$ and, moreover, $e^{-\frac{\alpha}{2}\cdot} f \in L^c(\mathbb{R}_+; X)$ and $e^{-\frac{\alpha}{2}\cdot} g \in L^d(\mathbb{R}_+; Y)$ provided (48) holds. Then $f \underset{q}{*} g \in L^2_\alpha(\mathbb{R}_+; Z)$ with*

$$\|f \underset{q}{*} g\|_{L^2_\alpha(\mathbb{R}_+;Z)} \leq K\, \|e^{-\frac{\alpha}{2}\cdot} f\|_{L^c(\mathbb{R}_+;X)} \cdot \|e^{-\frac{\alpha}{2}\cdot} g\|_{L^d(\mathbb{R}_+;Y)} \tag{50}$$

and the assertion of Theorem 3 is true.

Proof. This assertion follows directly from the Lemma 5 provided $p = 2$. □

For $p \in [1, \infty]$ we will consider linear spaces $l^p(X)$ that are composed of sequences $\mathbf{f} = (f_0,\, f_1, \ldots,\, f_n, \ldots)^\top \in X^{\mathbb{N}_0}$ such that $\sum_{n=0}^\infty \|f_n\|^p_X < \infty$ if $p \in [1, \infty)$ and $\sup\limits_{n\in\mathbb{N}_0} \|f_n\|_X < \infty$ in the case $p = \infty$. This spaces are equipped by norms

$$\|\mathbf{f}\|_{l^p(X)} := \Big[\sum_{n=0}^\infty \|f_n\|^p_X \Big]^{1/p} \quad \text{for} \quad p \in [1, \infty), \quad \text{and} \quad \|\mathbf{f}\|_{l^\infty(X)} := \sup_{n\in\mathbb{N}_0} \|f_n\|_X. \tag{51}$$

It easily seen that
$$l^1(X) \subset l^p(X), \quad p \in (1,\infty].$$
Indeed, if $\mathbf{f} \in l^1(X)$, that is $\sum_{n=0}^{\infty} \|f_n\|_X < \infty$, then $\lim_{n\to\infty} \|f_n\|_X = 0$. Hence there exists $n_0 \in \mathbb{N}_0$ such that $\|f_n\|_X \leq 1$ if $n \geq n_0$. But then $\|f_n\|_X^p \leq \|f_n\|_X$ for every $n \geq n_0$, that means the series $\sum_{n=0}^{\infty} \|f_n\|_X^p$ is convergent, if $p \in (1,\infty)$, and $\sup_n \|f_n\|_X < \infty$ in the case $p = \infty$.

Lemma 6. *Suppose that* $\mathbf{f} \in l^2(X)$ *and* $\mathbf{g} \in l^1(Y) \subset l^2(Y)$. *Then* $\mathbf{f} \underset{q}{\circ} \mathbf{g} \in l^2(Z)$ *and*
$$\|\mathbf{f} \underset{q}{\circ} \mathbf{g}\|_{l^2(Z)} \leq K \|\mathbf{f}\|_{l^2(X)} \|\mathbf{g}\|_{l^1(Y)}. \tag{52}$$

Proof. Let $N \in \mathbb{N}$. By (17) and the Cauchy–Bunyakovsky inequality we deduce that
$$\begin{aligned}\sum_{n=0}^{N} \left\|(\mathbf{f} \underset{q}{\circ} \mathbf{g})_n\right\|_Z^2 &= \sum_{n=0}^{N} \left\| \sum_{k=0}^{n} q(f_k, g_{n-k}) \right\|_Z^2 \leq \\ \leq K^2 \sum_{n=0}^{N} \left[\sum_{k=0}^{n} \|f_k\|_X \|g_{n-k}\|_Y \right]^2 &= K^2 \sum_{n=0}^{N} \left[\sum_{k=0}^{n} \|f_k\|_X \|g_{n-k}\|_Y^{1/2} \|g_{n-k}\|_Y^{1/2} \right]^2 \leq \\ \leq K^2 \sum_{n=0}^{N} \left[\sum_{k=0}^{n} \|f_k\|_X^2 \|g_{n-k}\|_Y \right] \left[\sum_{k=0}^{n} \|g_{n-k}\|_Y \right] &\leq K^2 \left[\sum_{k=0}^{\infty} \|g_k\|_Y \right] \sum_{n=0}^{N} \sum_{k=0}^{n} \|f_k\|_X^2 \|g_{n-k}\|_Y.\end{aligned} \tag{53}$$

Changing the summation order gives
$$\sum_{n=0}^{N} \sum_{k=0}^{n} \|f_k\|_X^2 \|g_{n-k}\|_Y = \sum_{k=0}^{N} \|f_k\|_X^2 \sum_{n=k}^{N} \|g_{n-k}\|_Y \leq \left[\sum_{k=0}^{\infty} \|f_k\|_X^2 \right] \left[\sum_{k=0}^{\infty} \|g_k\|_Y \right]. \tag{54}$$

Combining (53) and (54) we obtain
$$\sum_{n=0}^{N} \|(\mathbf{f} \underset{q}{\circ} \mathbf{g})_n\|_Z^2 \leq K^2 \|\mathbf{f}\|_{l^2(X)}^2 \big(\|\mathbf{g}\|_{l^1(Y)}\big)^2,$$

whence (52) directly follows. □

Theorem 5. *Suppose* $\mathbf{f} \in l^2(X)$ *and* $\mathbf{g} \in l^1(Y) \subset l^2(Y)$. *Then*
$$h = f \underset{q}{*} g \in L^2_\alpha(\mathbb{R}_+; Z), \tag{55}$$
where
$$f := \mathcal{L}^{-1}\mathbf{f}, \quad g := \mathcal{L}^{-1}\mathbf{g}, \quad h := \mathcal{L}^{-1}(\mathbf{f}^* \underset{q}{\circ} \mathbf{g}), \tag{56}$$
and $\mathbf{f}^*$ *is obtained from* $\mathbf{f}$ *by the rule* (27) *(by Lemma 6 we have* $\mathbf{f}^* \underset{q}{\circ} \mathbf{g} \in l^2(Z)$*).*

Proof. Given any $m \in \mathbb{N}$ define

$$\mathbf{f}^{\langle m\rangle} := (f_0,\ f_1, ...,\ f_m, 0, ...\,)^\top, \quad \mathbf{g}^{\langle m\rangle} := (g_0,\ g_1, ...,\ g_m, 0, ...\,)^\top.$$

Then

$$\mathbf{f}^{\langle m\rangle,*} = \frac{1}{\sqrt{\sigma}}(f_0,\ f_1 - f_0, ...,\ f_m - f_{m-1}, -f_m, 0, ...\,)^\top.$$

It is easy to check that

$$\mathbf{f}^{\langle m\rangle} \underset{m\to\infty}{\to} \mathbf{f} \text{ in } l^2(X), \quad \mathbf{f}^{\langle m\rangle,*} \underset{m\to\infty}{\to} \mathbf{f}^* \text{ in } l^2(X), \quad \mathbf{g}^{\langle m\rangle} \underset{m\to\infty}{\to} \mathbf{g} \text{ in } l^1(Y) \subset l^2(Y). \tag{57}$$

Let us show that

$$\mathbf{f}^{\langle m\rangle,*} \underset{q}{\circ} \mathbf{g}^{\langle m\rangle} \underset{m\to\infty}{\to} \mathbf{f}^* \underset{q}{\circ} \mathbf{g} \quad \text{in} \quad l^2(Z). \tag{58}$$

Indeed, by Lemma 6 we have

$$\mathbf{f}^* \underset{q}{\circ} \mathbf{g} \in l^2(Z), \quad \mathbf{f}^{\langle m\rangle,*} \underset{q}{\circ} \mathbf{g}^{\langle m\rangle} \in l^2(Z), \quad m \in \mathbb{N}. \tag{59}$$

Then, using (52), we deduce that

$$\begin{aligned}\|\mathbf{f}^* \underset{q}{\circ} \mathbf{g} - \mathbf{f}^{\langle m\rangle,*} \underset{q}{\circ} \mathbf{g}^{\langle m\rangle}\|_{l^2(Z)} &= \|\mathbf{f}^* \underset{q}{\circ} \mathbf{g} - \mathbf{f}^{\langle m\rangle,*} \underset{q}{\circ} \mathbf{g} + \mathbf{f}^{\langle m\rangle,*} \underset{q}{\circ} \mathbf{g} - \mathbf{f}^{\langle m\rangle,*} \underset{q}{\circ} \mathbf{g}^{\langle m\rangle}\|_{l^2(Z)} \le \\ &\le \|(\mathbf{f}^* - \mathbf{f}^{\langle m\rangle,*}) \underset{q}{\circ} \mathbf{g}\|_{l^2(Z)} + \|\mathbf{f}^{\langle m\rangle,*} \underset{q}{\circ} (\mathbf{g} - \mathbf{g}^{\langle m\rangle})\|_{l^2(Z)} \le \\ &\le K\|\mathbf{f}^* - \mathbf{f}^{\langle m\rangle,*}\|_{l^2(X)}\|\mathbf{g}\|_{l^1(Y)} + K\|\mathbf{f}^{\langle m\rangle,*}\|_{l^2(X)}\|\mathbf{g} - \mathbf{g}^{\langle m\rangle}\|_{l^1(Y)},\end{aligned} \tag{60}$$

whence (58) follows due to the limits (57).

Let us denote

$$\begin{gathered} f^{\langle m\rangle} := \mathcal{L}^{-1}(\mathbf{f}^{\langle m\rangle}) \in L^2(\mathbb{R}_+; X) \subset L^2_\alpha(\mathbb{R}_+; X),\ g^{\langle m\rangle} := \mathcal{L}^{-1}(\mathbf{g}^{\langle m\rangle}) \in L^1(\mathbb{R}_+; Y) \cap L^2_\alpha(\mathbb{R}_+; Y), \\ h^{\langle m\rangle} := \mathcal{L}^{-1}(\mathbf{f}^{\langle m\rangle,*} \underset{q}{\circ} \mathbf{g}^{\langle m\rangle}) \in L^2(\mathbb{R}_+; X) \subset L^2_\alpha(\mathbb{R}_+; Z). \end{gathered} \tag{61}$$

The inclusions mentioned here follow from the fact that $l_n \in L^p(\mathbb{R}_+)$ for each $n \in \mathbb{N}_0,\ p \in [1,\infty]$, and functions $f^{\langle m\rangle}, g^{\langle m\rangle}, h^{\langle m\rangle}$ are finite linear combinations of functions $l_n \in L^p(\mathbb{R})$, $n \in \mathbb{N}_0$, in particular, $f^{\langle m\rangle}(t) = \sum_{j=0}^m f_j l_j(t),\ t \in \mathbb{R}_+$. Since operator $\mathcal{L}$ is isometric, taking into account (57), we obtain

$$f^{\langle m\rangle} \underset{m\to\infty}{\to} f \text{ in } L^2_\alpha(\mathbb{R}_+; X), \quad g^{\langle m\rangle} \underset{m\to\infty}{\to} g \text{ in } L^2_\alpha(\mathbb{R}_+; Y), \quad h^{\langle m\rangle} \underset{m\to\infty}{\to} h \text{ in } L^2_\alpha(\mathbb{R}_+; Z). \tag{62}$$

By Lemma 5, using the definitions (61), we have

$$f^{\langle m\rangle} \underset{q}{*} g^{\langle m\rangle} \in L^2(\mathbb{R}_+; Z) \subset L^2_\alpha(\mathbb{R}_+; Z), \tag{63}$$

whence by Theorem 3 it follows that

$$\mathcal{L}(f^{\langle m\rangle} \underset{q}{*} g^{\langle m\rangle}) = \mathbf{f}^{\langle m\rangle,*} \underset{q}{\circ} \mathbf{g}^{\langle m\rangle},$$

therefore

$$f^{\langle m\rangle} \underset{q}{*} g^{\langle m\rangle} = \mathcal{L}^{-1}(\mathbf{f}^{\langle m\rangle,*} \underset{q}{\circ} \mathbf{g}^{\langle m\rangle}).$$

Whence, taking into account the definition of $h^{\langle m\rangle}$ (see (61)), we have

$$h^{\langle m\rangle} = f^{\langle m\rangle} \underset{q}{*} g^{\langle m\rangle}, \quad m \in \mathbb{N}. \tag{64}$$

Now we prove equality (55). Let $T > 0$ be an arbitrary number. Since $(f \underset{q}{*} g)(t) = \int\limits_0^t q\big(f(t-s), g(s)\big)\, ds,\ t \in \mathbb{R}_+$, we have $f \underset{q}{*} g \in L^2(0,T;Z)$. Consider

$$\int\limits_0^T \|h(t) - (f \underset{q}{*} g)(t)\|_Z^2 dt = \int\limits_0^T \|h(t) - h^{\langle m\rangle}(t) + h^{\langle m\rangle}(t) - (f \underset{q}{*} g)(t)\|_Z^2 dt \leq$$

$$\leq 2\Bigg[\int\limits_0^T \|h(t) - h^{\langle m\rangle}(t)\|_Z^2 dt + \int\limits_0^T \|h^{\langle m\rangle}(t) - (f \underset{q}{*} g)(t)\|_Z^2 dt\Bigg]. \tag{65}$$

By (64) we deduce that

$$\int\limits_0^T \|h^{\langle m\rangle}(t) - (f \underset{q}{*} g)(t)\|_Z^2\, dt = \int\limits_0^T \|(f^{\langle m\rangle} \underset{q}{*} g^{\langle m\rangle})(t) - (f \underset{q}{*} g)(t)\|_Z^2\, dt =$$

$$= \int\limits_0^T \|(f^{\langle m\rangle} \underset{q}{*} g^{\langle m\rangle})(t) - (f^{\langle m\rangle} \underset{q}{*} g)(t) + (f^{\langle m\rangle} \underset{q}{*} g)(t) - (f \underset{q}{*} g)(t)\|_Z^2\, dt \leq$$

$$\leq 2\Bigg[\int\limits_0^T \|(f^{\langle m\rangle} \underset{q}{*} \{g^{\langle m\rangle} - g\})(t)\|_Z^2\, dt + \int\limits_0^T \|(\{f^{\langle m\rangle} - f\} \underset{q}{*} g)(t)\|_Z^2\, dt\Bigg]. \tag{66}$$

Using the Cauchy–Bunyakovsky inequality, we obtain

$$\int\limits_0^T \|(f^{\langle m\rangle} \underset{q}{*} \{g^{\langle m\rangle} - g\})(t)\|_Z^2\, dt = \int\limits_0^T \Bigg\|\int\limits_0^t q\big(f^{\langle m\rangle}(t-s), g^{\langle m\rangle}(s) - g(s)\big)\, ds\Bigg\|_Z^2 dt \leq$$

$$\leq K^2 \int\limits_0^T \Bigg[\int\limits_0^t \|f^{\langle m\rangle}(t-s)\|_X \cdot \|g^{\langle m\rangle}(s) - g(s)\|_Y\, ds\Bigg]^2 dt \leq$$

$$\leq K^2 \int\limits_0^T \Bigg[\int\limits_0^t \|f^{\langle m\rangle}(t-s)\|_X^2\, ds\Bigg]\Bigg[\int\limits_0^t \|g^{\langle m\rangle}(s) - g(s)\|_Y^2\, ds\Bigg] dt \leq$$

$$\leq K^2 T\Bigg[\int\limits_0^T \|f^{\langle m\rangle}(s)\|_X^2\, ds\Bigg]\Bigg[\int\limits_0^T \|g^{\langle m\rangle}(s) - g(s)\|_Y^2\, ds\Bigg] \underset{m\to\infty}{\rightarrow} 0, \tag{67}$$

where K is the constant from the inequality (17).

In the same manner we can see that

$$\int\limits_0^T \|\, (\{f^{\langle m\rangle} - f\} \underset{q}{*} g)(t)\, \|_Z^2\, dt \underset{m\to\infty}{\rightarrow} 0. \tag{68}$$

So, if we go to the limit provided $m \to \infty$ in the inequality (65), then, using (62), (66)–(68), we get

$$h(t) = (f \underset{q}{*} g)(t), \quad t \in [0, T],$$

whence (55) directly follows due to the arbitrariness of T. The proof is complete. $\square$

5. Conclusions. The obtained results make it possible to efficiently apply the Laguerre transform to the convolution of vector-valued functions in applied research. In particular, these results may serve as the mathematical foundation in numerical analysis of the evolutionary problems of mathematical physics, as well as time-dependent boundary integral equations.

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