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A REVISIT TO N-NORMED SPACES THROUGH ITS QUOTIENT SPACES

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In this paper, we define several types of continuous mapping in n-normed spaces with respect to the norms of its quotient spaces. Then, we show that all types of the continuity are equivalent. We also study contractive mappings on n-normed spaces using these norms. In particular, we prove a fixed point theorem for contractive mappings on a closed and bounded set in the n-normed space with respect to the norms of its quotient spaces. In the last section we prove a fixed point theorem and give some remarks on the p-summable sequence space as an n-normed space.

1. Introduction. Let n be a nonnegative integer and X be a real vector space with $\dim(X) \geq n$. The pair $(X, \|\cdot,\dots,\cdot\|)$ is called an *n-normed space* where $\|\cdot,\dots,\cdot\|\colon X^n \to \mathbb{R}$ is an *n-norm* on X which satisfies the following conditions:

(i) $\|x_1,\dots,x_n\| \geq 0$; $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ linearly dependent;

(ii) $\|x_1,\dots,x_n\|$ is invariant under permutation;

(iii) $\|\alpha x_1,\dots,x_k\| = |\alpha|\ \|x_1,\dots,x_k\|$ for any $\alpha \in \mathbb{R}$;

(iv) $\|x_1 + x_1', x_2,\dots,x_k\| \leq \|x_1, x_2,\dots,x_k\| + \|x_1', x_2,\dots,x_k\|$.

The concept of n-normed spaces was developed by S. Gähler in [5, 6, 7, 8]. Some researchers studied further various aspects of these spaces [4, 9, 10, 12, 13, 14, 15]. In particular, Ekariani *et al.* [3] proved a fixed point theorem of a contractive mapping in ℓ^p as an n-normed space. They used a norm which is equivalent to the usual norm on ℓ^p. This norm is derived from the n-norm of ℓ^p using a linearly independent set consisting of n vectors.

In this paper, we prove a fixed point theorem in the n-normed space using more general approach. First, we define some quotient spaces of an n-normed space. In each quotient space we define a norm which is derived from the n-norm in a certain way. Using these norms, we investigate continuous mappings and contractive mappings in the n-normed space. Then we prove a fixed point theorem for contractive mappings on a closed and bounded set in the n-normed space. Moreover, we show that the norm that Ekariani *et al.* defined is equivalent with a norm of one of the class collections that we provide. This will be a bridge to prove a fixed point theorem in ℓ^p as an n-normed space. We also give some remarks on ℓ^p as an n-normed space.

2. Preliminaries. Let us begin this section with the construction of quotient spaces of an n-normed space. Let $(X, \|\cdot,\dots,\cdot\|)$ be an n-normed space, $m \in \{1,\dots,n\}$ and $Y = \{y_1,\dots,y_n\}$

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be a linearly independent set in X. For $\{i_1,\dots,i_m\}\subset\{1,\dots,n\}$, consider $Y\setminus\{y_{i_1},\dots,y_{i_m}\}$. We define:

$$Y^0_{i_1,\dots,i_m} := \operatorname{span} Y\setminus\{y_{i_1},\dots,y_{i_m}\} := \left\{\sum_{i\notin\{i_1,\dots,i_m\}} \alpha_i y_i \ ; \ \alpha_i\in\mathbb{R}.\right\}.$$

Then $Y^0_{i_1,\dots,i_m}$ is a subspace of X. For each $u\in X$, the corresponding coset with respect to $Y^0_{i_1,\dots,i_m}$ is defined by

$$\overline{u} := \left\{u+\sum_{i\notin\{i_1,\dots,i_n\}} \alpha_i y_i \colon \alpha_i\in\mathbb{R}\right\}.$$

One can see that if $\overline{u}=\overline{v}$, then $u-v\in\operatorname{span}Y\setminus\{y_{i_1},\dots,y_{i_m}\}$. Moreover, we define the quotient space

$$X^*_{i_1,\dots,i_m} := X/Y^0_{i_1,\dots,i_m} := \{\overline{u}\colon u\in X\}.$$

The addition and scalar multiplication apply in this space. Furthermore, we define the following norm on $X^*_{i_1,\dots,i_m}$:

$$\|\overline{u}\|^*_{i_1,\dots,i_m} := \|u,y_1,\dots,y_{i_1-1},y_{i_1+1},\dots,y_n\|+\cdots+\|u,y_1,\dots,y_{i_m-1},y_{i_m+1},\dots,y_n\|. \quad (1)$$

Using the above construction, we get $\binom{n}{m}$ quotient spaces. For an $m\in\{1,\dots,n\}$, the set that contains all quotient spaces constructed above is called **class-m collection** [2]. One can see that, equation (1) can be rewritten as

$$\|\overline{u}\|^*_{i_1,\dots,i_m} = \|\overline{u}\|^*_{i_1}+\cdots+\|\overline{u}\|^*_{i_m},$$

since each term of (1) is a norm of quotient spaces in class-1 collection.

For any $m\in\{1,\dots,n\}$ we will use the phrase 'norms of class-m collection' to mean 'all norms of each quotient space in the class-m collection'. Next, using the norms of class-m collection, we investigate some topology characteristics of an n-normed space as presented in the following.

Definition 1 ([1])**.** Let $(X,\|\cdot,\dots,\cdot\|)$ be an n-normed space and $m\in\{1,\dots,n\}$. We say a sequence $\{x_k\}\subset X$ *converges with respect to the norms of class-m collection to x* if for any $\epsilon>0$, there exists an $N\in\mathbb{N}$ such that for $k\geq N$ we have

$$\|\overline{x_k-x}\|^*_{i_1,\dots,i_m}<\epsilon,$$

for every $\{i_1,\dots,i_m\}\subset\{1,\dots,n\}$. In this case we also say

$$\lim_{n\to\infty}\|\overline{x_k-x}\|^*_{i_1,\dots,i_m}=0,$$

for every $\{i_1,\dots,i_m\}\subset\{1,\dots,n\}$. We say x is the limit point of $\{x_k\}$. If $\{x_k\}$ does not converge, we say it *diverges.*

The above definition leads to this following theorem.

Theorem 1 ([1])**.** *Let $(X,\|\cdot,\dots,\cdot\|)$ be an n-normed space and $m\in\{1,\dots,n\}$. A sequence $\{x_k\}\subset X$ is convergent with respect to the norms of class-1 collection if and only if it is convergent with respect to the norms of class-m collection.*

We obtain this following corollary as a direct consequence of Theorem 1.

Corollary 1. *Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space and $m_1, m_2 \in \{1, \dots, n\}$. Then, if a sequence in X converges with respect to the norms class-m_1 collection, then it also converges with respect to the norms of class-m_2 collection.*

Now we move to define Cauchy sequence using norms of class-m collection, for any $m \in \{1, \dots, n\}$.

Definition 2 ([1])**.** Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space and $m \in \{1, \dots, n\}$. A sequence $\{x_k\} \subset X$ is called a *Cauchy sequence with respect to the norms of class-m collection* if for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for every $k, l \geq N$, we have

$$\|\overline{x_k - x_l}\|^*_{i_1,\dots,i_m} < \epsilon,$$

for every $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. In other words

$$\lim_{k,l\to\infty} \|\overline{x_k - x_l}\|^*_{i_1,\dots,i_m} = 0,$$

for every $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$.

From this definition, we obtain the following theorem and corollary.

Theorem 2 ([1])**.** *Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space and $m \in \{1, \dots, n\}$. If $\{x_k\}$ is convergent with respect to the norms of class-1 collection, then $\{x_k\}$ is Cauchy with respect to the norms of class-m collection.*

Corollary 2. *Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space and $m_1, m_2 \in \{1, \dots, n\}$. Then, if a sequence in X is a Cauchy sequence with respect to the norms class-m_1 collection, then it is also a Cauchy sequence with respect to the norms of class-m_2 collection.*

Remark 1. For $m_1, m_2 \in \{1, \dots, n\}$, we find that all types of convergent sequence with respect to the norms of class-m_1 collection and to the norms of class-m_2 colection are equivalent. The equivalence also applies to all types of Cauchy sequence with respect to the norms of class-m_1 collection and to the norms of class-m_2 collection. In this regard, we may simply use the word 'converges or Cauchy' instead of 'converges or Cauchy with respect to the norms of class-m collection'. Furthermore if every Cauchy sequence in X converges, then X is *complete*. By the word 'complete', we mean 'complete with respect to the norms of class-m collection', for some $m \in \{1, \dots, n\}$.

Next, we define closed and bounded sets with respect to the norms of class-m collection, for any $m \in \{1, \dots, n\}$.

Definition 3 ([1])**.** Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space and $K \subseteq X$. The set K is called *closed* if for any sequence $\{x_k\}$ in K that converges in X, its limit belongs to K.

Note that by saying 'closed' we mean 'closed with respect to class-m collection', for some $m \in \{1, \dots, n\}$. Next is the definition of bounded sets.

Definition 4 ([1]). Let $(X, \|\cdot,\dots,\cdot\|)$ be an n-normed space, $m \in \{1,\dots,n\}$ and $K \subseteq X$ be a nonempty set. The set K is called *bounded with respect to the norms of class-m collection* if and only if for any $x \in K$ there exists an $M > 0$ such that

$$\|\overline{x}\|^*_{i_1,\dots,i_m} \leq M,$$

for every $\{i_1,\dots,i_m\} \subset \{1,\dots,n\}$.

We also have the following theorem which says that all types of bounded set are equivalent.

Theorem 3 ([1]). *Let $(X, \|\cdot,\dots,\cdot\|)$ is an n-normed space, $m \in \{1,\dots,n\}$ and $K \subset X$ nonempty. The set K is bounded with respect to the norms of class-1 collection if and only if it is bounded with respect to class-m collection.*

Furthermore, for the completeness of the n-normed space with respect to the norms of class-m collections we have the following theorem.

Theorem 4. *Let $(X, \|\cdot,\dots,\cdot\|)$ be an n-normed space and $m \in \{1,\dots,n\}$. Then X is complete with respect to the norms of class-1 collection if and only if X is complete with respect to the norms of class-m collection.*

Proof. Suppose that X is complete with respect to the norms of class-1 collection and $m \in \{1,\dots,n\}$. Take any Cauchy sequence $\{x_k\}$ with respect to the norms of class-m collection in X. Then by Theorems 1 and 2 we have $\{x_k\}$ is a convergent sequence with respect to the norms of class-m collection in X. This tells us that X is complete with respect to the norms of class-m collection. The converse is similar. □

Remark 2. As in Corollary 2, one can see that the equivalence also applies to boundedness and completeness of a set. From now on, we may simply say 'complete' instead of 'complete with respect to class-m collection', for any $m \in \{1,\dots,n\}$. We also will use the word 'bounded' instead of the phrase 'bounded with respect to the norms of class-m collection', for any $m \in \{1,\dots,n\}$.

Remark 3 ([2]). For a fixed $m \in \{1,\dots,n\}$, the convergence of a sequence, the closedness and the boundedness of a set with respect to the norms of class-m collection may be investigated with respect to some norms $\|\cdot\|^*_{i_1,\dots,i_m}$ we choose such that

$$\bigcup \{i_1,\dots,i_m\} \supseteq \{1,\dots,n\}.$$

Moreover, the least number of norms that can be used to investigate these notions is $\lceil \frac{n}{m} \rceil$. Next, by using a similar approach we will study continuous mappings, contractive mappings, and also prove a fixed point theorem of contractive mappings in a closed and bounded set in an n-normed space.

3. Continuous mappings with respect to the norms of class-m collection. We now discuss the continuity of a mapping with respect to the norms of class-m collection in an n-normed space.

Definition 5. Let $(X, \|\cdot,\dots,\cdot\|^*)$ be an n_1-normed space and $(Z, \|\cdot,\dots,\cdot\|^{**})$ be an n_2-normed space, $l \in \{1,\dots,n_1\}$, and $m \in \{1,\dots,n_2\}$. Suppose that $T\colon X \to Z$.

(i) We say that T is *continuous with respect to the norms of class-(l,m) collections at* $a \in X$ if and only if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for $x \in X$ with $\|\overline{x-a}\|^*_{i_1,\dots,i_l} < \delta$ for every $\{i_1,\dots,i_l\} \subset \{1,\dots,n_1\}$, we have $\|\overline{Tx-Ta}\|^{**}_{j_1,\dots,j_m} < \epsilon$ for every $\{j_1,\dots,j_m\} \subset \{1,\dots,n_2\}$.

(ii) We say that T is *continuous with respect to the norms of class-(l,m) collections on* X if and only if T is continuous with respect to the norms of class-(l,m) collection at each $x \in X$.

If $Z = X$, then we say 'T is continuous with respect to the norms of class-m collections' instead of 'continuous with respect to the norms of class-(m,m) collections'. The norms $\|\cdot\|^*_{i_1,\dots,i_l}$, $\|\cdot\|^{**}_{j_1,\dots,j_m}$ are the norms of class-(l,m) collections in X and Z respectively. Note that class-(l,m) collections is a pair of class-l colelction of X and class-m collection of Z. We define class-(l,m) collections by using a linearly independent set consisting of n_1 and n_2 vectors in X and Z, respectively. Based on the definition of continuous mapping, we will have several types of continuity with respect to the norms of class collections we used. To avoid confusion in choosing what class collection to use later, we provide the following theorem and corollary.

Theorem 5. *Let $(X, \|\cdot,\dots,\cdot\|^*)$ be an n_1-normed space, $(Z, \|\cdot,\dots,\cdot\|^{**})$ be an n_2-normed space, $l \in \{1,\dots,n_1\}$ and $m \in \{1,\dots,n_2\}$. A mapping $f\colon X \to Z$ is continuous with respect to class-1 collections if and only if T is continuous with respect to class-(l,m) collections.*

Proof. Let T be a continuous mapping with respect to the norms of class-1 collections at $a \in X$, $l \in \{1,\dots,n_1\}$ and $m \in \{1,\dots,n_2\}$. Given an $\epsilon > 0$, we choose $\delta > 0$ such that for any $x \in X$ with $\|\overline{x-a}\|^*_i < \delta$ for every $i \in \{1,\dots,n_1\}$, we have $\|\overline{Tx-Ta}\|^{**}_j < \frac{\epsilon}{m}$ for every $j \in \{1,\dots,n_2\}$. Then, for any $x \in X$ with

$$\|\overline{x-a}\|^*_{i_1,\dots,i_l} = \|\overline{x-a}\|^*_{i_1} + \cdots + \|\overline{x-a}\|^*_{i_l} < \delta,$$

for every $\{i_1,\dots,i_l\} \subset \{1,\dots,n_1\}$, we have $\|\overline{x-a}\|^*_i < \delta$ for every $i \in \{1,\dots,n_1\}$, implying that

$$\|\overline{Tx-Ta}\|^{**}_{j_1,\dots,j_m} = \|\overline{Tx-Ta}\|^{**}_{j_1} + \cdots + \|\overline{Tx-Ta}\|^{**}_{j_m} < m \cdot \frac{\epsilon}{m} = \epsilon,$$

for every $\{j_1,\dots,j_m\} \subset \{1,\dots,n_2\}$. This means T is continuous with respect to the norms of class-(l,m) collection.

Conversely, let T be a continous mapping with respect to the norms of class-(l,m) collection. Given an $\epsilon > 0$, we choose $\delta > 0$ such that for any $x \in X$ with

$$\|\overline{x-a}\|^*_{i_1} + \cdots + \|\overline{x-a}\|^*_{i_l} = \|\overline{x-a}\|^*_{i_1,\dots,i_l} < \delta, \tag{2}$$

for every $\{i_1,\dots,i_l\} \subset \{1,\dots,n_1\}$, we have

$$\|\overline{Tx-Ta}\|^{**}_{j_1} + \cdots + \|\overline{Tx-Ta}\|^{**}_{j_m} = \|\overline{Tx-Ta}\|^{**}_{j_1,\dots,j_m} < \epsilon, \tag{3}$$

for every $\{j_1,\dots,j_m\} \subset \{1,\dots,n_2\}$. Hence, for any $x \in X$ with $\|\overline{x-a}\|^*_i < \frac{\delta}{l}$ for every $i \in \{1,\dots,n_1\}$, we have $\|\overline{x-a}\|^*_{i_1,\dots,i_l} < \delta$, implying that $\|\overline{Tx-Ta}\|^{**}_j \le \|\overline{Tx-Ta}\|^{**}_{j_1,\dots,j_m} < \epsilon$ for every $j \in \{1,\dots,n_1\}$. (Here we can choose any set $\{j_1,\dots,j_m\}$ that contains j.) This means T is continous with respect to the norms of class-1 collection. □

Theorem 5 says that the continuity with respect to the norms of class-1 collection is equivalent to the continuity with respect to the norms of class-(l,m) collection for any

$l \in \{1, \ldots, n_1\}$ and $m \in \{1, \ldots, n_2\}$. By using the continuity with respect to the norms of class-1 collection as a brigde, we have the following corollary that shows the relation between continuity with respect to any two class collections.

Corollary 3. *Let $(X, \|\cdot, \ldots, \cdot\|^*)$ and $(Z, \|\cdot, \ldots, \cdot\|^{**})$ be an n_1-normed space and an n_2-normed space respectively, $l_1, l_2 \in \{1, \ldots, n_1\}$ and $m_1, m_2 \in \{1, \ldots, n_2\}$. If a mapping $T \colon X \to Z$ is continuous with respect to class-(l_1, m_1), then T is continuous with respect to the norms of class-(l_2, m_2).*

Remark 4. Corollary 3 is a direct result of Theorem 5. By this corollary, we can see that all types of continuity of a mapping with respect to the norms of class-(l, m) collections are equivalent. Then from now on we will use the word 'continuous' instead of 'continuous with respect to the norms of class-(l, m) collection'.

Now we present a proposition that gives a relation between a convergent sequence and a continuous mapping with respect to the norms of class-(l, m) collection.

Proposition 1. *Let $(X, \|\cdot, \ldots, \cdot\|^*)$ and $(Z, \|\cdot, \ldots, \cdot\|^{**})$ be an n_1-normed space and an n_2-normed space respectively, $l \in \{1, \ldots, n_1\}$, $m \in \{1, \ldots, n_2\}$. Suppose that $T \colon X \to Z$ is continuous. If $\{x_k\}$ converges to x, then*

$$\lim_{k \to \infty} Tx_k = Tx.$$

Proof. Let $l \in \{1, \ldots, n_1\}$, $m \in \{1, \ldots, n_2\}$, and $\epsilon > 0$. Since T is continuous, there is a $\delta > 0$ such that if $\|\overline{y - x}\|^*_{i_1, \ldots, i_m} < \delta$ for every $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n_1\}$, then $\|\overline{Ty - Tx}\|^{**}_{j_1, \ldots, j_l} < \epsilon$. Since $\{x_k\}$ converges to x with respect to norms of class-m, choose $N \in \mathbb{N}$ such that for $k \geq N$ we have $\|x_k - x\|_{i_1, \ldots, i_m} \leq \delta$. Then it follows that $\|\overline{Tx_k - Tx}\|^{**}_{j_1, \ldots, j_m} < \epsilon$ whenever $k \geq N$. Therefore Tx_k converges to Tx, or we simply write $\lim_{k \to \infty} Tx_k = Tx$. □

4. Fixed point theorem for contractive mappings with respect to the norms of class-m collection. In this section, we shall discuss contractive mappings with respect to the norms of class-m collection in an n-normed space and its fixed point theorem.

Definition 6. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space and $m \in \{1, \ldots, n\}$. A mapping $T \colon X \to X$ is called *contractive with respect to the norms of class-m collection* if there is a $C \in (0, 1)$ such that for any $x, y \in X$ we have

$$\|\overline{Tx - Ty}\|^*_{i_1, \ldots, i_m} \leq C \, \|\overline{x - y}\|^*_{i_1, \ldots, i_m},$$

for every $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ with $i_1 < \cdots < i_m$.

Following the definitions of continuous mappings and contractive mappings, we have the proposition and theorem below.

Proposition 2. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space and $m \in \{1, \ldots, n\}$. If T is a contractive mapping with respect to the norms of class-m collection, then T is continuous.*

Proof. For an $m \in \{1, \ldots, n\}$, let T be a contractive mapping with respect to the norms of class-m collection, then there is a $C \in (0, 1)$ such that for any $x, y \in X$ we have

$$\|\overline{Tx - Ty}\|^*_{i_1, \ldots, i_m} \leq C \, \|\overline{x - y}\|^*_{i_1, \ldots, i_m},$$

for every $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$.

For any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{C}$. Then, for $x, y \in X$ where $\|\overline{x-y}\|^*_{i_1,\dots,i_m} < \delta$ for every $\{i_1, \dots, i_m\} \in \{1, \dots, n\}$, we have

$$\|\overline{Tx - Ty}\|^*_{i_1,\dots,i_m} < C \cdot \frac{\epsilon}{C} = \epsilon,$$

for every $\{i_1, \dots, i_m\} \in \{1, \dots, n\}$. Therefore, T is continuous with respect to the norms of class-m collection on X. Since Remark 4 says that all types of continuity of a mapping with respect to the norms of any class colection are equivalent, we simply say that T is continuous. □

Moreover, we have the relation between the contractivity with respect to the norms of class-1 colection and class-m collection for any $m \in \{1, \dots, n\}$ as stated in the following theorem.

Theorem 6. *Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space, $m \in \{1, \dots, n\}$ and $T\colon X \to X$. If T is a contractive mapping with respect to the norms of class-1 collection, then T is a contractive mapping with respect to the norms of class-m collection.*

Proof. Suppose that T is a contractive mapping with respect to the norms of class-1 collection, then there is a $C \in (0, 1)$ such that for any $x, y \in X$ we have

$$\|\overline{Tx - Ty}\|^*_j \leq C\|\overline{x-y}\|^*_j, \tag{4}$$

for every $j \in \{1, \dots, n\}$. Let $m \in \{1, \dots, n\}$, by (4) we have

$$\begin{aligned} \|\overline{Tx - Ty}\|^*_{i_1} &\leq C\|\overline{x-y}\|^*_{i_1}, \\ &\vdots \\ \|\overline{Tx - Ty}\|^*_{i_m} &\leq C\|\overline{x-y}\|^*_{i_m}, \end{aligned}$$

for every $\{i_1, \dots, i_m\} \in \{1, \dots, n\}$. Therefore, It follows from the above inequalities that

$$\|\overline{Tx - Ty}\|^*_{i_1,\dots,i_m} = \|\overline{Tx - Ty}\|^*_{i_1} + \ldots + \|\overline{Tx - Ty}\|^*_{i_m} \leq C\left(\|\overline{x-y}\|^*_{i_1} + \cdots + \|\overline{x-y}\|^*_{i_m}\right) = \\ = C\|\overline{x-y}\|^*_{i_1,\dots,i_m},$$

for every $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. If we consider the norms of class-m collection, then this means T is a contractive mapping with respect to the norms of class-m collection. □

We also have this following theorem which will be used later.

Theorem 7. *Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space, $m \in \{1, \dots, n\}$ and $T\colon X \to X$. If T is a contractive mapping with respect to the norms of class-m collection, then T is also a contractive mapping with respect to the norm of class-n collection.*

Proof. Let $T\colon X \to X$ be a contractive mapping with respect to the norms of class-m collection for an $m \in \{1, \dots, n\}$. Then there is a $C \in (0, 1)$ such that for every $x, y \in X$ we have

$$\|\overline{Tx - Ty}\|^*_{i_1,\dots,i_m} \leq C\|\overline{x-y}\|^*_{i_1,\dots,i_m}, \tag{5}$$

for every $\{i_1,\dots,i_m\}\subset\{1,\dots,n\}$. One can see that (5) contains $\binom{n}{m}$ inequalities. From these inequalities, we have

$$(n-1)\|\overline{Tx-Ty}\|^*_{1,\dots,n}\le C(n-1)\|\overline{x-y}\|^*_{1,\dots,n},$$

or

$$\|\overline{Tx-Ty}\|^*_{1,\dots,n}\le C\|\overline{x-y}\|^*_{1,\dots,n},$$

which means that T is a contractive mapping with respect to the norm of class-n collection. □

Finally, we provide a fixed point theorem for a contractive mapping in a closed and bounded set on an n-normed space, with respect to the norms of class-m collection.

Theorem 8. *Let $(X,\|\cdot,\dots,\cdot\|)$ be an n-normed space, $m\in\{1,\dots,n\}$ and $K\subset X$ is nonempty, closed and bounded. Suppose that X is complete (with respect to the norms of class-m collection). If T: $K\to K$ is a contractive mapping with respect to the norms of class-m collection, then T has a unique fixed point.*

Proof. Fix an $m\in\{1,\dots,n\}$. Let $x_0\in K$ and $\{x_k\}$ be a sequence in K such that

$$x_k=Tx_{k-1}=T^k(x_0);\;\; k=1,2,\dots$$

Since T is a contractive mapping, there is a $C\in(0,1)$ such that for $x_0,x_1\in K$ we have

$$\begin{aligned}\|\overline{T^2x_0-T^2x_1}\|^*_{i_1,\dots,i_m}&=\|\overline{T(Tx_0)-T(Tx_1)}\|^*_{i_1,\dots,i_m}\le\\ &\le C\|\overline{Tx_0-Tx_1}\|^*_{i_1,\dots,i_m}\le C^2\|\overline{x_0-x_1}\|^*_{i_1,\dots,i_m},\end{aligned}$$

for every $\{i_1,\dots,i_m\}\subset\{1,\dots,n\}$. By using induction, we have

$$\|\overline{T^k(x_0)-T^k(x_1)}\|^*_{i_1,\dots,i_m}\le C^k\|\overline{x_0-x_1}\|^*_{i_1,\dots,i_m},$$

for every $\{i_1,\dots,i_m\}\subset\{1,\dots,n\}$. Now we show that $\{x_n\}$ is a Cauchy sequence in K. Let $k,l\in\mathbb{N}$. Without loss of generality, we take $l>k$ and $l=k+p$, with $p\in\mathbb{N}$. Then we have

$$\begin{aligned}\|\overline{x_k-x_l}\|^*_{i_1,\dots,i_m}&=\|\overline{x_k-x_{k+p}}\|^*_{i_1,\dots,i_m}\le\|\overline{x_k-x_{k+1}}\|^*_{i_1,\dots,i_m}+\cdots+\|\overline{x_{k+p-1}-x_{k+p}}\|^*_{i_1,\dots,i_m}=\\ &=\|\overline{T^k(x_0)-T^k(x_1)}\|^*_{i_1,\dots,i_m}+\cdots+\|\overline{T^{k+p-1}(x_0)-T^{k+p-1}(x_1)}\|^*_{i_1,\dots,i_m}\le\\ &\le\left(C^k+\cdots+C^{k+p-1}\right)\|\overline{x_0-x_1}\|^*_{i_1,\dots,i_m},\end{aligned}$$

for every $\{i_1,\dots,i_m\}\subset\{1,\dots,n\}$. Also, since K is bounded, for any $x_0,x_1\in K$ there exists an $M>0$ such that $\|\overline{x_0-x_1}\|^*_{i_1,\dots,i_m}\le M$, for every $\{i_1,\dots,i_m\}\subset\{1,\dots,n\}$. Then we have

$$\|\overline{x_k-x_l}\|^*_{i_1,\dots,i_m}\le\left(C^k+\cdots+C^{k+p-1}\right)M=\left(C^k+\cdots+C^{l-1}\right)M,$$

for every $\{i_1,\dots,i_m\}\subset\{1,\dots,n\}$. Since $C\in(0,1)$, we have

$$\lim_{k,l\to\infty}\|\overline{x_k-x_l}\|^*_{i_1,\dots,i_m}=0,$$

for every $\{i_1,\dots,i_m\}\subset\{1,\dots,n\}$, which means that $\{x_k\}$ is a Cauchy sequence.

Since X is complete with respect to the norms of class-m collection and K is closed then $x_k \to x$, with $x \in K$. Propositions 1 and 2 imply that

$$Tx = \lim_{k\to\infty} Tx_k = \lim_{k\to\infty} x_{k+1} = x.$$

Therefore, T has a fixed point in K with respect to the norms of class-m collection. Next, we want to show the uniqueness of the fixed point with respect to the norms of class-m collection. Assume that $x' \in K$ is another fixed point of T. Since T is a contractive mapping, there is a $C \in (0,1)$ such that

$$\|\overline{x - x'}\|^*_{i_1,\ldots,i_m} = \|\overline{Tx - Tx'}\|^*_{i_1,\ldots,i_m} \leq C\|\overline{x - x'}\|^*_{i_1,\ldots,i_m},$$

for every $\{i_1,\ldots,i_m\} \subset \{1,\ldots,n\}$. One can see that this is true only for $\|\overline{x - x'}\|^*_{i_1,\ldots,i_m} = 0$, for every $\{i_1,\ldots,i_m\} \subset \{1,\ldots,n\}$. This means that $x = x'$ or T has a unique fixed point. Therefore, for any $m \in \{1,\ldots,n\}$, a contractive mapping T has a unique fixed point. □

5. Concluding Remarks. Let us consider the p-summable sequences ℓ^p (for $1 \leq p \leq \infty$) containing all sequences of real numbers $x = (x_j)$ for which $\sum |x_j|^p < \infty$. As in [11], one may equip this space with the following n-norm

$$\|x_1,\ldots,x_n\|_p := \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\left|\det\left[\xi_{ij_k}\right]_{i,k}\right|^p\right]^{\frac{1}{p}},$$

with $x_i = (\xi_{ij}) \in \ell^p, i = 1,\ldots,n$.

Now, one can see that if $\{y_1,\ldots,y_n\}$ is a linearly independent set in ℓ^p, the norm of class-n collection $\|\cdot\|^*_{1,\ldots,n}$ can be writen as

$$\|x\|^*_{1,\ldots,n} = \sum_{\{i_2,\ldots,i_n\}\subset\{1,\ldots,n\}} \|x, y_{i_2},\ldots,y_{i_n}\|_p,$$

This is simpler than norm that used by [3], namely

$$\|x\|^*_p := \left[\sum_{\{i_2,\ldots,i_m\}\subseteq\{1,\ldots,n\}} \|x, y_{i_2},\ldots,y_{i_n}\|^p_p\right]^{\frac{1}{p}}. \tag{6}$$

The norm in (6) will be a bridge to prove the fixed point theorem on $(\ell^p, \|\cdot,\ldots,\cdot\|)$. Recall that the usual norm on ℓ^p is defined by

$$\|x\|_p = \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}}.$$

We have some equivalence relations between norm $\|\cdot\|_p$ and $\|\cdot\|^*_p$ as written in the following theorem.

Theorem 9 ([3])**.** *Let $\{y_1,\ldots,y_n\}$ be a linearly independent set on ℓ^p. Then the norm $\|\cdot\|^*_p$ defined in (6) is equivalent to the usual norm $\|\cdot\|_p$ on ℓ^p. Precisely, we have*

$$\frac{n\|y_1,\ldots,y_n\|_p}{(2n-1)\left[\|y_1\|_p + \cdots + \|y_n\|_p\right]}\|x\|_p \leq \|x\|^*_p \leq (n!)^{1-\frac{1}{p}}\left[\sum_{\{i_2,\ldots,i_n\}\subset\{1,\ldots,n\}} \|y_{i_2}\|^p_p \cdots \|y_{i_n}\|^p_p\right]^{\frac{1}{p}} \|x\|_p,$$

for every $x \in \ell^p$.

Since $(\ell^p, \|\cdot\|_p)$ is a Banach space, by Theorem 9 we have the following corollary.

Corollary 4 ([3]). *The normed space $(\ell^p, \|\cdot\|_p^*)$ is a Banach space.*

Comparing norms of class-n collection and $\|\cdot\|_p^*$, we find that these two norms are equivalent. We write it in the proposition below.

Proposition 3. *Let $\{y_1, \ldots, y_n\}$ be a linearly independent set on ℓ^p. Then the norm of class n-collection which is defined in (1) is equivalent with the norm $\|\cdot\|_p^*$ in (6). Precisely, we have*

$$\|\cdot\|_p^* \leq \|\cdot\|_{1,\ldots,n}^* \leq (n)^{1-\frac{1}{p}} \|\cdot\|_p^*.$$

Proof. Consider the n-normed space $(\ell^p, \|\cdot, \ldots, \cdot\|_p)$ and $\{y_1, \ldots, y_n\}$ be a linearly independent set on ℓ^p. For any $x \in X$, we have

$$\|x\|_p^* = \left(\sum_{\{i_2,\ldots,i_n\} \subset \{1,\ldots,n\}} \|x, y_{i_2}, \ldots, i_n\|_p^p \right)^{\frac{1}{p}} \leq \sum_{\{i_2,\ldots,i_n\} \subset \{1,\ldots,n\}} \|x, y_{i_2}, \ldots, i_n\|_p = \|x\|_{1,\ldots,n}^*.$$

From Theorem 9 and by using Hölder inequality we have

$$\|x\|_{1,\ldots,n}^* = \sum_{\{i_2,\ldots,i_n\} \subset \{1,\ldots,n\}} \|x, y_{i_2}, \ldots, i_n\|_p \leq$$

$$\leq (n)^{1-\frac{1}{p}} \left(\sum_{\{i_2,\ldots,i_n\} \subset \{1,\ldots,n\}} \|x, y_{i_2}, \ldots, i_n\|_p^p \right)^{\frac{1}{p}} = (n)^{1-\frac{1}{p}} \|x\|_p^*.$$

Then we have the inequality we want. □

Note that, we write $\|x\|_{1,\ldots,n}^*$ instead of $\|\overline{x}\|_{1,\ldots,n}^*$, because the element of class-n collection of an n-normed space is the n-normed space itself. One might notice that the formula of the norm of class-n collection is simpler than the norm $\|\cdot\|_p^*$ that Ekariani *et al.* used and both norms are equivalent. Since the norm $\|\cdot\|_p^*$ equivalent to the usual norm $\|\cdot\|_p^*$ on ℓ^p, we have the following corollaries.

Corollary 5. *Let $\{y_1, \ldots, y_n\}$ be a linearly independent set on ℓ^p. Then the norm $\|\cdot\|_{1,\ldots,n}^*$ is equivalent to the usual norm $\|\cdot\|_p$ on ℓ^p. Precisely, we have*

$$\frac{n\|y_1, \ldots, y_n\|_p}{(2n-1)\left[\|y_1\|_p + \cdots + \|y_n\|_p\right]} \|x\|_p \leq \|x\|_{1,\ldots,n}^* \leq$$

$$\leq (n \cdot n!)^{1-\frac{1}{p}} \left[\sum_{\{i_2,\ldots,i_n\} \subset \{1,\ldots,n\}} \|y_{i_2}\|_p^p \ldots \|y_{i_n}\|_p^p \right]^{\frac{1}{p}} \|x\|_p,$$

Corollary 6. *The normed space $(\ell^p, \|\cdot\|_{1,\ldots,n}^*)$ is a Banach space.*

Now, we present a more general fixed point theorem on $(\ell^p, \|\cdot, \ldots, \cdot\|_p)$ as follows.

Theorem 10. *Consider the n-normed space $(\ell^p, \|\cdot, \ldots, \cdot\|_p)$ and let $K \subseteq \ell^p$ be nonempty, closed and bounded. If $T \colon K \to K$ is a contractive mapping with respect to norms of class-m collection for an $m \in \{1, \ldots, n\}$, then T has a unique fixed point.*

Proof. Fix an $m \in \{1, \dots, n\}$ and let $T\colon K \to K$ be a contractive mapping with respect to norms of class-m collection on ℓ^p. By Theorem 7, T is contractive with respect to the norm of class-n collection on ℓ^p. By Corollary 6 ℓ^p is complete with respect to the norm of class-n collection. Hence, T has a unique fixed point. □

Remark 5. We can also prove Theorem 10 using Theorem 8, noting the fact that ℓ^p is complete with respect to the norms of class-m collection (see Theorem 4).

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