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ISOLATED SINGULARITIES OF MAPPINGS WITH THE INVERSE POLETSKY INEQUALITY

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The manuscript is devoted to the study of mappings with finite distortion, which have been actively studied recently. We consider mappings satisfying the inverse Poletsky inequality, which can have branch points. Note that mappings with the reverse Poletsky inequality include the classes of conformal, quasiconformal, and quasiregular mappings. The subject of this article is the question of removability an isolated singularity of a mapping. The main result is as follows. Suppose that f is an open discrete mapping between domains of a Euclidean n-dimensional space satisfying the inverse Poletsky inequality with some integrable majorant Q. If the cluster set of f at some isolated boundary point x_0 is a subset of the boundary of the image of the domain, and, in addition, the function Q is integrable, then f has a continuous extension to x_0. Moreover, if f is finite at x_0, then f is logarithmic Hölder continuous at x_0 with the exponent $1/n$.

1. Introduction. In our joint paper [1], we obtained a continuous extension of homeomorphisms, the inverses of which satisfy the weight Poletsky inequality, to an isolated boundary point (see Theorem 5.1). The main purpose of this manuscript is to transfer the specified result to mappings with branching. More precisely, we consider open discrete mappings between two domains of the extended Euclidean space and assume that they satisfy some weight estimate of the distortion of the modulus of families of paths with integrable majorant. Note that the studies of this paper are in the context of studying mappings with bounded and finite distortion (see, e.g., [2]–[8]). The conditions concerning distortion of modulus of families of paths are well known in the theory of quasiconformal mappings and their generalizations (see, for example, [3, Theorem 3.2], [5, Theorem 8.5] and [7, Theorem 6.7.II]).

Let us turn to the definitions. Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A(y_0, r_1, r_2) = \{y \in \mathbb{R}^n \colon r_1 < |y - y_0| < r_2\},$$
$$B(y_0, r) = \{y \in \mathbb{R}^n \colon |y - y_0| < r\}, S(y_0, r) = \{y \in \mathbb{R}^n \colon |y - y_0| = r\}.$$

Set $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$. Given sets E, $F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ denote by $\Gamma(E, F, D)$ the family of all paths $\gamma\colon [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in [a, b]$. Given a domain $D \subset \mathbb{R}^n$, or $D \subset \overline{\mathbb{R}^n}$, a *mapping* $f\colon D \to \mathbb{R}^n$ is an arbitrary continuous transformation $x = (x_1, \dots, x_n) \mapsto f(x) = (f_1(x), \dots, f_n(x))$. Let $f\colon D \to \mathbb{R}^n$, let $y_0 \in f(D)$

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and let $0 < r_1 < r_2 < d_0 = \sup_{y\in f(D)} |y - y_0|$. Now, we denote by $\Gamma_f(y_0, r_1, r_2)$ the family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Everywhere below, $M(\Gamma)$ denotes the modulus of a family Γ of paths γ in $\mathbb{R}^n$, the definition and basic properties of which we assume to be known (see, for example, [8, section 6]). Let $Q\colon \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function such that $Q(x) \equiv 0$ for $\mathbb{R}^n \setminus D$. We will say that f *satisfies the inverse Poletsky inequality* at a point $y_0 \in f(D)$, if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{f(D)\cap A(y_0,r_1,r_2)} Q(y)\cdot \eta^n(|y - y_0|)dm(y) \qquad (1)$$

holds for any Lebesgue measurable function $\eta\colon (r_1, r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)dr \geqslant 1. \qquad (2)$$

Note that the inequalities (1) are well known in the theory of quasiregular mappings and hold for $Q = N(f,D)\cdot K$, where $N(f,D)$ is the maximal multiplicity of f in D, and $K \geqslant 1$ is some constant that may be calculated in the following way: $K = \operatorname{ess\,sup} K_O(x,f)$, $K_O(x,f) = \|f'(x)\|^n/|J(x,f)|$ for $J(x,f) \neq 0$; $K_O(x,f) = 1$ for $f'(x) = 0$, and $K_O(x,f) = \infty$ for $f'(x) \neq 0$, where $J(x,f) = 0$ (see, e.g., [3, Theorem 3.2] or [7, Theorem 6.7.II]). A mapping $f\colon D \to \mathbb{R}^n$ is called a *discrete*, if the pre-image $\{f^{-1}(y)\}$ consists of isolated points for any $y \in \mathbb{R}^n$, and an *open*, if $f(U)$ is open for any open set $U \subset D$. As usual, we put

$$C(x,f) := \{y \in \overline{\mathbb{R}^n}\colon \exists x_k \in D\colon x_k \to x, f(x_k) \to y, k \to \infty\}.$$

Hereinafter, the boundary ∂D and the closure $\overline{D}$ are understood in the topology of the extended Euclidean space $\overline{\mathbb{R}^n}$. The following statement holds.

Theorem 1. *Let D and D' be domains in $\overline{\mathbb{R}^n}$, $n \geqslant 2$, $x_0 \in D$, and let f be an open and discrete mapping of $D \setminus \{x_0\}$ onto D', such that the relation (1) holds at least one finite point $y_0 \in C(x_0,f)$. Let $C(x_0,f) \subset \partial D'$. If $Q \in L^1(D')$, then f has a continuous extension $f\colon D \to \overline{D'}$. Moreover, if $x_0 \neq \infty \neq f(x_0)$, then*

$$|f(x) - f(x_0)| \leqslant \frac{C_n\cdot(\|Q\|_1)^{1/n}}{\log^{1/n}\left(1 + \frac{r_0}{|x-x_0|}\right)} \qquad (3)$$

for any $0 < 2r_0 < \operatorname{dist}(x_0, \partial D)$ and every $x \in B(x_0, r_0)$, where $\|Q\|_1$ is the norm of the function Q in $L^1(D')$.

2. Proof of Theorem 1. Without a loss of generality, we may assume that $x_0 \neq \infty$. Everywhere further $h(x,y)$ denotes the chordal distance between the points $x, y \in \overline{\mathbb{R}^n}$ (see e.g. [8, Definition 12.1]). Due to the discreteness of the mapping f, there is $0 < \varepsilon_0 < \operatorname{dist}(x_0, \partial D)$ such that $\infty \notin f(S(x_0,\varepsilon))$ (if $\partial D = \varnothing$, we choose any $\varepsilon_0 > 0$ with the condition mentioned above). Put $g := f|_{B(x_0,\varepsilon_0)\setminus\{x_0\}}$.

Suppose the opposite, namely, that the mapping f does not have a continuous boundary extension to a point x_0. Then in the same way the mapping g does not have a continuous boundary extension to the same point. Since the space $\overline{\mathbb{R}^n}$ is compact, $C(x_0,f) = C(x_0,g) \neq$

$\varnothing$. Then there are $y_1, y_2 \in C(x_0, f)$, $y_1 \neq y_2$, and at least two sequences $x_m, x'_m \in B(x_0, \varepsilon_0) \setminus \{x_0\}$ such that $x_m, x'_m \to x_0$ as $m \to \infty$, and $z_m := g(x_m) \to y_1$, $z'_m = g(x'_m) \to y_2$ as $m \to \infty$. We may consider that $y_1 \neq \infty$.

Let $D_* := f(B(x_0, \varepsilon_0) \setminus \{x_0\})$. We show that there exists $\varepsilon_1 > 0$ such that

$$B(y_1, \varepsilon_1) \cap f(S(x_0, \varepsilon_0)) = \varnothing. \tag{4}$$

Observe that $y_1 \in \partial D_*$. Indeed, if y_1 is an inner point of D_*, then y_1 is also inner for D', because $D_* \subset D'$. The latter contradicts the condition $C(x_0, f) \subset \partial D'$. Next, since $S(x_0, \varepsilon_0)$ is a compact set in D, then $f(S(x_0, \varepsilon_0))$ is compact in D', therefore $h(f(S(x_0, \varepsilon_0)), y_1) > \delta > 0$, where $h(A, B) = \inf\limits_{x \in A, y \in B} h(x, y)$ is the chordal distance between sets $A, B \subset \overline{\mathbb{R}^n}$. Hence

$$\operatorname{dist}(y_1, f(S(x_0, \varepsilon_0))) > \delta_1 > 0, \tag{5}$$

where $\operatorname{dist}(A, B)$ denotes the Euclidean distance between sets A and B in $\mathbb{R}^n$. By (5), the relation (4) holds for $\varepsilon_1 := \delta_1$.

Now we will reason as follows. Let $B_*(y_2, \varepsilon_2) = B(y_2, \varepsilon_2)$ for $y_2 \neq \infty$ and $B_*(y_2, \varepsilon_2) = \{x \in \overline{\mathbb{R}^n} \colon h(x, \infty) < \varepsilon_2\}$ for $y_2 = \infty$. Arguing similarly to the proof of the relation (4), we may show that there is $\varepsilon_2 > 0$ such that $B_*(y_2, \varepsilon_2) \cap f(S(x_0, \varepsilon_0)) = \varnothing$. Without loss of the

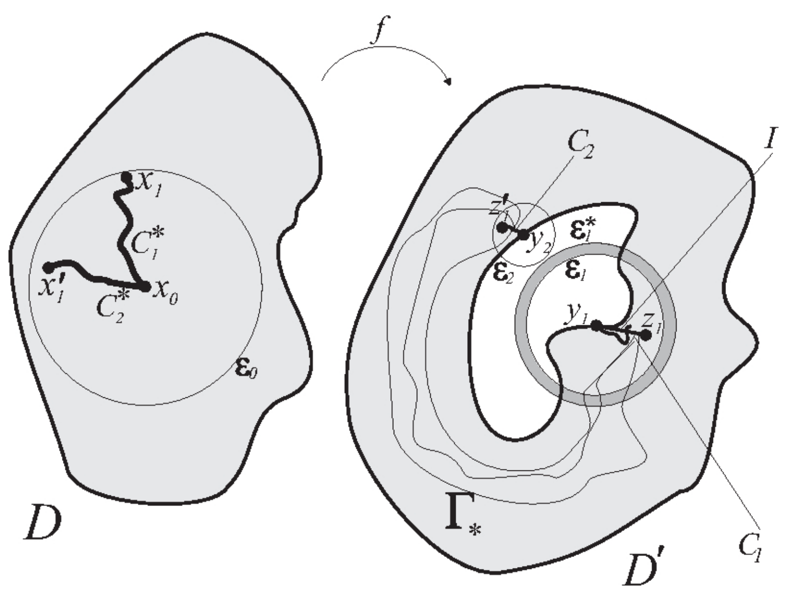

Figure 1: To the proof of Theorem 1

generality, we may assume that $\overline{B(y_1, \varepsilon_1)} \cap \overline{B_*(y_2, \varepsilon_2)} = \varnothing$, in addition, $z_m \in B(y_1, \varepsilon_1)$ and $z'_m \in B_*(y_2, \varepsilon_2)$ (see Figure 1).

Note that, $B(y_1, \varepsilon_1)$ is convex, and $B_*(y_2, \varepsilon_2)$ is linearly path connected. In this case, the points z_1 and y_1 may be joined by the segment $I(t) = z_1 + t(y_1 - z_1)$, $t \in (0, 1)$, which lies entirely in $B(y_1, \varepsilon_1)$. Similarly, points z'_1 and y_2 may be joined by a path $J = J(t)$, $t \in [0, 1]$, which lies in the "ball" $B_*(y_2, \varepsilon_2)$.

Observe that, by the construction, $|I| \cap \partial D_* \neq \varnothing \neq |J| \cap \partial D_*$.
Set $t_* := \sup\{t \colon t \in [0, 1],\ I(t) \in D_*\}$, $\quad p_* := \sup\{t \colon t \in [0, 1], J(t) \in D_*\}$.
Let $C_1 := I_{[0, t_*)}$, $\quad C_2 := J_{[0, p_*)}$.

By [4, Lemma 3.12], C_1 and C_2 have maximal f-liftings $C_1^* \colon [0, c_1) \to B(x_0, \varepsilon_0) \setminus \{x_0\}$ and $C_2^* \colon [0, c_2) \to B(x_0, \varepsilon_0) \setminus \{x_0\}$ starting at the points x_1 and x'_1, respectively. Note that the case $C_1(t) \to z_0$ as $t \to c_1 - 0$, where $z_0 \in B(x_0, \varepsilon_0) \setminus \{x_0\}$, is impossible. Indeed, by [4, Lemma 3.12], $c_1 = t_*$ and $I(t) \to f(z_0) \in D_*$, that contradicts the definition of t_*. Now, by [4, Lemma 3.12]

$$h(C_1^*(t), \partial(B(x_0, \varepsilon_0) \setminus \{x_0\})) \to 0, \qquad t \to c_1 - 0. \tag{6}$$

We show that the case $h(C_1^*(t), S(x_0, \varepsilon_0)) \to 0$ as $t \to c_1 - 0$ is also impossible. Indeed, otherwise $h(C_1^*(t_k), S(x_0, \varepsilon_0)) \to 0$ as $k \to \infty$ for some sequence $t_k \to c - 0$. Due to the compactness of the sphere $S(x_0, \varepsilon_0)$ there is a sequence $w_k \in S(x_0, \varepsilon_0)$ such that

$$h(C_1^*(t_k), S(x_0, \varepsilon_0)) = h(C_1^*(t_k), w_k).$$

Again, since the sphere $S(x_0, \varepsilon_0)$ is compact, we may assume that $w_k \to w_0$ as $k \to \infty$. Then $C_1^*(t_k) \to w_0$ as $k \to \infty$. By the continuity of f in D we obtain that

$$f(C_1^*(t_k)) = C_1(t_k) \to f(w_0) \in f(S(x_0, \varepsilon_0)) \text{ as } k \to \infty.$$

The latter contradicts the condition (4), because simultaneously $f(w_0) \in f(S(x_0, \varepsilon_0))$ and $f(w_0) \in |I| \subset B(y_1, \varepsilon_1)$. Then, it follows from (6) that,

$$h(C_1^*(t), x_0) \to 0, \qquad t \to c_1 - 0. \tag{7}$$

Applying similar considerations to the path $C_2^*(t)$, we may show that

$$h(C_2^*(t), x_0) \to 0, \qquad t \to c_2 - 0. \tag{8}$$

By (7) and (8), and by [8, Theorem 10.12] we obtain that

$$M(\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\})) = \infty. \tag{9}$$

We show that (9) contradicts the condition (1) at the point $y_0 = y_1$. Since $\overline{B(y_1, \varepsilon_1)} \cap \overline{B_*(y_2, \varepsilon_2)} = \varnothing$, we may found $\varepsilon_1^* > \varepsilon_1$, for which we still have $\overline{B(y_1, \varepsilon_1^*)} \cap \overline{B_*(y_2, \varepsilon_2)} = \varnothing$. Let $\Gamma_* = \Gamma(|C_1|, |C_2|, D_*)$. Observe that

$$\Gamma_* > \Gamma(S(y_1, \varepsilon_1^*), S(y_1, \varepsilon_1), A(y_1, \varepsilon_1, \varepsilon_1^*)). \tag{10}$$

Indeed, let $\gamma \in \Gamma_*$, $\gamma\colon [a, b] \to \mathbb{R}^n$. Since $\gamma(a) \in |C_1| \subset B(y_1, \varepsilon_1)$ and $\gamma(b) \in |C_2| \subset \overline{\mathbb{R}^n} \setminus B(y_1, \varepsilon_1)$, by [9, Theorem 1.I.5.46] there is $t_1 \in (a, b)$ such that $\gamma(t_1) \in S(y_1, \varepsilon_1)$. Without loss of generality, we may assume that $|\gamma(t) - y_1| > \varepsilon_1$ for $t > t_1$. Since $\gamma(t_1) \in B(y_1, \varepsilon_1^*)$ and $\gamma(b) \in |C_2| \subset \mathbb{R}^n \setminus B(y_1, \varepsilon_1^*)$, by [9, Theorem 1.I.5.46] there is $t_2 \in (t_1, b)$ such that $\gamma(t_2) \in S(y_1, \varepsilon_1^*)$. Without loss of generality, we may assume that $|\gamma(t) - y_1| < \varepsilon_1^*$ for $t_1 < t < t_2$. Therefore, $\gamma|_{[t_1, t_2]}$ is a subpath of γ, which belongs to $\Gamma(S(y_1, \varepsilon_1^*), S(y_1, \varepsilon_1), A(y_1, \varepsilon_1, \varepsilon_1^*))$. Thus, the relation (10) is proved.

Let us prove that

$$\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\}) > \Gamma_f(y_1, \varepsilon_1, \varepsilon_1^*). \tag{11}$$

Indeed, if $\gamma\colon [a, b] \to B(x_0, \varepsilon_0) \setminus \{x_0\}$ belongs to $\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\})$, then $f(\gamma)$ belongs to D_*, in addition, $f(\gamma(a)) \in |C_1|$ and $f(\gamma(b)) \in |C_2|$, i.e., $f(\gamma) \in \Gamma_*$. Then, according to the above and by (10), $f(\gamma)$ has a subpath $f(\gamma)^* := f(\gamma)|_{[t_1, t_2]}$, $a \leqslant t_1 < t_2 \leqslant b$, belonging to $\Gamma(S(y_1, \varepsilon_1^*), S(y_1, \varepsilon_1), A(y_1, \varepsilon_1, \varepsilon_1^*))$. Then $\gamma^* := \gamma|_{[t_1, t_2]}$ is a subpath of γ and it belongs to $\Gamma_f(y_1, \varepsilon_1, \varepsilon_1^*)$, which was required to prove. Put $\eta(t) = \begin{cases} 1/(\varepsilon_1^* - \varepsilon_1), & t \in [\varepsilon_1, \varepsilon_1^*], \\ 0, & t \in \mathbb{R} \setminus [\varepsilon_1, \varepsilon_1^*]. \end{cases}$

Observe that η satisfies (2) for $r_1 = \varepsilon_1$ and $r_2 = \varepsilon_1^*$. Applying (1) at y_1, and taking into account the relation (11), we obtain that

$$M(\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\})) \leqslant M(\Gamma_f(y_1, \varepsilon_1, \varepsilon_1^*)) \leqslant \|Q\|_1/(\varepsilon_1^* - \varepsilon_1)^n < \infty,$$

where $\|Q\|_1$ denotes L^1-norm of Q in D'. This relation and relation (9) contradict each other, which refutes the assumption of the existence of different y_1 and y_2 in $C(x_0, f)$.

Finally, if $x_0 \neq \infty$, we consider a domain $D_1 := D \setminus \{f^{-1}(\infty)\}$. Note that, due to the discreteness of the mapping f, the set $\{f^{-1}(\infty)\}$ is at most countable. Thus, D_1 is a domain, and the point x_0 is its inner point. Arguing similarly to the second part of the proof of [10, Theorem 6.4], one can show that the mapping $f\colon D_1 \to \mathbb{R}^n$ is also open and discrete. In this case, the relation (3) holds by [11, Theorem 1.1].

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